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Exact solutions of a model for granular avalanches

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Abstract. – We present exact solutions of the non-linear BCRE model for granular avalanches without diffusion. We assume a generic sandpile profile consisting of two regions of constant but different slopes. Our solution is constructed in terms of characteristic curves from which several novel predictions for experiments on avalanches are deduced. Analytical results are given for the shock condition, shock coordinates, universal quantities at the shock, slope relaxation at large times, and velocities of the active region and of the sandpile profile.

Introduction and model. – The study of avalanches and surface flows in granular materials has attracted much attention recently, both from a theoretical [1] and an experimental point of view [2]. A simple model, thought to capture some of the essential phenomena, has been proposed in [3–5], and further discussed in [6]. It is based on the assumption that a strict separation between rolling grains and static grains can be made. Dynamical "BCRE" equations for these two species, based on phenomenological arguments, can then be derived. Calling R the local density of rolling grains and h the height of the static grains, the simplest form of these equations reads:

$$H_t = -\gamma R H_x, \tag{1}$$
$$R_t = R_x + \gamma R H_x, \tag{2}$$

where H is the height of the static grains relative to the repose slope of angle θ_r: $h(t,x) = H(t,x) + x\tan(\theta_\mathrm{r})$ (the heap is sloping upwards from left to right). In the above equations, distance is measured in units of the grain diameter, while the units of time are chosen such that the downhill grain velocity v is unity. The term $\gamma R H_x$ describes the conversion of static grains into rolling grains if $H_x > 0$, or vice versa if $H_x < 0$. γ is a grain collision frequency, typically of the order of 100 Hz.

Many important phenomena are left out of the above description and can be included by adding more terms. For example, diffusion terms (such as $D_1 R_{xx}$ or $D_2 H_{xx}$ describing, *e.g.*, nonlocal dislodgement effects) will generically be present and will qualitatively change the structure of the solutions [5]. Another feature not described by the linear conversion term above is the expected saturation of rolling grains with time, rather than the exponential growth predicted by eq. (2) for a constant positive slope H_x. Non-linear saturation terms, as well as a dependence of the velocity of the rolling grains on R, are thus expected in general, and can lead to important differences from the above equations [7, 8].

Recently, these equations have been studied by Mahadevan and Pomeau (MP) [9]. They found a conservation law that relates the solutions $R(t,x)$ and $H(t,x)$ in a frame moving with the velocity of the grains. From this law, they concluded that the BCRE equations have characteristics that are straight lines, along which both $R(t,x)$ and $H(t,x)$ are constant. Independent of the initial profile $H_0(x)$, they found that a shock forms at time $t_s = -1/(\gamma R'_{0,\max})$ where $R'_{0,\max}$ is the maximum (in absolute value) of the initial gradient of rolling grains $R_0(x)$. Whereas our exact solution fulfills the same conservation law, our results for the characteristics and the shock time disagree with the results of MP. As we will discuss below, the reason for this disagreement is their implicit assumption of a very restrictive relation between the initial profiles $R_0(x)$ and $H_0(x)$.

Characteristic coordinates. – The basis of the method [10] that we used to solve eqs. (1), (2) consists of a replacement of the original equations by an equivalent system of four partial differential equations for the functions t, x, R and H, but now considered as functions of new coordinates μ and ν, which will be defined below([1]). These new equations are particularly simple in that each equation has derivatives only with respect to either μ or ν, although the mapping between the coordinates (t,x) and (μ,ν) will in general be complicated. To define the characteristic coordinates (μ,ν), we first have to specify the characteristic curves of the system (1), (2). For practical reasons, we introduce new functions $u(t,x) = 1 - R(t,x)/\alpha$ and $v(t,x) = (\alpha + x - R(t,x) - H(t,x))/\alpha$ instead of $H(t,x)$ and $R(t,x)$, where $R_0(x) = \alpha$ is a constant, see below. For these new functions, the differential equations become

$$L_1[u,v] = -u_t - \gamma\alpha(1-u)u_x + v_t + \gamma\alpha(1-u)v_x - \gamma(1-u) = 0, \tag{3}$$

$$L_2[u,v] = u_t + [-1 + \gamma\alpha(1-u)]u_x - \gamma\alpha(1-u)v_x + \gamma(1-u) = 0. \tag{4}$$

