Chapter Five

Propagation in periodic media : Bloch waves and evanescent waves

5.1 Bloch Wave Theory

In this section, we study the fields that can exist in an infinite metamaterial. Considering the infinite structure allows us to obtain a very precise and elegant way of characterising the (photonic) band structure and the dispersion curves of the medium.

5.1.1 *The periodic structure*

The structure is defined by repeating periodically an elementary cell Y along its basis vectors $\mathbf{a}_i$, where according to the dimension i belongs to $\{1\}, \{1,2\}$ or $\{1,2,3\}$. The underlying structure is thus an integer lattice with basis $\mathbf{a}_i$. In one-dimension, the metamaterial is simply characterized by its period $[0,d[$. In higher dimensions, the period is made of all the points M such that: $\mathbf{OM} = x^i \mathbf{a}_i, x^i \in [0,1[$ (where a sum is implied over each pair of repeated index). Generically, a vector belonging to the lattice is denoted by $\mathbf{T} = n^i \mathbf{a}_i$ with integer coefficients n^i. We also define the so-called reciprocal lattice, which is a lattice whose basis vectors $\mathbf{a}^i$ are defined by: $\mathbf{a}^i \cdot \mathbf{a}_j = 2\pi\delta^i_j$. where δ^j_i is the Kronecker symbol. The basic cell of this lattice is denoted by Y^*, and it is the so-called Brillouin zone. Explicitly, it is defined as the set of points P such that $\mathbf{OP} = y_i \mathbf{a}^i, y_i \in [-1/2, 1/2[$. Generically a vector belonging to the reciprocal lattice is denoted by $\mathbf{G} = n_i \mathbf{a}^i$ for some integers n_i.

5.1.2 *Waves in a homogeneous space*

We want to be able to characterize the waves that can exist in an infinite periodic medium. Let us first consider the case of a homogeneous medium and the scalar wave equation for harmonic waves: $\Delta u + k^2 u = 0$. We want to find all the bounded functions u satisfying this equation, for all values of k. Let us pretend for the moment that we do not know that a basis of solutions is the plane waves of the form $\exp(i\mathbf{k}\cdot\mathbf{y})$. If we rewrite the problem in the following form: Find a function u and a positive number E such that: $-\Delta u = Eu$, it appears as a spectral one: the

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point is now to determine the eigenvalues and eigenvectors of some linear operator (here the Laplacian). In order to do so, let us Fourier transform the function $u(\mathbf{y})$:

$$u(\mathbf{y}) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{y}} d\mathbf{k}. \tag{5.1}$$

This leads to the relation: $(\|\mathbf{k}^2\| - E)\widehat{u}(\mathbf{k}) = 0$. This shows that $\widehat{u}(\mathbf{k})$ is not a function but a Schwartz distribution, in fact: $\widehat{u} = A(\mathbf{k})\delta(\|\mathbf{k}^2\| - E)$, that is, it is proportional to the Dirac distribution whose support is a spherical shell of radius E.[1] We then obtain $u(\mathbf{y}) = (2\pi)^{-N/2} \int_{\mathbb{S}_E^{N-1}} A(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{y}} d\mathbf{k}$.[2] A solution to the spectral problem (**??**) is thus a continuous sum of plane waves with some amplitude factor: the spectral problem is parametrized by plane waves. The very reason why this decomposition works is the fact that all translations of space $T_{\mathbf{u}}, \mathbf{u} \in \mathbb{R}^N$,[3] commute with the Laplacian, and therefore the translations and the Laplacian have a common basis of eigenvectors. This basis is formed with plane waves $(T_{\mathbf{u}}(e^{i\mathbf{k}\cdot\mathbf{y}}) = e^{-i\mathbf{k}\cdot\mathbf{u}} e^{i\mathbf{k}\cdot\mathbf{y}})$.

