Semiclassical Systems

A. Semiclassical approximations in the coherent-state representation

A first scheme of quantisation, going back the early days of Quantum Mechanics, is the WKB method. This is usually introduced by considering quantum waves of very short wavelength compared to the spatial variations of the potential, with wavefront direction and wavelength given by the velocities obtained by solving the classical ('geometric optics') problem [1].

This program is usually implemented in configuration (as opposed to phase) space. Working in phase-space has an obvious aesthetic appeal – preferring configurations to momenta amounts to 'configuration chauvinism' (in the words of M. Berry) – and it has the pleasing technical advantage of avoiding the return points, where the velocity cancels.

The problem is that in the original formulation of quantum mechanics, wavefunctions are configuration-space objects. Several authors resorted to the use of the Wigner representation of wavefunctions, an object defined in phase-space.

In our work [5], in collaboration with P. Leboeuf and M. Saraceno, we followed an alternative strategy: we used the coherent-state representation, also a natural phase-space representation of quantum mechanics. This has an extra advantage: coherent states can be defined for groups other than the (Heisenberg) group generated by coordinates and momenta. In our paper we considered also the $SU(2)$ rotation group:

$$[J_z, J_\pm] = \pm J_\pm \quad ; \quad [J_-, J_+] = -2J_z \tag{1}$$

and the $SU(1,1)$ group:

$$[K_z, K_\pm] = \pm K_\pm \quad ; \quad [K_-, K_+] = 2K_z \tag{2}$$

The latter group appears naturally in spherically symmetric problems in arbitrary dimensions, since:

$$\sum_i x_i^2 \propto K_+ \qquad \sum_i p_i^2 \propto K_- \qquad \sum_i (x_i p_i + p_i x_i) \propto K_z \tag{3}$$

The semiclassical limit is achieved in the limit of high angular momentum, or high dimensions in (3), respectively. As an example of application to the rotation group, we considered

[1] LD Landau and EM Litshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977)

the WKB quantization of the assymmetric quantum rotor in the high angular momentum limit:

$$H = \frac{J_x^2}{2} + \chi \frac{J_y^2}{2} - \frac{J_z^2}{2} \tag{4}$$

Figure 1 shows the classical trajectories, while figure 2 shows the wavefunctions obtained with our method. Wavefunctions clearly peak on trajectories.

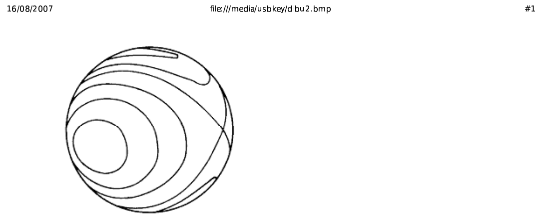

FIG. 1 Classical trajectories of the rotor in the moving frame.

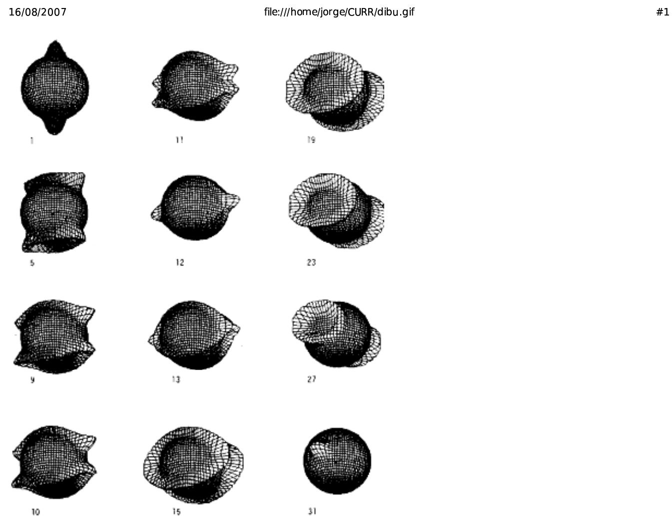

FIG. 2 Semiclassical wavefunctions of different energies for the rotor.

This work that has been generalised in several directions, and on occasion rediscovered.

