

# Comportement Élastique

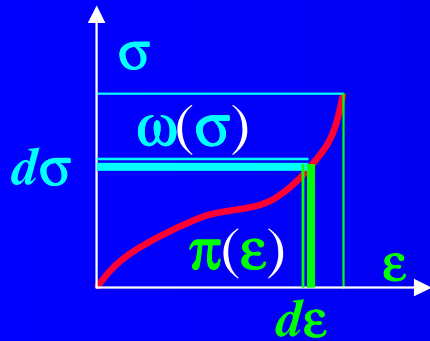
- **I** Élasticité Cristalline et Haute Élasticité
- **II** Les Modules Élastiques
- **III** Solutions Élastiques en Statique
- **IV** Théorèmes Énergétiques
- **V** La Propagation du Son
- **VI** Les Problèmes Plans
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# I Élasticité Cristalline et Haute Élasticité

- I-1 Les Potentiels d'Élasticité
- I-2 Les Deux Élasticités
- I-3 Élasticité Linéaire des Milieux Isotropes
- I-4 Potentiels en Élasticité Linéaire



# I-1 Les Potentiels d'Élasticité



$$F = E - TS$$

$$\Phi = F - \text{Tr}(\sigma \epsilon)$$

$E$  densité volumique d'énergie interne

$F$  densité volumique d'énergie libre

$S$  densité volumique d'entropie

$\Phi$  densité volumique de potentiel thermodynamique

**Réversibilité Mécanique : Travail de Déformation  $\text{Tr}(\sigma \epsilon)$  indépendant du chemin suivi**

$\pi(\epsilon)$  densité volumique de Potentiel d'Élasticité

$$d\pi = \text{Tr}(\sigma d\epsilon)$$

$$\bar{\sigma} = \frac{\partial \pi}{\partial \bar{\epsilon}} \quad \bar{\epsilon} = \frac{\partial \omega}{\partial \bar{\sigma}}$$

$\omega(\sigma)$  densité volumique de Potentiel Complémentaire

$$d\omega = \text{Tr}(\epsilon d\sigma)$$

$$d(\pi + \omega) = d\text{Tr}(\sigma \epsilon) \Rightarrow \pi + \omega = \text{Tr}(\sigma \epsilon) \quad \text{à partir d'un état naturel } \sigma = 0 \quad \epsilon = 0$$

**Élasticité parfaite**

$$\text{Cauchy} \Rightarrow \frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \epsilon_{kl}}{\partial \sigma_{ij}}$$

$$dE = \text{Tr}(\sigma d\epsilon) + T dS$$

$$\bar{\sigma} = \frac{\partial \pi}{\partial \bar{\epsilon}} = \frac{\partial E}{\partial \bar{\epsilon}} \Big|_S$$

$$dF = \text{Tr}(\sigma d\epsilon) - S dT$$

$$\bar{\sigma} = \frac{\partial \pi}{\partial \bar{\epsilon}} = \frac{\partial F}{\partial \bar{\epsilon}} \Big|_T$$

$$d\Phi = -\text{Tr}(\epsilon d\sigma) - S dT$$

$$\bar{\epsilon} = \frac{\partial \omega}{\partial \bar{\sigma}} = -\frac{\partial \Phi}{\partial \bar{\sigma}} \Big|_T$$

A partir d'un état **naturel**  $\sigma = 0 \quad \epsilon = 0$  **isotherme**  $T = \text{Cte}$

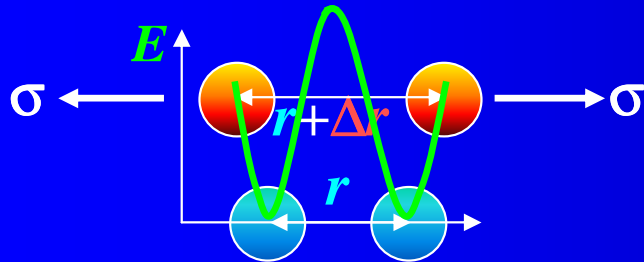
$$\pi \equiv F \quad \text{et} \quad \omega \equiv -\Phi$$

# I-2 Les Deux Élasticités

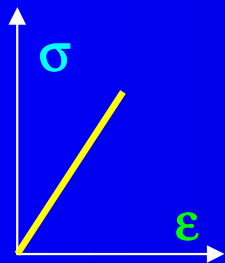
En régime **isotherme**  $T=T_0$   $F=E-T_0S \Rightarrow$

$$\bar{\sigma} = \frac{\partial \pi}{\partial \bar{\epsilon}} = \frac{\partial F}{\partial \bar{\epsilon}} \Big|_{T_0} = \frac{\partial E}{\partial \bar{\epsilon}} \Big|_{T_0} - T_0 \frac{\partial S}{\partial \bar{\epsilon}} \Big|_{T_0}$$