Both operators L_1 and L_2 contain linear combinations $au_t + bu_x$ of the derivatives of u (and the same holds for v). This combination means that u is differentiated in the direction given by the ratio $t/x = a/b$. Since the coefficients a and b differ for u and v and also for L_1 and L_2, the functions u and v are differentiated in each of the operators in different directions in the (t,x)-plane. Notice that the directions depend also on u itself, and therefore on the solution under consideration, which is a typical feature of non-linear systems. As noted above, our goal is to find equivalent differential equations which each contain derivatives in only one (local) direction corresponding to one of the new coordinates μ and ν. Therefore we take a linear combination $L = \lambda_1 L_1 + \lambda_2 L_2$ of the operators in eqs. (3), (4) such that the derivatives of u and v in L combine to give derivatives in the same direction, which is called a characteristic direction. Moreover, we assume that these local directions change smoothly as functions of t and x, and are given by the tangential vectors $(t_\sigma(\sigma), x_\sigma(\sigma))$ of a smooth path $(t(\sigma), x(\sigma))$ with σ as a parameter. The functions u and v along this path depend only on σ and we have, *e.g.*, $u_\sigma = u_t t_\sigma + u_x x_\sigma$. We thus obtain four homogeneous linear equations for the coefficients λ_1 and λ_2 with coefficients depending on t, x, u, v and their derivatives with respect to σ. For non-trivial solutions all possible determinants of the matrix of these coefficients have to vanish, leading to three independent equations or characteristic relations (CR). The first one can be written as a quadratic equation for the local direction $\zeta = x_\sigma/t_\sigma$ of differentiation, the

([1])The theory used here is actually more general, and can be used in the presence of non-linear saturation terms or for ripple models [11].

solutions of which are $\zeta_+ = -1$ and $\zeta_- = \gamma\alpha(1-u)$. Now, for a fixed solution u, the equations $\mathrm{d}x/\mathrm{d}t = \zeta_+$ and $\mathrm{d}x/\mathrm{d}t = \zeta_-$ are ordinary differential equations, which define two families of paths with the starting position x_0 at $t = 0$ as a parameter. These families of paths are the characteristics C_+ and C_- of the system (1), (2). From a physical point of view, they are simply the paths along which single grains (ζ_+) and $H(t,x)$ (ζ_-) freely evolve with time.

The new curved coordinate frame (μ,ν) is now defined such that the two one-parametric families of characteristics are mapped by the coordinate transformation onto the usual Cartesian coordinate frame in the (μ,ν)-plane; *i.e.*, along the characteristics the coordinate functions $\mu(t,x)$ and $\nu(t,x)$, respectively, are constant. We choose to map the line $t = 0$ onto the line given by $\mu = -\nu$. In terms of the new coordinates, we find

$$x_\nu + t_\nu = 0, \quad x_\mu - \gamma\alpha(1-u)t_\mu = 0. \tag{5}$$

Now we make use of another CR that, when evaluated along C_+ and C_- by identifying σ with ν and μ, respectively, yields the conditions

$$u_\nu + \gamma\alpha(1-u)v_\nu + \gamma(1-u)t_\nu = 0, \quad u_\mu - v_\mu + \gamma(1-u)t_\mu = 0. \tag{6}$$

These equations, together with eqs. (5), form the desired set of four equations mentioned above. Every solution of this new system satisfies the original equations (1), (2), since the Jacobian $t_\nu x_\mu - t_\mu x_\nu \sim 1+\gamma R(\mu,\nu)$ of the coordinate map does not vanish since $\gamma R(\mu,\nu) > 0$.

General solution. – Before we can construct a solution to the equivalent system (5), (6), we have to specify the initial condition along the line $\mu = -\nu$ corresponding to $t = 0$. We choose a general profile $H_0(x)$, perturbed at $t = 0$ by a uniform "rain" of rolling grains: $R_0(x) = \alpha$. In terms of the new coordinates, the initial conditions become $t_0(\mu) = 0$, $x_0(\mu) = -\mu$, $u_0(\mu) = 0$, and $v_0(\mu) = -(\mu + H_0(-\mu))/\alpha$. By introducing the function $\Delta(\mu,\nu) = -1-\gamma\alpha(1-u(\mu,\nu))$, one can show that the problem of solving the system given by eqs. (5), (6) can be reduced to the task of finding a solution to the equation $\Delta_\nu = \gamma H_0'(\nu)(1+1/\Delta)$, with initial condition $\Delta(\mu,-\mu) = -1-\gamma\alpha$. The solution of this equation can be simply expressed in terms of the so-called Lambert function, W [12]:

$$\Delta(\mu,\nu) = -1 - W\left\{\alpha\gamma \exp[\alpha\gamma + \gamma(H_0(-\mu) - H_0(\nu))]\right\}. \tag{7}$$