If the wavenumber $k = \|\mathbf{k}\|$ is given, the set of parameters is a spherical shell of dimension $N-1$ (note that a shell of dimension 0 is just a pair of points symmetric with respect to the origin). The decomposition can also be taken in reverse order: we can begin by fixing the wavevector $\mathbf{k}$. Then for the plane wave with wavevector $\mathbf{k}$ the eigenvalue is k^2 and the associated frequency is $\omega = ck$. From this point of view, with one frequency is associated only one energy. Still, we can remark that it is possible to decompose any $\mathbf{k} \in \mathbb{R}^N$, in the following form:

$$\mathbf{k} = \mathbf{k}_b + 2\pi\mathbf{p},$$

where $\mathbf{p}$ is a vector with integer components (i.e. $\mathbf{p} \in \mathbb{Z}^N$) and $\mathbf{k}$ belongs to[4] $Y^* = [-\pi, \pi[^N$. If we use only Y^* and not the entire space $\mathbb{R}^N$ to parametrize the spectral problem, then with a wavevector $\mathbf{k}_b \in Y^*$ is now associated an infinite set of frequencies $\omega_p = c|\mathbf{k}_b + 2\pi\mathbf{p}|$ and an infinite set of eigenvectors, the so-called Bloch waves:

$$\psi_{\mathbf{p}}(\mathbf{k}_b, \mathbf{y}) = \exp(i\mathbf{k}_b \cdot \mathbf{y})\phi_{\mathbf{p}}(\mathbf{k}_b, \mathbf{y}), \tag{5.2}$$

where $\phi_{\mathbf{p}}(\mathbf{y}) = \exp(2i\pi\mathbf{p}\cdot\mathbf{y})$ (note that it is a $Y-$periodic function). Using this formulation, the Fourier integral of $u(\mathbf{y})$ can be written:

$$u(\mathbf{y}) = \int_{Y^*} \sum_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{y}) e^{i\mathbf{k}_b\cdot\mathbf{y}} \phi_{\mathbf{p}}(\mathbf{y}) d\mathbf{y}, \tag{5.3}$$

where $u_{\mathbf{p}}(\mathbf{y}) = (2\pi)^{-N/2}\widehat{u}(\mathbf{k} - 2\pi\mathbf{p})$. This expression can be extended so as to deal with non-homogeneous media.

[1] its action on a regular test function ϕ is $\langle \delta(\|\mathbf{k}\|^2 - E), \psi \rangle = \int_{\mathbb{S}_E^{N-1}} \phi(s) ds$, where $\mathbb{S}_E^{N-1}$ is the sphere of radius $\sqrt{E}$ in $\mathbb{R}^N$

[2] For $N = 2$ it reads as:

$$u(y_1, y_2) = (2\pi)^{-1} \int_{-\sqrt{E}}^{\sqrt{E}} e^{ik_1 y_1} \left(A^+(k_1) e^{i\sqrt{E-k_1^2}y_2} + A^-(k_1) e^{-i\sqrt{E-k_1^2}y_2} \right) dk_1.$$

In this formula, y_1 and y_2 can be exchanged.

[3] A translation acts on a function f of the variable $\mathbf{y} \in \mathbb{R}^N$ in the following way: $T_{\mathbf{u}}(f)(\mathbf{y}) = f(\mathbf{y} - \mathbf{u})$.

[4] The notation is not innocent. What we do amounts to decomposing the space $\mathbb{R}^N$ into cubic boxes of side 1, which endows it with a lattice structure of basic cell $Y = [0, 1[^N$, whose corresponding Brillouin zone is Y^*.

5.1.3 Bloch modes

Let us now consider a metamaterial with basic cell Y and a partial differential operator with periodic coefficient $\mathcal{L}$ that describes wave propagation in the medium. For instance, operator $\mathcal{L}$ can be the Helmholtz-like operator: $-\varepsilon(\mathbf{y})^{-1}\Delta$ or $-\mathrm{div}(\varepsilon(\mathbf{y})^{-1}\,\mathbf{grad}(\cdot))$, or else for the full Maxwell system: $\mathbf{curl}(\varepsilon(\mathbf{y})^{-1}\,\mathbf{curl}(\cdot))$. In such a situation, the Fourier transform cannot lead easily to the solution, because of the inhomogeneity of space. The idea behind Bloch waves is to find a way to generalize the Fourier transform to operators with periodic coefficients. The first point at issue is that the space is now really periodic and not homogeneous. It is no longer invariant under an arbitrary translation but only by those of the form $n^i\mathbf{a}_i, n^i \in \mathbb{Z}$. The consequence is that plane waves are no longer solutions of the propagation equation. However, the form of Bloch waves given in Eq. 5.2, where now $\phi_{\mathbf{p}}(\mathbf{k},\mathbf{y})$ is an unknown Y-periodic function, shows that they transform in the following way under a translation of the direct lattice:

$$\psi_{\mathbf{p}}(\mathbf{k}_b,\mathbf{y}+\mathbf{T}) = \exp(i\mathbf{k}_b\cdot\mathbf{T})\psi_{\mathbf{p}}(\mathbf{k}_b,\mathbf{y}) \tag{5.4}$$

The function is then said to be pseudo-periodic. This suggests that we look for a decomposition such as that in Eq. 5.23, where the plane waves of the homogeneous space are now replaced by Bloch waves (i.e. the product of a plane wave by a $Y-$periodic function).