Élasticité Cristalline  
Petites Déformations

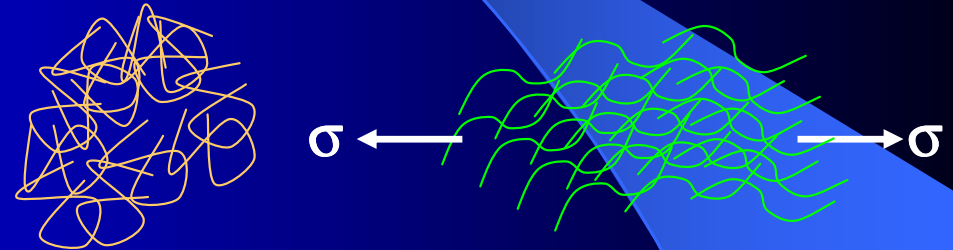


petite variation de la distance interatomique  
 $\Rightarrow$  forte variation de l'énergie interne

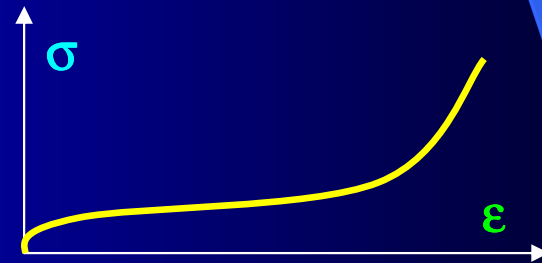


Élasticité Linéaire

Haute Élasticité Caoutchoutique  
Grandes Déformations

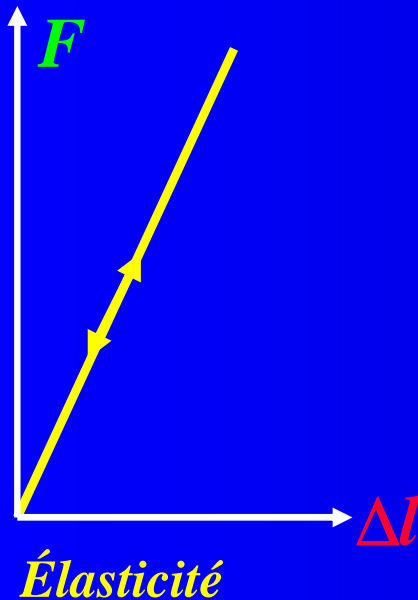
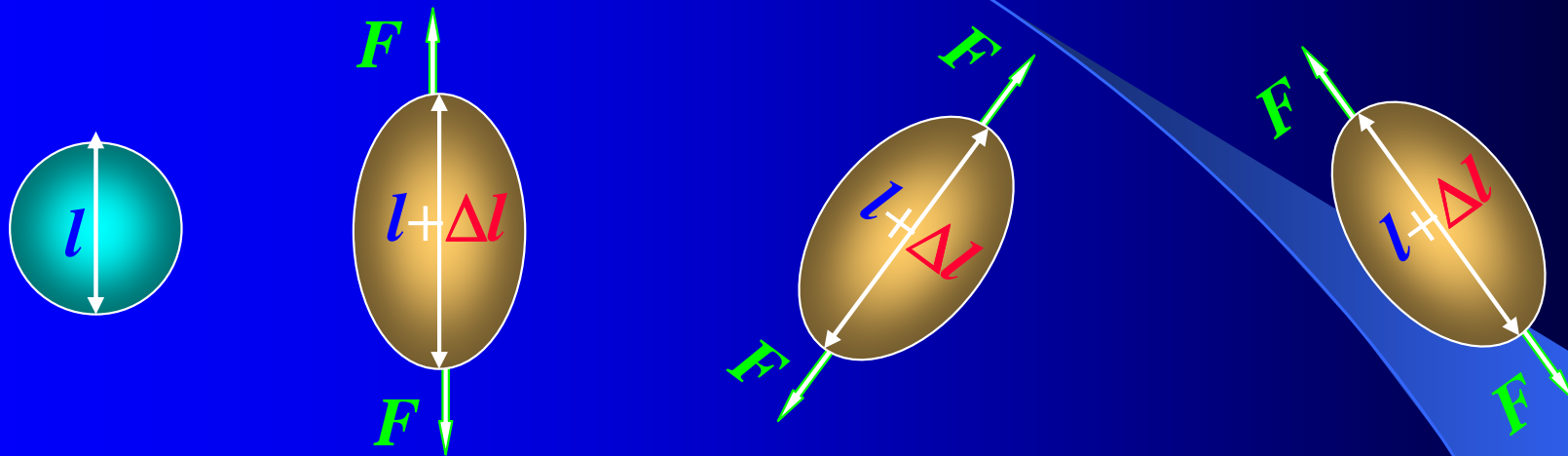


forte variation de conformation des chaînes  
 $\Rightarrow$  diminution de l'entropie désordre  $\rightarrow$  ordre



Élasticité Non Linéaire

# I-3 Élasticité Linéaire des Milieux Isotropes



Isotropie :