With this solution at hand, the solution to the system (5), (6) is determined by

$$\begin{aligned} t(\mu,\nu) &= \int_{-\nu}^{\mu} \frac{\mathrm{d}s}{\Delta(s,\nu)} = -\mu-\nu+\int_{-\nu}^{\mu}\frac{\Delta_\mu(s,\nu)\mathrm{d}s}{\gamma H_0'(-s)}, \quad x(\mu,\nu) = -\mu - t(\mu,\nu)\,, \\ R(\mu,\nu) &= -\frac{1+\Delta(\mu,\nu)}{\gamma}, \quad H(\mu,\nu) = H_0(\nu), \end{aligned} \tag{8}$$

where we have expressed the original fields $R(\mu,\nu)$ and $H(\mu,\nu)$ in terms of the functions u and v. To get the fields as functions of t and x, one has to invert the coordinate map. This can be done by using $\mu(t,x) = -t-x$ and integrating the equation for $t(\mu,\nu)$ to obtain $\nu(t,x)$. As mentioned above, the height profile $H(t,x) = H_0(\nu(t,x))$ turns out to be constant along the characteristics C_-.

Generic shape for $H(t,x)$. – In the following, we will consider a generic situation for sandpile surfaces. Suppose that one starts with a sandpile profile that consists of two regions with constant but different slopes matching with a kink at $x = 0$, and again with a constant amount of rolling grains. The slopes may be either larger or smaller than the angle of repose

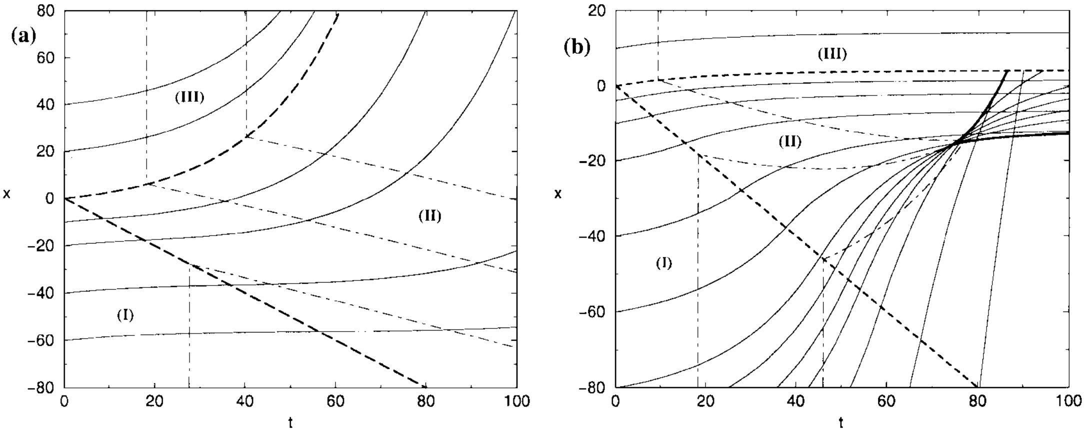

Fig. 1 – Characteristics C_- for the cases (a) $\theta_> = 0.1$, $\theta_< = -0.1$; and (b) $\theta_> = -0.1$, $\theta_< = 0.1$. For both cases, we have taken $\gamma = 0.5$, $\alpha = 0.4$. The dashed lines represent the boundaries between the different regions I, II and III explained in the text. The bold line in (b) is the envelope of the characteristics. It presents a kink at the shock position, where the characteristics cross for the first time. Along the dot-dashed lines, $R(t, x)$ is constant.

θ_r. If we denote the slope to the right (left) by $\tan(\theta_\mathrm{r} + \theta_>) \approx \theta_\mathrm{r} + \theta_>$ $(\theta_\mathrm{r} + \theta_<)$, we have $H_0(x) = \theta_> x$ for $x > 0$ and $H_0(x) = \theta_< x$ for $x < 0$. In the case of a piecewise constant $H_0'(x)$ one can easily integrate the equation for $t(\mu, \nu)$ as can be seen from eq. (8). The structure of eq. (7) suggests that we distinguish between three regions given by $\mu > 0$, $\nu < 0$ (I), $\mu, \nu < 0$ (II) and $\mu < 0, \nu > 0$ (III) ([2]). In regions I and III, one can find the explicit expression $\nu(t, x) = x + \frac{\alpha}{\theta}(1 - e^{\gamma\theta t})$ with $\theta = \theta_<$ (I) or $\theta = \theta_>$ (III); *i.e.*, the characteristics C_- are in these regions simple exponential curves. As a consequence, no shocks, *i.e.*, crossings of characteristics can appear in these two regions and the corresponding solutions are particularly simple:

$$R_{\mathrm{I(III)}}(t, x) = \alpha e^{\gamma\theta_{<(>)} t}, \quad H_{\mathrm{I(III)}}(t, x) = H_0(x) + \alpha - R_{\mathrm{I(III)}}(t, x). \tag{9}$$

