Stress response function of a two-dimensional ordered packing of frictional beads

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Abstract. – We study the stress profile of an ordered two-dimensional packing of beads in response to the application of a vertical overload localized at its top surface. Disorder is introduced through the Coulombic friction between the grains which gives some indeterminacy and allows the choice of one constrained random number per grain in the calculation of the contact forces. The so-called "multi-agent" technique we use lets us deal with systems as large as 1000×1000 grains. We show that the average response profile has a double-peaked structure. At large depth z, the position of these peaks grows with cz, while their widths scales like $\sqrt{Dz}$. c and D are analogous to "propagation" and "diffusion" coefficients. Their values depend on that of the friction coefficient μ. At small μ, we get $c_0 - c \propto \mu$ and $D \propto \mu^{\beta}$, with $\beta \sim 2.5$, which means that the peaks get closer and wider as the disorder gets larger. This behavior is qualitatively what was predicted in a model where a stochastic relation between the stress components is assumed.

The statics of granular materials is subject to an active research, see, *e.g.*, [1]. One of the main issues concerns the link between the distribution of stresses in a granular medium and its "past history" which has induced, at the microscopic level, a certain texture to the packing. As a matter of fact, the mechanical properties of an assembly of grains depends on the way it has been prepared. A now famous example is that of the sandpile: when built from the source point of a hopper, the pressure profile at the bottom of a pile shows an unambiguous minimum right below its apex [2]. By contrast, this profile is almost flat with a slight hump when the pile has been prepared by successive horizontal layers [3]. The difference between the two sandpiles could be seen at the level of the grains whose contacts and forces are oriented in relation to the external solicitation applied: during the construction, grains tend to gain strong contacts —*i.e.* carrying a large force— in the direction of compression [4]. This strong texture can be visualized in photoelasticity experiments which show clear "force chains" structures [5]. The minimum of pressure at the bottom of a sandpile can then be interpreted as a screening effect of the weight of the grains by these chains.

Conceptually, the simplest test to probe the internal packing structure of a layer of grains is to perform a stress response function: the layer is submitted to a given force $\vec{F}$ localized at a point of its top surface, and the resulting additional stress is measured at some distance

from that point. $\vec{F}$ must be small and supportable by the packing, in order to avoid contact rearrangements. The interesting point is that the shape of the averaged response function gives some information on the grain packing.

Let us take few instructive examples. Recent experiments were performed with layers of natural sand submitted to a vertical overload [6]. The response pressure at the bottom of the layer presents a single centered peaked profile whose width scales linearly with the layer thickness. Such shape and scaling resemble that of the isotropic elastic response. However, the measurements have shown that these profiles are very dependent on the system preparation: compacted sand layers have a much wider response peak than loose ones [7]. Other experiments were performed on 2d pilings of photoelastic grains [8], showing the importance of the amount of disorder. The stress response function of a regular packing of monodiperse beads has a double-peaked structure localized on the two diagonals of the triangular lattice —the behavior in 3d ordered systems is similar [9]. Polydispersity in the distribution of the grain diameters make these two peaks move closer to each other. They eventually merge into a single one at large disorder. Numerical results are also available in 2d: the stress response function of a polydisperse assembly of frictionless discs was computed using the generic isostatic property of such systems [10]. The corresponding response profile has two peaks that become asymmetrical when the packing is initially sheared. By contrast, Contact Dynamics simulations of a disordered layer of frictional pentagons show a single-peaked response, although the distribution of contact orientation has two clear oblique preferred directions [11].

The stress response function can also be used as a test to discriminate between all the models which aim at describing the statics of granular materials. As a matter of fact, the different classes of models give qualitatively different response profiles. Models with equations of the elliptic class, like those from elastic theories, typically lead to single-peaked shapes, with peak widths scaling like the depth h. They can however also give double-peaked response profiles if some strong anisotropy is included [12], but these profiles keep a linear scaling with h [13]. In models which handle scalar variables like the q-model [14], response profiles also show a single peak, but narrower: its width grows like $\sqrt{h}$ only. Note that such a behavior was claimed to be observed, but on a quite particular and rather small-scale "brick" packing [15]. Initially proposed to explain the central minimum of pressure below a sandpile built from a hopper, the models "OSL" (for "Oriented Stress Linearity"), where one postulates, as a closure relation, a phenomenological linear relationship between the stress tensor components [16, 17] of the type $\sigma_{xx} = \eta\sigma_{zz} + \eta'\sigma_{xz}$, give by contrast hyperbolic equations. Such equations are formally identical to wave propagative equations and have so-called "characteristics lines" along which stresses are transmitted. The coefficients η and η' encode here the texture of the system they describe and thus depend on the way the packing was prepared —they are "history dependent". As a consequence, the corresponding response functions have two narrow peaks resulting from the two characteristics initiated at the overload point. Edwards and Grinev, who studied isostatic assemblies of grains with infinite friction, obtained a "stress-geometry" equation which, in the simplest case, is equivalent to the OSL relation written above [18], see also [19]. Tkachenko and Witten also deduced a large-scale relation of the OSL type from microscopic quantities, but this time on isostatic packings of frictionless beads [20]. This is consistent with the result of Head *et al.* cited above, but rather contradictory to the conclusions of the extensive work of Roux on the mechanical properties of isostatic systems [21].