Réponse indépendante de la direction du chargement

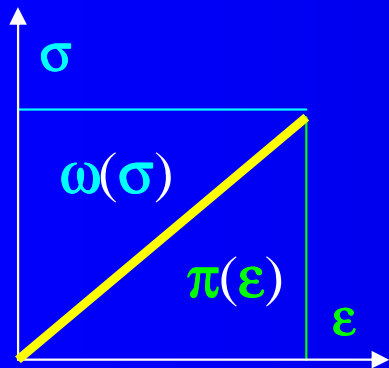
Linéarité :

Réponse proportionnelle à l'amplitude du chargement

Élasticité parfaite :

Réversibilité de la réponse avec le chargement

# I-4 Potentiels en Élasticité Linéaire



$\sigma$  proportionnel à  $\varepsilon \Rightarrow F(\varepsilon) = \pi(\varepsilon) = \omega(\sigma) = -\Phi(\sigma)$

$$\bar{\sigma} = \frac{\partial F}{\partial \bar{\varepsilon}} \leftarrow F = \frac{1}{2} \text{Tr}(\sigma \varepsilon) \rightarrow \bar{\varepsilon} = \frac{\partial F}{\partial \bar{\sigma}}$$

fonction homogène de degré 2 des composantes de  $\sigma$  ou de  $\varepsilon$

$$\text{Euler : } 2F = \text{Tr}(\bar{\sigma} \bar{\varepsilon}) = \text{Tr}(\bar{\sigma} \frac{\partial F}{\partial \bar{\sigma}}) = \text{Tr}(\bar{\varepsilon} \frac{\partial F}{\partial \bar{\varepsilon}})$$

$F$  fonction scalaire homogène de degré 2 est invariante par changement de repère

$$\left. \begin{array}{l} I_1^\varepsilon = \text{Tr}(\bar{\varepsilon}) \\ I_2^\varepsilon = \text{Tr}(\bar{\varepsilon}^2) \end{array} \right\} m_1^\varepsilon \text{Tr}^2(\bar{\varepsilon}) + m_2^\varepsilon \text{Tr}(\bar{\varepsilon}^2) = F = m_1^\sigma \text{Tr}^2(\bar{\sigma}) + m_2^\sigma \text{Tr}(\bar{\sigma}^2) \left\{ \begin{array}{l} I_1^\sigma = \text{Tr}(\bar{\sigma}) \\ I_2^\sigma = \text{Tr}(\bar{\sigma}^2) \end{array} \right.$$

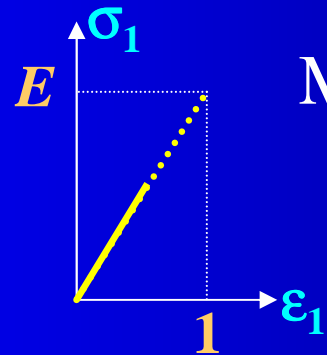
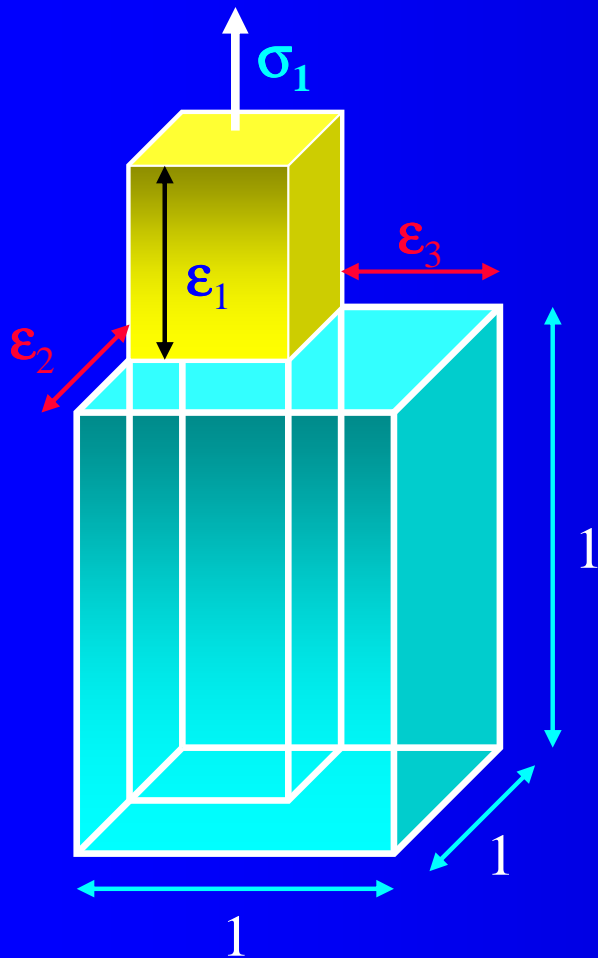
Le comportement élastique linéaire isotrope  
est entièrement caractérisé par un couple de 2 Modules Élastiques