The boundaries of the regions I and III in real space (t, x) are given by the conditions $x < -t$ and $x > x_1(t) = \frac{\alpha}{\theta_>}(e^{\gamma\theta_> t} - 1)$, corresponding to the $\mu = 0$ and $\nu = 0$ characteristics, respectively; see fig. 1. The boundary for region I has an obvious physical meaning: the information that there is a kink at $x = 0$ can only propagate to the left with the velocity of the moving grains, which is 1 in our rescaled units. Moreover, it is important to note that the "uphill" velocity with which the kink moves is only equal to $\gamma\alpha$ at small times, and later grows exponentially. As discussed in the introduction, this growth eventually saturates, as does the value R, or else the characteristic C_- quickly reaches the edge of the pile.

The range of x in between the above two regions corresponds to the intermediate region II. Within this range, one can obtain only an implicit solution for the coordinate map $\nu(t, x)$. It reads

$$\nu = x + \frac{1}{\gamma}\left[\frac{\Delta(-x - t, \nu) - \Delta(0, \nu)}{\theta_>} + \frac{\Delta(0, \nu) + 1 + \alpha\gamma}{\theta_<}\right], \tag{10}$$

where $\Delta(\mu, \nu) = -1 - W\left\{\alpha\gamma \exp[\alpha\gamma - \gamma(\theta_> \mu + \theta_< \nu)]\right\}$, as follows from eq. (7). The shape of $R(t, x)$ and $H(t, x)$ can be obtained directly from the last two equations of (8). In general,

([2])The region where $\mu, \nu > 0$ turns out to be mapped onto the half-space with $t < 0$, and is therefore not of physical interest.

eq. (10) has to be solved numerically, although several results can be obtained analytically. It turns out that the solutions of eq. (10) fall into two qualitatively different classes, according to the values of $\beta = \theta_>/\theta_<$ and $\theta_<$: for $\beta > 1 - \alpha\gamma$ or $\theta_< < 0$, both $R(t,x)$ and $H(t,x)$ remain continuous for all times, while for $\beta < 1 - \alpha\gamma$ and $\theta_< > 0$, the solutions develop a discontinuity in $R(t,x)$ and $H(t,x)$ beyond a finite shock time t_s. In contrast, MP predict that shocks are absent for all times in the present case $R_0(x) = \alpha$.

Examples. – The characteristics resulting from numerical solutions of eq. (10) are plotted in fig. 1. The left part of this figure was obtained for $\theta_> > 0$ and $\theta_< < 0$, corresponding to $\beta < 0$. In this case, the characteristics become more and more "diluted" as time increases, and therefore never cross —no shock. In the limit of large times, the argument of the Lambert W function becomes very large. Using the first two terms of the asymptotic expansion of W [12] we get $\nu(t,x) = [-\beta t + \ln(x + \frac{\beta t}{\beta - 1})/(\gamma\theta_<)]/(\beta - 1)$. The corresponding expression for $R(t,x)$ and $H(t,x)$ can be obtained from eq. (8). A particularly interesting quantity to look at is the local slope at, say, $x = 0$. In this limit the slope is negative and decays with time as $H_x(t, x=0) = 1/(\gamma\beta t)$(3). This means that the "true" slope h_x actually relaxes to the angle of repose θ_r at very large times. If L is the size of experimental system, then C_- reaches the boundary of the system at a time t^* such that $L \approx \frac{\alpha}{\theta_>} e^{\gamma\theta_> t^*}$. One should therefore measure a final slope $h_x \approx \theta_\mathrm{r} + \theta_</\ln(\theta_> L/\alpha)$ *smaller* than the repose angle. This result is consistent with the qualitative discussion of Boutreux and de Gennes for a similar situation [14].