In this context, Eloy and Clément have proposed few years ago to study the statics of a regular two-dimensional layer of beads [22]. Although in this system each grain has six neighbors, only upper and lower contacts were considered —see fig. 1 (left). For frictionless beads, all contact forces $\vec{f}_i$ must be along the normal at the contact point, and it is easy to see
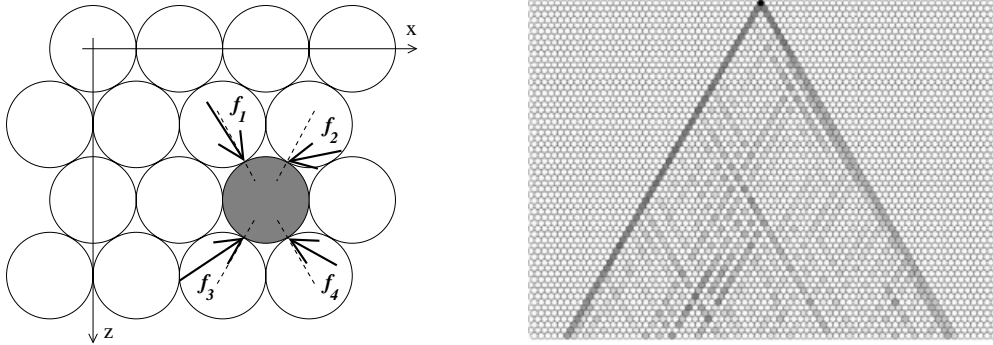
Fig. 1 – Left: grains are placed on a regular triangular lattice, but have four contacts with their upper and lower neighbors only. x is the horizontal axis, and z measures the depth from the surface. All lengths are measured in units of grains diameters. Only compressive forces are admissible for these cohesionless grains. The tangent of the angle θ_i that the force $\vec{f}_i$ makes with the contact direction (dashed lines) cannot exceed the Coulomb friction coefficient μ. These forces must satisfy the equilibrium relations $\vec{f}_1 + \vec{f}_2 + \vec{f}_3 + \vec{f}_4 = 0$ and $f_1 \sin \theta_1 + f_2 \sin \theta_2 + f_3 \sin \theta_3 + f_4 \sin \theta_4 = 0$, leaving one undetermined unknown per grain chosen at random. Right: forces on a 70×60 piling due to a vertical overload localized at its top surface. Forces are larger when the grains are darker. One can clearly see the double-peaked response. This picture has been computed with $\mu = 0.1$.

that they can all be uniquely computed layer by layer, from the force balance equations and the top boundary conditions. More interesting is the case where each bead contact is subject to a Coulombic friction condition $|f_i^{\mathrm{T}}| \leq \mu f_i^{\mathrm{N}}$, where f_i^{T} and f_i^{N} are, respectively, the tangential and the normal components of the contact force $\vec{f}_i$. $\mu = \tan \phi$ is the friction coefficient: all forces have to be in the Coulomb cone of angle ϕ. The force and torque balance equations give for each grain three relations, while there are four unknowns —note that each contact belongs to two adjacent beads. The idea is to explore the space of mechanically acceptable solutions by choosing one of the four unknowns at random among the values permitted by the friction conditions. Besides, we also impose that all contact forces have to be positive. Therefore, this simple toy model, where disorder is governed by friction only, gives a clear framework where one can compute contact forces.