# II Les Modules Élastiques

- II-1 Module d'Young et Coefficient de Poisson
- II-2 Module de Coulomb
- II-3 Modules de Lamé
- II-4 Modules de Compressibilité
- II-5 Modules de Cisaillement
- II-6 Loi de Hooke
- II-7 Application : Modules Effectifs
- II-8 Application : Flexion Faible



# II-1 Module d'Young et Coefficient de Poisson



Module d'Young  $E = \frac{\sigma_1}{\epsilon_1}$  [GPa]

- Acier  $E = 200$  GPa
- Plastiques  $E = 1$  GPa

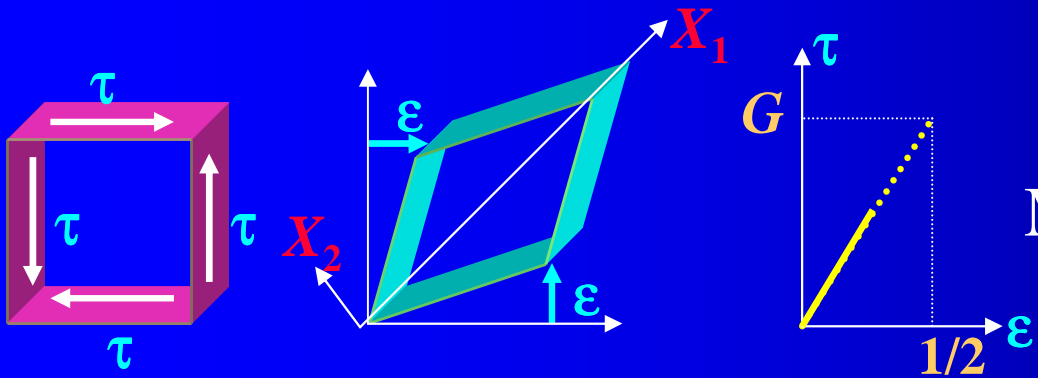
Coefficient de Poisson  $\eta = -\frac{\epsilon_2}{\epsilon_1} = -\frac{\epsilon_3}{\epsilon_1}$   $0 < \eta < \frac{1}{2}$

- Liège  $\eta = 0$
- Métaux...  $\eta = 1/4-1/3$
- Caoutchouc  $\eta = 1/2$

$$\sigma = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \epsilon = \begin{vmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{vmatrix} = \frac{\sigma_1}{E} \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\eta & 0 \\ 0 & 0 & -\eta \end{vmatrix}$$



# II-2 Module de Coulomb



Module de Coulomb  $G = \frac{\tau}{2\varepsilon}$  [GPa]

$$\sigma = \begin{vmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \varepsilon = \begin{vmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \frac{\tau}{2G} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\tau = 2G\varepsilon = G\gamma$$

$$\sigma = \begin{vmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \varepsilon = \begin{vmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{vmatrix} = \frac{\tau}{E} \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\eta & 0 \\ 0 & 0 & -\eta \end{vmatrix} + \frac{-\tau}{E} \begin{vmatrix} -\eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\eta \end{vmatrix}$$

$$\Rightarrow \varepsilon = \frac{1+\eta}{E} \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tau \Rightarrow \tau = \frac{E}{1+\eta} \varepsilon \Rightarrow G = \frac{E}{2(1+\eta)}$$

## II-3 Modules de Lamé

Équation d'Young  $\bar{\bar{\boldsymbol{\varepsilon}}} = \mathcal{F}\{\bar{\bar{\boldsymbol{\sigma}}}\}$

$$E\bar{\bar{\boldsymbol{\varepsilon}}} = -\eta \text{Tr}(\bar{\bar{\boldsymbol{\sigma}}})\bar{\delta} + (1+\eta)\bar{\bar{\boldsymbol{\sigma}}}$$

$$E\varepsilon_{ij} = -\eta\sigma_{kk}\delta_{ij} + (1+\eta)\sigma_{ij}$$

Équation de Lamé  $\bar{\bar{\boldsymbol{\sigma}}} = \mathcal{F}\{\bar{\bar{\boldsymbol{\varepsilon}}}\}$

$$\bar{\bar{\boldsymbol{\sigma}}} = \lambda \text{Tr}(\bar{\bar{\boldsymbol{\varepsilon}}})\bar{\delta} + 2\mu\bar{\bar{\boldsymbol{\varepsilon}}}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\sigma_{ij}$$

$$E\text{Tr}(\bar{\bar{\boldsymbol{\varepsilon}}}) = -3\eta\text{Tr}(\bar{\bar{\boldsymbol{\sigma}}}) + (1+\eta)\text{Tr}(\bar{\bar{\boldsymbol{\sigma}}}) = (1-2\eta)\text{Tr}(\bar{\bar{\boldsymbol{\sigma}}})$$