For such a decomposition to hold, we have to show that we can reduce the spectral problem by imposing the quasi-periodicity condition (5.4) and obtain an equivalent problem. In other words, we do not want to remove any solution by requesting that they be pseudo-periodic.

The first step consists in associating with any square integrable function u on $\mathbb{R}^N$ a family of pseudo-periodic functions indexed by $\mathbf{k}$. This is done by means of the Wannier transform:

$$\mathcal{W}(u)(\mathbf{k},\mathbf{y}) = \sum_{\mathbf{T}} u(\mathbf{y}-\mathbf{T})e^{i\mathbf{k}\cdot\mathbf{T}}\,, \tag{5.5}$$

where the sum runs over all vectors of the direct lattice, and $\mathbf{k}$ belongs to the Brillouin zone Y^*. It is easy to check that the transformed function is quasi-periodic with respect to $\mathbf{y}$:

$$\begin{aligned}\mathcal{W}(\mathbf{k},\mathbf{y}+\mathbf{T}') &= \textstyle\sum_{\mathbf{T}} u(\mathbf{y}+\mathbf{T}'-\mathbf{T})e^{i\mathbf{k}\cdot\mathbf{T}} &(5.6)\\ &= e^{i\mathbf{k}\cdot\mathbf{T}'}\textstyle\sum_{\mathbf{T}''} u(\mathbf{y}-\mathbf{T}'')e^{i\mathbf{k}\cdot\mathbf{T}''} &(5.7)\\ &= e^{i\mathbf{k}\cdot\mathbf{T}'}\mathcal{W}(\mathbf{k},\mathbf{y}+\mathbf{T}'). &(5.8)\end{aligned}$$

We can get back the original function by applying the inverse transform:

$$\mathcal{W}^*(\psi)(\mathbf{y}) = \frac{1}{|Y^*|}\int_{Y^*}\psi(\mathbf{k},\mathbf{y})d\mathbf{k}\,. \tag{5.9}$$

Let us apply the inverse Wannier transform to $\mathcal{W}(u)$:

$$\begin{aligned}\mathcal{W}^*\left(\mathcal{W}(u)(\mathbf{k},\mathbf{y})\right) &= \tfrac{1}{|Y^*|}\textstyle\int_{Y^*}\mathcal{W}(u)(\mathbf{k},\mathbf{y})d\mathbf{k} &(5.10)\\ &= \tfrac{1}{|Y^*|}\textstyle\int_{Y^*}\sum_{\mathbf{T}} u(\mathbf{y}-\mathbf{T})e^{i\mathbf{k}\cdot\mathbf{T}}d\mathbf{k} &(5.11)\\ &= \textstyle\sum_{\mathbf{T}} u(\mathbf{y}-\mathbf{T})\tfrac{1}{|Y^*|}\int_{Y^*}e^{i\mathbf{k}\cdot\mathbf{T}}d\mathbf{k}. &(5.12)\end{aligned}$$

The conclusion follows by the identity: $\frac{1}{|Y^*|}\int_{Y^*} e^{i\mathbf{k}\cdot\mathbf{T}}d\mathbf{k} = \delta_{\mathbf{0}}^{\mathbf{T}}$. Conversely, starting with a pseudo-periodic function $\psi(\mathbf{k},\mathbf{y})$ we have:

$$\mathcal{W}\left(\mathcal{W}^*(\psi)\right)(\mathbf{k}',\mathbf{y}) = \frac{1}{|Y^*|}\sum_{\mathbf{T}}\int_{Y^*}\psi(\mathbf{k},\mathbf{y}-\mathbf{T})e^{i\mathbf{k}'\cdot\mathbf{T}}d\mathbf{k}$$