The coherent-state representation is an extremely powerful and beautiful tool: we have recently used it again to treat the hydrodynamic limit of a particle-diffusion model [74].

B. Topological aspects of quantum chaos

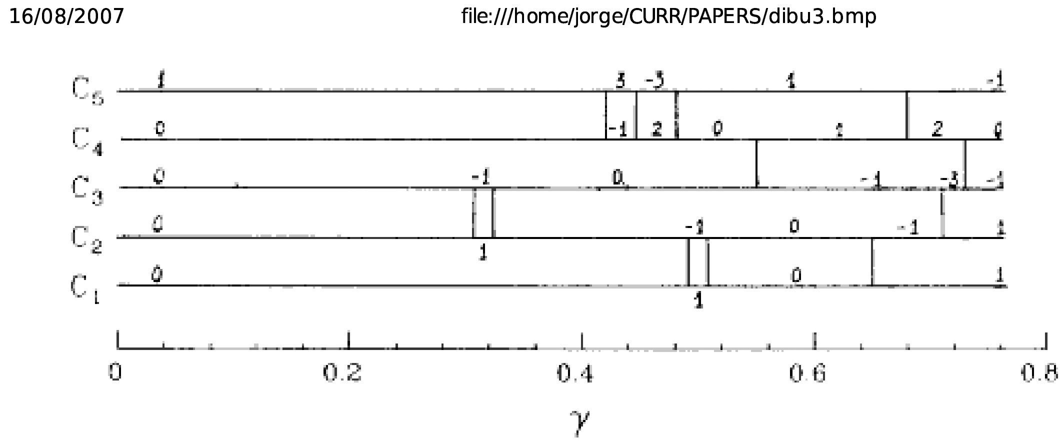

FIG. 3 Evolution of Chern indices as chaoticity is increased.

The WKB method fails for systems that are classically not integrable, since in those cases the wavefront folds and superposes hopelessly. For strongly chaotic cases, one can still retrieve information from the classical system using the knowledge of periodic orbits. These yield modulations of the *coarse grained* spectrum of energy levels, and 'scars' of the wavefunctions [2].

If one considers a *single* wavefunction, interference effects can change its form dramatically as soon as any parameter is varied: one is at the heart of chaos. An interesting question is how to characterise such an extreme sensitivity.

In Refs [10,12], together with D. Arovas, M. Feingold and P. Leboeuf, we proposed a method based on techniques developed for the Quantum Hall Effect [3]. We considered a model

$$H = -\frac{\gamma}{2\pi}\left[\cos(2\pi p) + \cos(2\pi p) K(t)\right] \qquad (5)$$

[2] Berry, M V, 1991, 'Some quantum-to-classical asymptotics', in Les Houches Lecture Series LII (1989), eds. M-J Giannoni, A Voros and J Zinn-Justin, North-Holland, Amsterdam, 251-304.

[3] D J Thouless 1981 J. Phys. C: Solid State Phys. 14 3475-3480

where $K(t)$ is a 'kick' function having a delta impulse every integer time. The parameter γ determines the degree of chaoticity, the classical system is integrable at small γ, and becomes very chaotic by $\gamma = 10$. The wavefunctions of the chaotic counterpart are periodic in q and in p, up to phases (θ_q, θ_p) – just as in the theory of bands (but whith now *two* angles). As the angles are varied, the wavefunctions change. How much they do depends on the level of chaoticity: if the system is nearly integrable one expects the wavefunctions to stay concentrated on classical trajectories (just as in Figure 1), while if it is very chaotic there are plenty of classical structures over which the wavefunction may resonate, and one expects it to vary wildly as the parameters change.

This can be quantified by the following topological number: as the parameters span the torus (θ_q, θ_p), the zeroes of the wavefunction move over the phase-space torus (q, p). How many times the zeroes wind around the latter is by definition the Chern index of the eigenfunction (or, more precisely, the band). Figure 3 shows the increase of the Chern indices of the first few wavefunctions as the chaoticity is turned on, starting from zero for the integrable case. A high Chern index means that the zeroes of the wavefunction are swept violently as the parameters are changed, thus confirming the picture.

These papers have been cited and to a certain degree generalised. the problem remains how to apply this method for a generic Hamiltonian.