$$\bar{\bar{\boldsymbol{\sigma}}} = \frac{\eta E}{(1-2\eta)(1+\eta)} \text{Tr}(\bar{\bar{\boldsymbol{\varepsilon}}})\bar{\delta} + \frac{E}{1+\eta} \bar{\bar{\boldsymbol{\varepsilon}}}$$

$$\lambda = \frac{\eta E}{(1-2\eta)(1+\eta)}$$

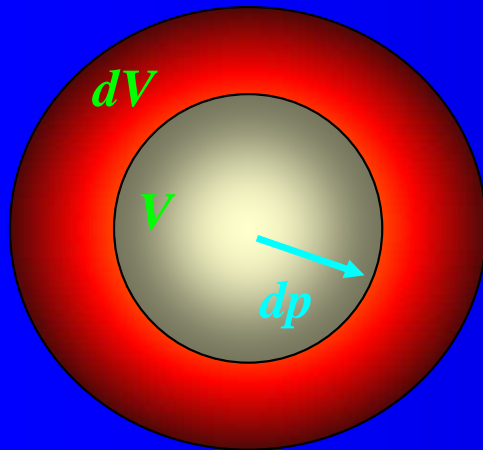
$$\mu = G = \frac{E}{2(1+\eta)}$$

# II-4 Module de Compressibilité

Équation d'Young  $E\bar{\bar{\epsilon}} = -\eta\text{Tr}(\bar{\bar{\sigma}})\bar{\bar{\delta}} + (1+\eta)\bar{\bar{\sigma}}$

Équation de Lamé  $\bar{\bar{\sigma}} = \lambda\text{Tr}(\bar{\bar{\epsilon}})\bar{\bar{\delta}} + 2\mu\bar{\bar{\epsilon}} \quad \text{Tr}(\bar{\bar{\sigma}}) = (3\lambda + 2\mu)\text{Tr}(\bar{\bar{\epsilon}})$

Matériau isotrope et Surpression hydrostatique  $dp$



$$\bar{\bar{\sigma}} = -dp \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \bar{\bar{\epsilon}} = \frac{dV}{3V} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Coefficient de compressibilité  $\chi = -\frac{1}{V} \frac{dV}{dp} = \frac{3\text{Tr}(\bar{\bar{\epsilon}})}{\text{Tr}(\bar{\bar{\sigma}})}$

$$\bar{\bar{\sigma}} = -dp\bar{\bar{\delta}} \Rightarrow E\bar{\bar{\epsilon}} = -dp(1-2\eta)\bar{\bar{\delta}}$$

$$K = \frac{1}{\chi} = \frac{E}{3(1-2\eta)} = \lambda + \frac{2}{3}\mu$$

Module de compressibilité

$$\text{Tr}(\bar{\bar{\sigma}}) = -3dp \quad \text{Tr}(\bar{\bar{\epsilon}}) = \frac{dV}{V} = -dp \frac{3(1-2\eta)}{E}$$

$$\sigma_m = -dp \quad \text{et} \quad 3\epsilon_m = \frac{dV}{V} \Rightarrow \sigma_m = 3K\epsilon_m$$

Relation entre les parties sphériques de  $\bar{\bar{\sigma}}$  et de  $\bar{\bar{\epsilon}}$

# II-5 Module de Cisaillement

$$\bar{\bar{\sigma}} = \sigma_m \bar{\delta} + \sigma_d \bar{\pi}_\sigma$$

$$\text{Tr}(\bar{\pi}_\sigma) = 0$$

Équation de Lamé

$$\bar{\bar{\sigma}} = \lambda \text{Tr}(\bar{\bar{\epsilon}}) \bar{\delta} + 2\mu \bar{\bar{\epsilon}}$$

$$\bar{\bar{\epsilon}} = \epsilon_m \bar{\delta} + \epsilon_d \bar{\pi}_\epsilon$$

$$\text{Tr}(\bar{\pi}_\epsilon) = 0$$

$$\begin{aligned} \bar{\bar{\sigma}} &= \sigma_m \bar{\delta} + \sigma_d \bar{\pi}_\sigma = \lambda \text{Tr}(\epsilon_m \bar{\delta} + \epsilon_d \bar{\pi}_\epsilon) \bar{\delta} + 2\mu (\epsilon_m \bar{\delta} + \epsilon_d \bar{\pi}_\epsilon) \\ &= (3\lambda + 2\mu) \epsilon_m \bar{\delta} + 2\mu \epsilon_d \bar{\pi}_\epsilon + \cancel{\lambda \epsilon_d \text{Tr}(\bar{\pi}_\epsilon) \bar{\delta}} \end{aligned}$$

$$\boxed{\sigma_m \bar{\delta}} + \sigma_d \bar{\pi}_\sigma = \boxed{3K \epsilon_m \bar{\delta}} + 2\mu \epsilon_d \bar{\pi}_\epsilon$$