Using the pseudo-periodicity of ψ, we get

$$\mathcal{W}\left(\mathcal{W}^*(\psi)\right)(\mathbf{k}',\mathbf{y}) = \frac{1}{|Y^*|}\int_{Y^*}\psi(\mathbf{k},\mathbf{y})\sum_{\mathbf{T}}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{T}}d\mathbf{k}$$

where the conclusion follows from the identity:

$$\frac{1}{|Y^*|}\sum_{\mathbf{T}}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{T}} = \sum_{\mathbf{G}}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{G})\,.$$

In order to be clearer, we state explicitly the spaces involved: the Wannier transform is defined on the space $\mathcal{H} = L^2(\mathbb{R}^N)$ and is into $\mathcal{V}$ which is the set of functions defined on $\mathbb{R}^N \times Y^*$ such that

$$\|\psi\|^2 = \int_{Y^*\times Y}|\psi(\mathbf{y},\mathbf{k})|^2 d\mathbf{k}d\mathbf{y} < +\infty\,,$$

$\mathcal{V}$ is a Hilbert space for the scalar product:

$$(\psi_1,\psi_2) = \int_{Y^*\times Y}\psi_1(\mathbf{k},\mathbf{y})\overline{\psi_2(\mathbf{k},\mathbf{y})}\,d\mathbf{k}d\mathbf{y}.$$

We have $\mathcal{W}\mathcal{W}^* = I_{\mathcal{V}}$ and $\mathcal{W}^*\mathcal{W} = I_{\mathcal{H}}$. We can obtain all functions[5] of $\mathcal{V}$ by fixing first the wavevector $\mathbf{k}$ and then by considering the functions $u_{\mathbf{k}}$ such that:

$$u_{\mathbf{k}}(\mathbf{y}+\mathbf{T}) = e^{i\mathbf{k}\cdot\mathbf{T}}u_{\mathbf{k}}(\mathbf{y}).$$

This set of functions is denoted by $\mathcal{H}_{\mathbf{k}}$:

$$\mathcal{H}_{\mathbf{k}} = \left\{u, u(\mathbf{y}+\mathbf{T}) = e^{i\mathbf{k}\cdot\mathbf{T}}u(\mathbf{y}), \|u\|^2 = \frac{1}{|Y|}\int_Y |u(\mathbf{y})|^2 d\mathbf{y} < +\infty\right\}. \tag{5.13}$$

We now have a mathematical set-up that shows that any square integrable function can be considered as a sum of quasi-periodic functions, by using the Wannier transform. In order to be able to use this transform, it should commute with $\mathcal{L}$. Indeed, we want to find u such that : $\mathcal{L}(u) = Eu$. If the commutator $[\mathcal{W},\mathcal{L}] = \mathcal{W}\mathcal{L} - \mathcal{L}\mathcal{W} = 0$, then we get, by applying $\mathcal{W}$ to this equation

$$\mathcal{L}(\mathcal{W}(u)) = E\mathcal{W}(u)\,.$$

The commutation is due to the invariance of the medium, and hence that of the permittivity, by translation along the vectors $\mathbf{T}$ of the direct lattice. The eigenvalues and eigenvectors of $\mathcal{W}$ can thus be obtained by solving the equation in $\mathcal{H}_{\mathbf{k}}$, then by varying $\mathbf{k}$ in Y^*. For each $\mathbf{k} \in Y^*$, we therefore look for functions $u \in \mathcal{H}_{\mathbf{k}}$ such that $\mathcal{L}(u) = E(\mathbf{k})u$. Once the eigenvalues $E(\mathbf{k})$ are obtained the corresponding frequencies are $\omega/c = \sqrt{E(\mathbf{k})}$. In the Hilbert space $\mathcal{H}_{\mathbf{k}}$, the operator $\mathcal{L}$ has a set of

[5] Mathematically speaking, the space $\mathcal{V}$ can be identified with a direct Hilbertian integral: $\mathcal{V} = \int_{Y^*}^{\oplus}\mathcal{H}_{\mathbf{k}}d\mathbf{k}$, which corresponds to the notion of a "continuous" sum of Hilbert spaces.