The stochastic calculation begins at the surface $z = 0$, where overload forces are given, and continues deeper and deeper, one layer after another. Sometimes, the interval of admissible values in which the random number must be chosen is empty and thus makes the choice of this random number impossible. It means that for given upper forces on such a grain, at least one of the lower forces would be out of its Coulomb cone, or that a contact force is negative. A rearrangement has then to occur. In the original work of [22], the random numbers of the n previous layers were simply recomputed until the calculation could go on, where n was chosen proportionally to the number of failures encountered in the current layer. Such a procedure is of course very slow and actually makes the total computation time increase exponentially with both the size of the system and the value of the friction. The pile could then be entirely rebuilt hundreds of times before having the chance to generate no impossibilities at all. In practice, it was not possible to build, in a reasonable CPU time, piles larger than 50 grains for $\mu \leq 0.6$, which makes statistical and large-scale studies difficult. On these rather small pilings, OSL features were evidenced [22].

A much more sophisticated method called GranuSolve [23] involving a so-called "multi-agent" representation and eco-solving algorithms [24] has been used here to improve the

computation of the model. In this method, we consider each bead as an "agent" whose "goal" is to reach its static equilibrium with respect to its mechanical conditions, *i.e.* to "solve itself". When a grain is solved, it communicates its contact forces to its lower neighbors. The lower bead-agents continue the process, and solve themselves until a failure is encountered by one of them. In such a case, the bead-agent asks its upper neighbors to change their own values for the contact forces they have in common. The rearrangements are therefore treated locally by the grains and we do not need to recompute all random numbers from the upper layers but only in the local area from where the failure occurred. When averaging the data, all configurations are taken with the same weight. To avoid any bias in the scan of the space of solutions, we were particularly careful in the choice that the simulation makes for the next bead-agent that needs to get priority treatment. The choice that we finally kept was to solve grain layers from top to bottom, starting on a new random grain on each layer, and treating unsolved grains as "clusters". With this technique, the computation time is linear with the number of grains, and a complete resolution of a 1000×1000 pile takes few minutes on a G4 400 MHz machine, up to friction coefficients as large as $\mu \sim 1.7$. We shall, however, restrict our discussion to relatively small values of μ for which averaged data confidence is high.

In this letter we would like to present features of the stress response function of this model. Gravity is switched off. All grains of the top surface $z = 0$ are overloaded with a unitary vertical force $f_z = 1$, except for the central bead at $x = 0$ on which we apply $f_z = 2$, see fig. 1 (right). For a given value of the friction coefficient, all contact forces are computed and averaged over typically few thousands of samples. The uniform unitary confining overload is subtracted. We call $P(x, z)$ the resulting vertical pressure at point x and depth z. Note that this is not the standard response procedure, which should have been the following: i) apply a uniform overload at the top and solve for all contact forces in the layer; ii) add a small extra force at $x = 0$, and solve for the new forces, keeping the *same* random numbers for each grain; iii) subtract the two previous stress profiles to get the response. Of course, this would have been much slower. To justify our alternative procedure, we checked that, as long as μ is not too large, the percentage of grains that have to change their random number in step ii) in order to satisfy the friction conditions with the new forces is reasonably small —of the order of 0.5% for $\mu = 0.1$, 2.4% for $\mu = 0.3$. Besides, these rearrangements are rather localized and concern preferentially forces that are small compared to the additional overload.

Two pressure profiles are shown in fig. 2. They have been computed on 500×200 systems with $\mu = 0.3$, and measured at 30 (left) and 90 (right) grain layers depth. They have a double-peaked structure which can be well fitted by a double symmetrical Gaussian profile:

$$G(x) = \frac{1}{2\sqrt{2\pi W^2}} \left[e^{-\frac{(x+x_\mathrm{p})^2}{2W^2}} + e^{-\frac{(x-x_\mathrm{p})^2}{2W^2}} \right], \tag{1}$$

where $\pm x_\mathrm{p}$ are the positions of the peaks and W their widths. As the depth z increases, the peaks move away from each other, and they also get smaller and wider. Although residual fluctuations are large —they regress like $1/\sqrt{N_\mathrm{r}}$, where N_r is the number of realizations— we are able to extract a response whose amplitude decreases like $1/\sqrt{z}$.