Relation entre les Sphériques

Relation entre les Déviateurs

$$\boxed{\sigma_d \bar{\pi}_\sigma = 2\mu \epsilon_d \bar{\pi}_\epsilon}$$

$$\sigma_d = 2\mu \epsilon_d$$

$\mu$  Module de Cisaillement

$$\bar{\pi}_\sigma = \bar{\pi}_\epsilon$$

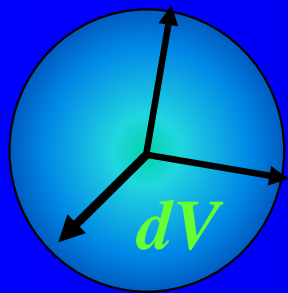
Égalité des Directeurs

# II-6 Loi de Hooke

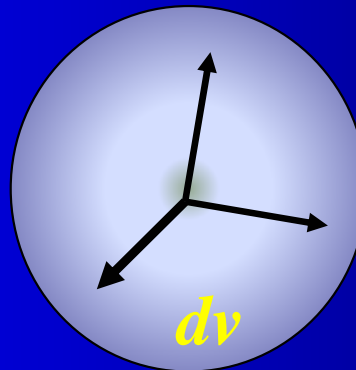
Relation entre les  
**Sphériques** de  $\sigma$  et de  $\varepsilon$

Relation entre les  
**Déviateurs** de  $\sigma$  et de  $\varepsilon$

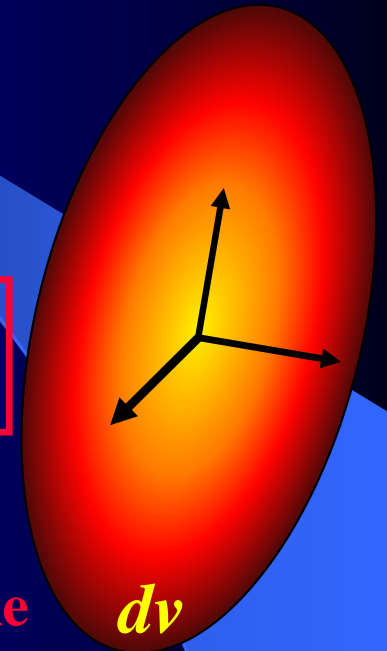
$\sigma$



$$\sigma_m \bar{\delta} = 3K \varepsilon_m \bar{\delta}$$



$$\sigma_d \bar{\pi}_\sigma = 2\mu \varepsilon_d \bar{\pi}_\varepsilon$$



**K** **Changement de Volume**  
à Forme Constante

**$\mu$**  **Changement de Forme**  
à Volume Constant

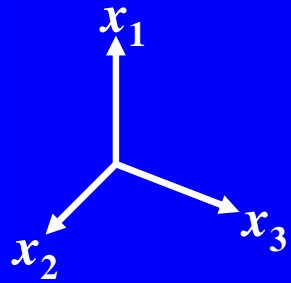
**Égalité des Directeurs**

$$\bar{\pi}_\sigma = \bar{\pi}_\varepsilon$$

$\sigma$  et  $\varepsilon$  ont mêmes  
Directions Principales

# II-7 Application : Modules Effectifs

$$E\bar{\bar{\epsilon}} = -\eta \text{Tr}(\bar{\bar{\sigma}})\bar{\bar{\delta}} + (1+\eta)\bar{\bar{\sigma}}$$



$$\begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix}$$



$$\begin{vmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{vmatrix}$$

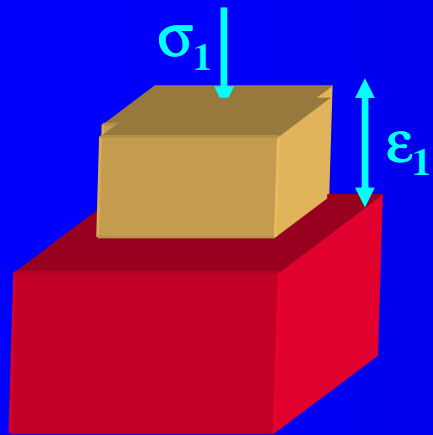
Module d'Young Effectif  $E^* = \frac{\sigma_1}{\epsilon_1}$

Coeff de Poisson Effectif  $\eta^* = -\frac{\epsilon_2}{\epsilon_1}$

## Déformations

### Libre

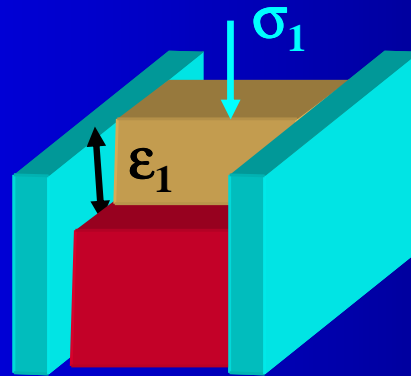
$$\begin{aligned} \sigma_1 & & \sigma_2 = \sigma_3 = 0 \\ \epsilon_1 = \frac{1}{E} \sigma_1 & & \epsilon_2 = \epsilon_3 = \frac{-\eta}{E} \sigma_1 \end{aligned}$$