quasi-periodic eigenfunctions that form a Hilbert-basis, the so-called Bloch waves. They are numbered by an integer[6] p, and are of the form:

$$\psi_p(\mathbf{k}, \mathbf{y}) = e^{i\mathbf{k}\cdot\mathbf{y}}\phi_p(\mathbf{k}, \mathbf{y}), \tag{5.14}$$

where ϕ_p is a $Y-$periodic function. They are associated with a set of eigenvalues $E_p(\mathbf{k})$ that are ordered in ascending order: $E_1 < E_2 < \ldots < E_p < \ldots$ By varying $\mathbf{k}$ in Y^*, we obtain all of the eigenvalues as a set of surfaces indexed by p.
We have in fact obtained a new way of decomposing a square integrable function[7] u. Indeed,for a given $\mathbf{k}$, $\mathcal{W}(u)(\cdot, \mathbf{k})$ belongs to $\mathcal{H}_{\mathbf{k}}$, and therefore it can be expanded on the basis $\{\psi_p\}_p$:

$$\mathcal{W}(u)(\mathbf{k}, \mathbf{y}) = \sum_p W_p(\mathbf{k})\psi_p(\mathbf{k}, \mathbf{y})$$

where $W_p(\mathbf{k}) = \int_Y \mathcal{W}(u)(\mathbf{k}, \mathbf{y})\overline{\phi_p(\mathbf{k}, \mathbf{y})}e^{-i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}$, which can be written:

$$W_p(\mathbf{k}) = \int_Y \sum_{\mathbf{T}} u(\mathbf{y} - \mathbf{T})e^{i\mathbf{k}\cdot\mathbf{T}}\overline{\phi_p(\mathbf{k}, \mathbf{y})}e^{-i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y} \tag{5.15}$$

$$= \sum_{\mathbf{T}} \int_{Y+\mathbf{T}} u(\mathbf{y})\overline{\phi_p(\mathbf{k}, \mathbf{y})}e^{-i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y} \tag{5.16}$$

$$= \int_{\mathbb{R}^N} u(\mathbf{y})\overline{\phi_p(\mathbf{k}, \mathbf{y})}e^{-i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y} \tag{5.17}$$

By using the inverse transform $\mathcal{W}^*$, we get:

$$u(\mathbf{y}) = \frac{1}{|Y^*|}\int_{Y^*} \sum_p W_p(\mathbf{k})\phi_p(\mathbf{k}, \mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{k}\,.$$

Finally, we can state the Bloch decomposition theorem, which is a generalization of the Fourier transform:

Theorem 5.1. *Let u be a function of $L^2(\mathbb{R}^N)$, where its p^{th} Bloch coefficient is defined by:*

$$\widehat{u}_p(\mathbf{k}) = \int_{\mathbb{R}^N} u(\mathbf{y})\overline{\phi_p(\mathbf{k}, \mathbf{y})}e^{-i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\ .$$

We then have the Bloch decomposition formula:

$$u(\mathbf{y}) = \frac{1}{|Y^*|}\int_{Y^*} \sum_p \widehat{u}_p(\mathbf{k})\phi_p(\mathbf{k}, \mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{k}$$

and the Parseval identity:

$$\int_{\mathbb{R}^N} |u(\mathbf{y})|^2 d\mathbf{y} = \frac{1}{|Y^*|}\int_{Y^*} \sum_p |\widehat{u}_p(\mathbf{k})|^2 d\mathbf{k}.$$

[6]This is the band index, and it labels the various allowed frequencies for a given wavevector $\mathbf{k}$.
[7]The reader can remark that in fact the decomposition is valid in $\mathbb{R}^N$ for an arbitrary N, not necessarily for $N = 1, 2, 3$.

5.2 Computation of band structures

5.2.1 *Two-dimensional metamaterials*

The previous section has shown that the bounded fields that can exist in an infinite crystal could be parametrized by the set Y^*. In order to describe in a concise way these fields, only a reduced part of the Brillouin zone is used. Indeed, the crystal is invariant under some group of symmetries, and hence it is not necessary to use the entire Brillouin zone in order to compute the spectrum. For instance, let us consider a $2D$ photonic crystal with a square lattice of side a. The crystal is made of rods of radius R and relative permittivity ε_2 embedded in a matrix of relative permittivity ε_1 (See Fig. 5.1.). The Brillouin zone is a square of side $2\pi/a$ (See Fig. 5.2.). It suffices to describe only $1/8^{th}$ of this square (namely, the triangle in bold lines on Fig. 5.2.) in order to characterize the spectrum entirely. The description can be further reduced by restricting $\mathbf{k}$ to the lines connecting the points of higher symmetries: Γ, X, M (a more detailed treatment of the symmetries can be found in [**?**]).