We have studied the evolution of the two parameters x_p and W of the pressure profiles as a function of depth. As evidenced in fig. 3, at large z the ratios x_p/z and W^2/z saturate to some asymptotic values. In other words, we can define, in analogy to wave propagation and diffusion, a coefficient of "propagation" c and "diffusion" D such that $x_\mathrm{p} = cz$ and $W = \sqrt{Dz}$. We emphasize that these scalings are not compatible with a homogeneous, anisotropic elasticity analysis. The coefficients c and D depend on the value of the friction μ. As a trivial example, $c = c_0 = \tan 30°$ and $D = 0$ at zero friction. c and D as functions of μ are plotted in the insets of the graphs in fig. 3. As one can expect, D increases with μ. Less intuitively, c is weakly
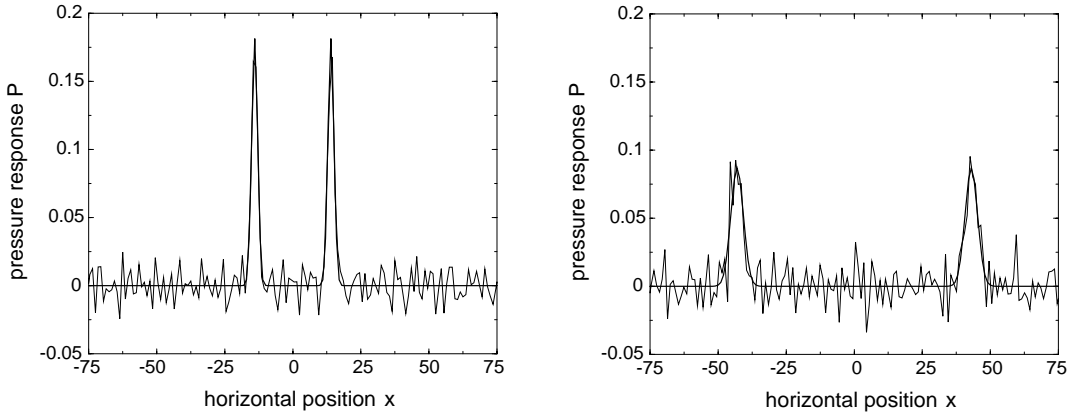
Fig. 2 – Vertical pressure profile $P(x)$ in response to a unitary force localized at $x = z = 0$, measured at 30 (left) and 90 (right) grain layers depth, *i.e.* at $z = 95.8$ and $z = 147.7$. The thin lines are numerical data averaged over 3461 different samples. The thick ones are the corresponding Gaussian fits $G(x)$ which give the peak positions $\pm x_{\mathrm{p}}$ and width W. As the depth z increases, the peaks move away from each other, and get smaller and wider. These data were computed with $\mu = 0.3$.

reduced by the friction. This last result is in agreement with the experimental observation [8]. More quantitatively, we get the following scalings: $c_0 - c \propto \mu$ and $D \propto \mu^\beta$, with $\beta \sim 2.5$.

The features of the pressure response profiles obtained in these calculations resemble very much those which were predicted in [25]. The goal of that paper was to study the role of the disorder on OSL equations, *i.e.* to close the equilibrium equations by the relation $\sigma_{xx} = \eta[1 + v(x,z)]\sigma_{zz}$, which represents the local heterogeneities of the granular packing. v is a random noise. Its mean value is zero and its correlation function is chosen to have the factorable form $\langle v(x,z)v(x',z') \rangle = \Delta^2 g_x(x - x')g_z(z - z')$, where the functions g_x and g_z are taken short-ranged. Δ is the amplitude of the disorder. When the disorder vanishes,
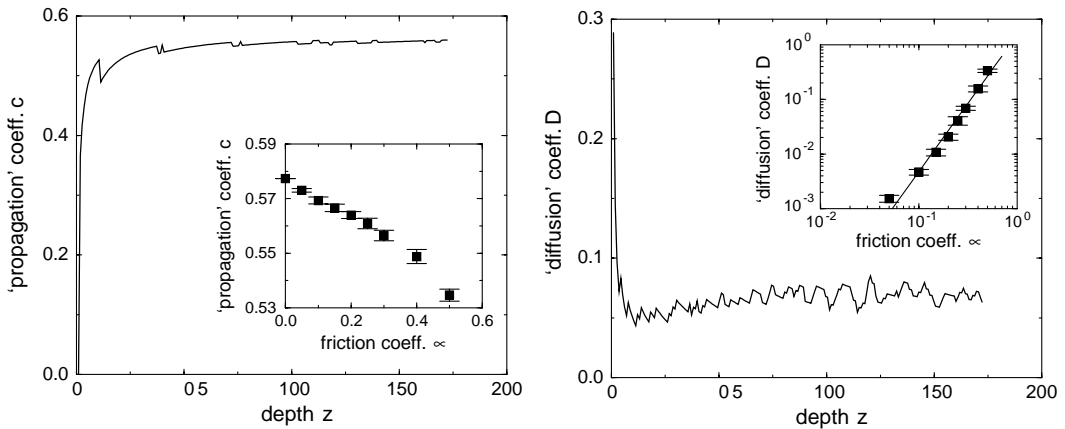