$$E^* = E \quad \eta^* = \eta$$

### Interdites selon $x_3$

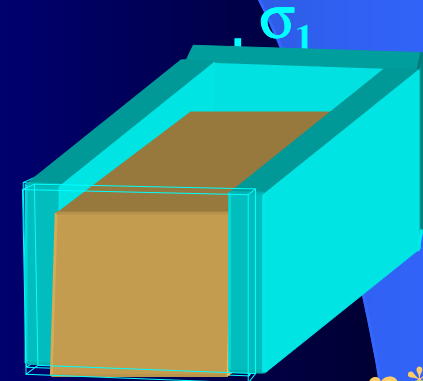
$$\begin{aligned} \sigma_1 & & \sigma_2 = 0 & & \epsilon_3 = 0 \\ \epsilon_1 = \frac{1-\eta^2}{E} \sigma_1 & & \epsilon_2 = \frac{-\eta(1+\eta)}{E} \sigma_1 & & \sigma_3 = \eta \sigma_1 \end{aligned}$$



$$E^* = \frac{E}{1-\eta^2} \quad \eta^* = \frac{\eta}{1-\eta}$$

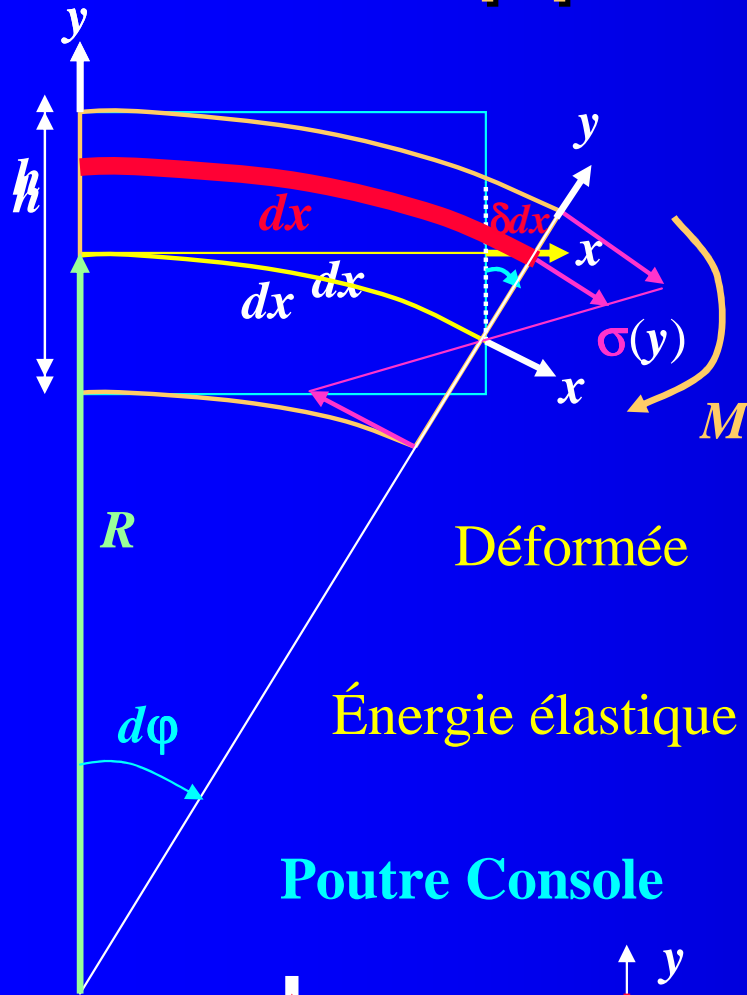
### Interdites selon $x_2$ et $x_3$

$$\begin{aligned} \sigma_1 & & \epsilon_2 = 0 & & \epsilon_3 = 0 \\ \epsilon_1 = \frac{(1+\eta)(1-2\eta)}{(1-\eta)E} \sigma_1 & & \sigma_2 = \sigma_3 = \frac{\eta}{1-\eta} \sigma_1 \end{aligned}$$



$$E^* = \frac{E(1-\eta)}{(1+\eta)(1-2\eta)} = \lambda + 2\mu \quad \eta^* = 0$$

# II-8 Application : Flexion Faible



Déformations  $Rd\phi = dx$   $y d\phi = \delta dx \Rightarrow \epsilon = \frac{\delta dx}{dx} = \frac{y}{R}$

Contraintes  $\sigma = E\epsilon = E \frac{y}{R}$   $\sigma_{\max} = \frac{Mh}{2I}$