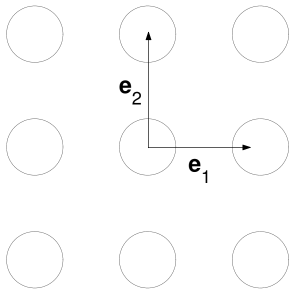

Figure 5.1. A few cells of a dielectric metamaterial with square lattice.

Let us characterize the $z-$independent fields that can exist in such a structure. The Maxwell system can be reduced to two fundamental cases of polarization:

(1) The $E_{||}$ case, in which the electric field is parallel to the z axis, and its z component E_z satisfies

$$-\varepsilon^{-1}(\mathbf{y})\Delta E_z = \left(\frac{\omega}{c}\right)^2 E_z \tag{5.18}$$

and:

(2) The $H_{||}$ case, in which the magnetic field is parallel to the z axis, and its z component H_z satisfies:

$$-\mathrm{div}(\varepsilon^{-1}(\mathbf{y})\mathbf{grad}H_z) = \left(\frac{\omega}{c}\right)^2 H_z. \tag{5.19}$$

In both cases, we have to compute the Fourier series of $\varepsilon^{-1}(\mathbf{y})$. We write:

$$\varepsilon^{-1}(\mathbf{y}) = \sum_{\mathbf{G}} \widehat{\varepsilon^{-1}}(\mathbf{G}) e^{i\mathbf{G}\cdot\mathbf{y}}.$$

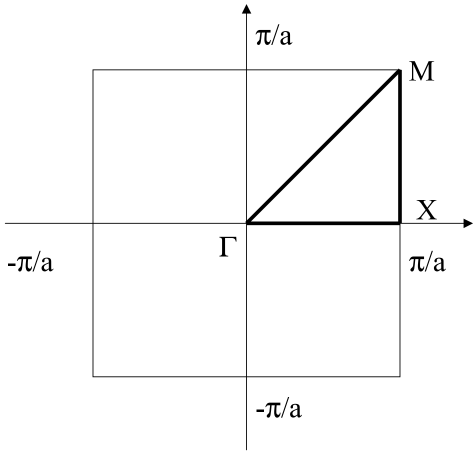

Figure 5.2. The Brillouin zone Y^*.

For a circular fibre, we have explicitly:

$$\widehat{\varepsilon^{-1}}(\mathbf{G}) = \begin{cases} \frac{1}{\varepsilon_1} f + \frac{1}{\varepsilon_2}(1-f), & \mathbf{G} = 0 \\ \left[\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right] f \frac{2J_1(\|\mathbf{G}\|R)}{\|\mathbf{G}\|R}, & \mathbf{G} \neq 0 \end{cases} \tag{5.20}$$

where $f = \frac{\pi R^2}{a^2}$ is the filling fraction and J_1 is the Bessel function of order 1.
We now choose a vector $\mathbf{k} \in Y^*$ and look for Bloch waves solving these equations. First, we expand any Bloch wave associated with E_z and H_z in Fourier series:

$$\begin{cases} E_z(\mathbf{k},\mathbf{y}) = \sum_{\mathbf{G}} \widehat{E}(\mathbf{k},\mathbf{G}) e^{i(\mathbf{G}+\mathbf{k})\cdot\mathbf{y}} \\ H_z(\mathbf{k},\mathbf{y}) = \sum_{\mathbf{G}} \widehat{H}(\mathbf{k},\mathbf{G}) e^{i(\mathbf{G}+\mathbf{k})\cdot\mathbf{y}} \end{cases}. \tag{5.21}$$

It suffices now to insert the expansions into Eqs. 5.18, 5.19, to obtain two eigenvalues problems:

$$\begin{cases} \sum_{\mathbf{G}'} (\mathbf{k}+\mathbf{G}) \cdot (\mathbf{k}+\mathbf{G}') \widehat{\varepsilon^{-1}}(\mathbf{G}-\mathbf{G}') \widehat{H}(\mathbf{k},\mathbf{G}') = \left(\frac{\omega}{c}\right)^2 \widehat{H}(\mathbf{k},\mathbf{G}) \\ \sum_{\mathbf{G}'} (\mathbf{k}+\mathbf{G}')^2 \widehat{\varepsilon^{-1}}(\mathbf{G}-\mathbf{G}') \widehat{E}(\mathbf{k},\mathbf{G}') = \left(\frac{\omega}{c}\right)^2 \widehat{E}(\mathbf{k},\mathbf{G}) \end{cases} \tag{5.22}$$

Solving these linear systems for a given value of $\mathbf{k}$ and keeping only the positive eigenvalues, we obtain the allowed frequencies $\frac{\omega_p}{c}$. By varying $\mathbf{k}$ along the lines connecting the points of high symmetry, we obtain the curves in Figs. 5.3. and **??** (a complete example of a triangular lattice is given in [**?**]).