Fig. 3 – Left: at large depth, the positions of the response peaks scale linearly with z: $x_{\mathrm{p}} = \pm cz$ where c is the "propagation" coefficient. This quantity decreases linearly with the friction coefficient μ. Right: similarly, the peak width grows as in a "diffusion" process: $W = \sqrt{Dz}$ at large depth. D varies like μ^β, with $\beta \sim 2.5$ (solid line) —see also fig. 4. The curves $c(z)$ and $D(z)$ have been computed with $\mu = 0.3$.
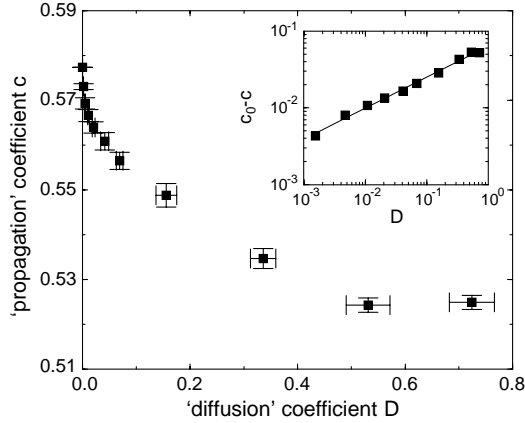
Fig. 4 – This plot shows the "propagation" coefficient c as a function of the "diffusion" one D. Each point corresponds to a different value of μ. As emphasized by the inset, $c_0 - c \propto D^\alpha$ at small D, were the best fit gives $\alpha = 1/\beta \sim 0.4 \pm 0.01$ (solid line).

the pressure response function is simply that of a wave-like equation, *i.e.* the sum of two δ-functions centered at $x_{\rm p} = \pm c_0 z$, where $c_0 = \sqrt{\eta}$. For finite disorder, although the exact shape of the response profile is more complicated than a double Gaussian like (1), it can also be characterized by the position of the peaks $x = \pm cz$ and the amplitude of their width $W = \sqrt{Dz}$. The calculation, carried out in the limit of small Δ, gave $c_0^2 - c^2 \propto \Delta^2$ and $D - D_0 \propto \Delta^2$. D_0 is the diffusion coefficient due to the underlying lattice on which the model is defined —in the case of the triangular lattice used here, $D_0 = 0$. In other words, the quantitative role of disorder that was predicted is the same as what is obtained here with this numerical model, but the scaling seems to be different. The theory in [25] predicts a linear relationship between c and D, at least at small D. Figure 4 shows the corresponding plot from the numerics. As evidenced in the inset, we rather get $c_0 - c \propto D^\alpha$, with $\alpha = 1/\beta \sim 0.4$.

How can we account for this difference? It is actually not easy to identify the clear correspondence between the random noise v of the theory in [25] and the disorder generated by a random choice of a contact force under Coulombic friction conditions. An important difference, however, lies in the fact that, in contrast to the noise v, the random variable implemented in these numerics certainly leads to a noise with a *finite mean value*. This bias is probably dominant in the behavior of c, but not on that of D.

To check this point, we plan to use one of the "microscopic justifications" of the stochastic OSL relation proposed in [25], which was called the "three-leg model". In this model, the beads lie on a rectangular lattice and transmit their forces to their three lower neighbors, the central force being random. Thus, although no friction condition was explicitly written, the spirit of the two modellings is very close. The advantage is that we know how to relate the mean of the random number to the mean propagation coefficient $\bar{c}_0$, which should allow us to extract the additional effect due to a zero mean noise. In the three-leg model, however, no backtracking calculation was implemented, and eventually negative forces (tractions) appeared at some finite depth, making the calculation invalid beyond this point. In the future, we plan to use the highly efficient multi-agent technique at work in the GranuSolve algorithm that we devised, to test the three-leg model on large-scale simulations with the self-consistent condition of positive forces.

Finally, as mentioned in the introduction of this letter, experiments showed that strongly disordered frictional granular systems actually have single-peaked response functions [6–9].

In these simulations, even at very high friction coefficient (up to 1.7), we were not able to observe any peak merging. A possible explanation may be that some additional geometrical disorder is required. We then plan to extend our work to packings where some random links between grains are opened and thus cannot transmit any force.

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