Another experimentally important quantity is the velocity v_R of the "active" region. Following [5], this region can be defined by the condition $R(t,x) > R_\mathrm{min}$, where R_min is a small threshold. v_R is then given by the slope of the curves of constant $R(t,x)$, which tends to a constant in the large-t limit as can be seen in fig. 1(a). Asymptotic analysis yields $v_R = \beta/(1-\beta)$. Since $\beta < 0$, $-1 < v_R < 0$, and the avalanche proceeds *downhill*, but slower than the grains themselves. This is an effect of the non-linear term in the BCRE equations, since the linearized theory yields $v_R = -1$ [5].

The situation where $\theta_> < 0$ and $\theta_< > 0$ is qualitatively different. In this case, the characteristics cross at some finite time: a shock occurs —see fig. 1(b). A crossing point of two characteristics means that at this point, two different values of R (or H) are possible and that these functions then become discontinuous. Strictly speaking, eqs. (1), (2) are no longer valid, and the diffusion terms left out of the analysis become important to smooth out this discontinuity. In fig. 2, we plotted snapshots of the h and R profiles at different times, with and without the occurrence of a shock. One can calculate the time t_s and location x_s at which the shock occurs. For that purpose, let us introduce the envelope of the characteristic curves $x(t,\nu)$, where ν is a label. The envelope can be represented in a parametric way as $(t_\mathrm{e}(\nu), x_\mathrm{e}(\nu))$. It has the property that for each of its points there exists a characteristic that touches it tangentially. It must then fulfill the conditions $x(t_\mathrm{e}(\nu),\nu) = x_\mathrm{e}(\nu)$, $x_\nu(t_\mathrm{e}(\nu),\nu) = 0$. After some calculations, one can find the *explicit* expression for the envelope,

$$x_\mathrm{e}(\nu) = \nu - \frac{1}{\gamma\theta_<}\left[1 + \alpha\gamma + \Delta(0,\nu)\left(1 + \frac{1}{1-\beta}\right)\right], \tag{11}$$

$$t_\mathrm{e}(\nu) = -x_\mathrm{e}(\nu) + \frac{\nu}{\beta} - \frac{1}{\gamma\theta_>}\left[\alpha\gamma + \ln(\alpha\gamma) + 1 + \frac{\Delta(0,\nu)}{1-\beta} - \ln\left(-1 - \frac{\Delta(0,\nu)}{1-\beta}\right)\right]. \tag{12}$$

This envelope has two branches, separated by a kink (see fig. 1(b)) given by $\nu = \nu_\mathrm{s} = [\alpha\gamma + \ln(\alpha\gamma) - 1 + \beta - \ln(1-\beta)]/(\gamma\theta_<)$. Whereas the upper branch is parameterized by

(3)Note that this t^{-1} relaxation of the slope has also been obtained in [13] within a very different model.

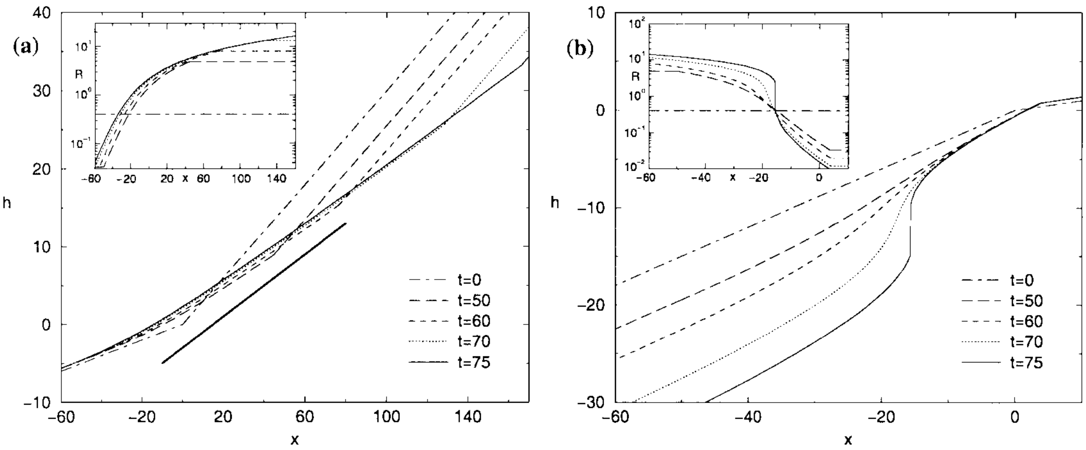

Fig. 2 – Total height profiles $h(t,x)$ for the two cases of fig. 1. The bold line in (a) shows the critical slope, which is chosen here as $\theta_r = 0.2$. In (b), the shock occurs at $t_s = 73.78$, $x_s = -16.05$. The insets depict the corresponding evolution of the amount of moving grains $R(t,x)$. For $t = 75$, the solutions $h(t,x)$ and $R(t,x)$ are not single-valued for $-15.61 < x < -15.20$, since each point within the range bounded by the envelope is covered three times by a characteristic C_-. As a visual guide, the limiting values at both boundaries are connected here by straight lines.