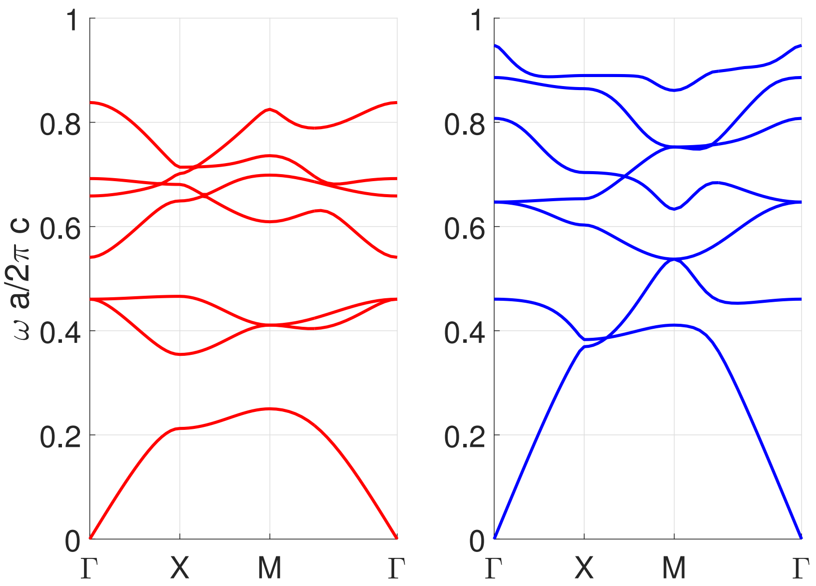

Figure 5.3. The band structure for the structure depicted in fig. 5.1.. The relative permittivity of the rods is 12 and the radius to period ratio is 1/4

5.3 Periodic Waveguides

5.3.1 *Bloch modes*

Although real structures are finite and one is often interested in the study of defects, the determination of modes in ideal periodic structures is of foremost importance. The Floquet-Bloch theory reduces the problem to the study of a single cell [?] as recalled in Sec. 5.1 p. 105 in this book. The purpose of this section is to show how to combine this feature with finite element modelling in order to obtain numerical solutions for propagating modes in periodic structures. We consider a structure still invariant along the z-axis but now also periodic in the xy-plane. Given two linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$ in the xy-plane, the set of points $n\mathbf{a} + m\mathbf{b}$ is called the *lattice*. The *primitive cell* Y is a subset of $\mathbb{R}^2$ such that for any point $\mathbf{r}'$ of $\mathbb{R}^2$ there exist unique $\mathbf{r} = x\varepsilon_x + y\varepsilon_y \in Y$ and $n, m \in \mathbb{Z}$ such that $\mathbf{r}' = \mathbf{r} + n\mathbf{a} + m\mathbf{b}$. A function $U(\mathbf{r})$ is *Y-periodic* if $U(\mathbf{r} + n\mathbf{a} + m\mathbf{b}) = U(\mathbf{r})$ for any $n, m \in \mathbb{Z}$. The waveguide is Y-periodic if $\varepsilon_r(x, y)$ and $\mu_r(x, y)$ are Y-periodic functions. Possible PECs and PMWs have boundaries that form a Y-periodic pattern.

The problem reduces to looking for *Bloch wave* solutions $\mathbf{U_k}$ that have the form (Bloch theorem, see Sec. 5.1 p. 109):

$$\mathbf{U_k}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{U}(\mathbf{r}) = e^{i(k_x x + k_y y)}\mathbf{U}(x, y)\ ,\ \forall\ (x, y) \text{ in } \mathbb{R}^2 \tag{5.23}$$

where $\mathbf{U}(x, y)$ is a Y-periodic function and $\mathbf{k} = k_x\varepsilon^x + k_y\varepsilon^y \in Y^* \subset \mathbb{R}^2$ is a parameter (the *Bloch vector* or quasi-momentum in solid state physics). $Y^* \subset \mathbb{R}^2$ is the *dual*