$-\infty < \nu < \nu_s$, the lower one corresponds to $\nu_s < \nu < \nu_c = [\beta + \alpha\gamma + \ln(-\alpha\gamma/\beta)]/(\gamma\theta_<)$. The resulting shock coordinates are

$$x_s = \frac{1}{\gamma\theta_<}\left[\ln(\alpha\gamma) - \ln(1-\beta) + 1 + \frac{1}{1-\beta}\right], \quad t_s = \frac{1}{\gamma\theta_<}\left[\left(1 - \frac{2}{\beta}\right)\ln(1-\beta) - \ln(\alpha\gamma)\right]. \quad (13)$$

The condition that (t_s, x_s) has to be located inside region II leads to the boundary between the classes with and without shock, as mentioned above. At the shock position, the amount of moving grains is *universal* (independent of the initial value α) and given by $R_s = 1/(\gamma(1-\beta))$, while $H_s = \theta_< \nu_s$. Since, typically, the grain velocity is $\sim \gamma d$ with d the grain diameter, we have in our rescaled units $\gamma \sim 1$. This shows that non-linear saturation terms can indeed be neglected at the shock if β is negative and rather large. The lower branch of the envelope saturates exponentially quickly for large t, with a characteristic time $1/(\gamma\theta_>)$ at $x_\infty = [1 + \ln(-\alpha\gamma/\beta)]/(\gamma\theta_<) > x_s$. This means that the shock stops propagating upwards. A large time expansion in the shock-free range $-t < x < x_\infty$ gives, taking the two leading terms of W, $\nu(t,x) = -(\alpha/\theta_<)\exp[\gamma\theta_< t - (\theta_</\alpha)xe^{-\gamma\theta_< t}]$. Thus, interestingly, the slope is non-monotonic within this range; after increasing for small times, it relaxes again to the initial value $\theta_<$ as $H_x(t,x) = \theta_< \exp[-(\theta_</\alpha)xe^{-\gamma\theta_< t}]$.

Discussion. – Let us summarize our major results, which could also be explored experimentally. Starting from an initial profile made up of two different slopes, we find that shocks can occur after a finite time, depending on the value of the two slopes and the initial density of rolling grains. When shocks are absent, we find that the evolution surface profile is characterized by different velocities: the kink moves upwards with a velocity of the order of $\alpha\gamma$ for early times, while the edge of the "active" region moves downwards at a velocity that only depends on the initial slopes, and is smaller than the velocity of the grains. The final slope is shown to be the angle of repose; however, for finite-size systems, one expects the final slope to be smaller by an amount that varies as $1/\ln L$. When a shock appears, we predict the time and position of this shock, as well as its density of rolling grains, which takes on a universal value. The shock is found to progress upwards only by a finite amount. Our results are in disagreement with those of MP. For the situation considered here, they predict that the initial profile is rigidly shifted along straight characteristics. Therefore, for example, the final slope would be given by $H_x(t, x = 0) = \theta_<$, which is completely different from our prediction of a decaying slope. The reason for this discrepancy comes from their implicit assumption that $R_0(x) + H_0(x) + \ln(R_0(x))/\gamma = \text{const}$, which does not hold in the cases considered here. The method presented here can be extended to more general situations. For example, each profile $H_0(x)$ can be approximated by a piecewise-linear function. Therefore, our analysis can be used to obtain analytical results for more complicated situations such as, *e.g.*, bumps or sinusoidal shapes. Another interesting situation is the case where $R_0(x)$ is localized in space. Applications of this method to the problem of ripple formation are underway. Two important physical phenomena have been neglected: diffusion terms, which are expected to be important in the presence of shocks or in the case of a localized initial $R_0(x)$ (see [5]); and non-linear effects, which lead to a saturation of the static/rolling grains conversion term. A simple way to account for the latter effect is to replace the characteristics by straight lines of velocity γR_∞ as soon as $R = R_\infty$. The influence of a density-dependent grain velocity would also be worth investigating [8].

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