Équilibre  $M = \int \sigma y dS = \frac{E}{R} \int y^2 dS = \frac{EI}{R}$

$\frac{EI}{R} = EI \frac{d\phi}{dx} = -EI \frac{d^2y}{dx^2} = M$

$\frac{d\phi}{dx} = \frac{1}{R} = \frac{M}{EI}$

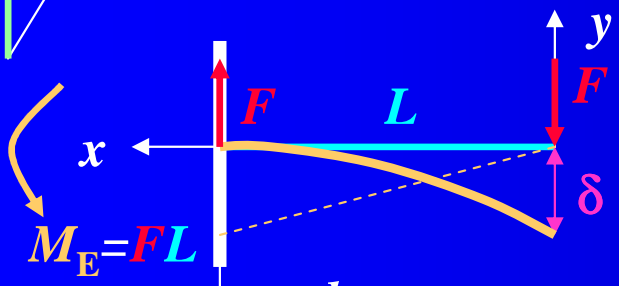
Énergie élastique  $\frac{dW}{dV} = \frac{1}{2} \sigma \epsilon$   $\frac{dW}{dx} = \frac{1}{2} \frac{M^2}{EI^2} \int y^2 dS = \frac{1}{2} M \frac{d\phi}{dx}$   $dW = \frac{1}{2} M d\phi$

Poutre Console

$-EI \frac{d^2y}{dx^2} = M = Fx \Rightarrow \frac{dy}{dx} = \frac{F}{2EI} (L^2 - x^2)$

$\delta = \frac{FL^3}{3EI} < h$

$y = \frac{F}{6EI} (-x^3 + 3Lx^2 - 2L^3) = -\frac{\delta}{2} \left( \frac{x^3}{L^3} - 3 \frac{x^2}{L^2} + 2 \right)$



Encastrement  $\frac{dy}{dx}(L) = 0$   $y(L) = 0$   $W = \frac{1}{2} \int M d\phi = \frac{1}{2} \int M \frac{d\phi}{dx} dx = \frac{FL^3}{6EI} = \frac{1}{2} F\delta = W$

# III Solutions Élastiques en Statique

- **III-1** Équation Élastique de Lamé
- **III-2** Forces de Volume Constantes





# III-1 Équation élastique de Lamé

Équilibre Dynamique

$$\text{Div}_D \bar{\sigma} + \rho \bar{X} = \rho \bar{\gamma}$$

$$\bar{\gamma} = \frac{\partial^2 \bar{u}}{\partial t^2}$$

$$\bar{\sigma} = \lambda \text{Tr}(\bar{\epsilon}) \bar{\delta} + 2\mu \bar{\epsilon}$$

Élasticité Linéaire

Déformation et Déplacement

$$2\bar{\epsilon} = \overline{\text{Grad}} \bar{u} + {}^t\overline{\text{Grad}} \bar{u}$$

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

$$(\lambda + \mu) \overline{\text{Grad}}(\text{Div} \bar{u}) + \mu \bar{\Delta} \bar{u} + \rho \bar{X} = \rho \frac{\partial^2 \bar{u}}{\partial t^2}$$

$$(\lambda + \mu) \overline{\text{Rot}}(\overline{\text{Rot}} \bar{u}) + (\lambda + 2\mu) \bar{\Delta} \bar{u} + \rho \bar{X} = \rho \frac{\partial^2 \bar{u}}{\partial t^2}$$

$$\frac{1}{1 - 2\nu} \overline{\text{Grad}}(\text{Div} \bar{u}) + \bar{\Delta} \bar{u} + \frac{\rho}{\mu} \bar{X} = \frac{\rho}{\mu} \frac{\partial^2 \bar{u}}{\partial t^2}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \frac{\partial u_i}{\partial x_j} + \mu \frac{\partial u_j}{\partial x_i} \right) = (\lambda + \mu) \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

# III-3 Forces de Volume Constantes

$$\vec{\gamma} = \frac{\partial^2 \vec{u}}{\partial t^2} = \text{Cte} \quad \text{et} \quad \vec{X} = \text{Cte}$$

Équation Élastique de Lamé  
Déformation

$$(\lambda + \mu) \text{Grad}(\text{Div} \vec{u}) + \mu \Delta \vec{u} + \rho \vec{X} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

$$\text{Div}[(\lambda + \mu) \text{Grad}(\text{Div} \vec{u}) + \mu \Delta \vec{u}] = \text{Div}[\rho(\vec{\gamma} - \vec{X})]$$

$$(\lambda + \mu) \text{Div}[(\text{Grad}(\text{Div} \vec{u}))] + \mu \Delta(\text{Div} \vec{u}) = 0 \quad \Rightarrow \quad \Delta(\text{Div} \vec{u}) = \Delta(\text{Tr}(\bar{\epsilon})) = 0$$

Équation de Lamé  
Contrainte

$$\bar{\sigma} = \lambda \text{Tr}(\bar{\epsilon}) \bar{\delta} + 2\mu \bar{\epsilon} \quad \Rightarrow \quad \text{Tr}(\bar{\sigma}) = (3\lambda + 2\mu) \text{Tr}(\bar{\epsilon})$$

En présence de Forces de Volume *Constantes* :

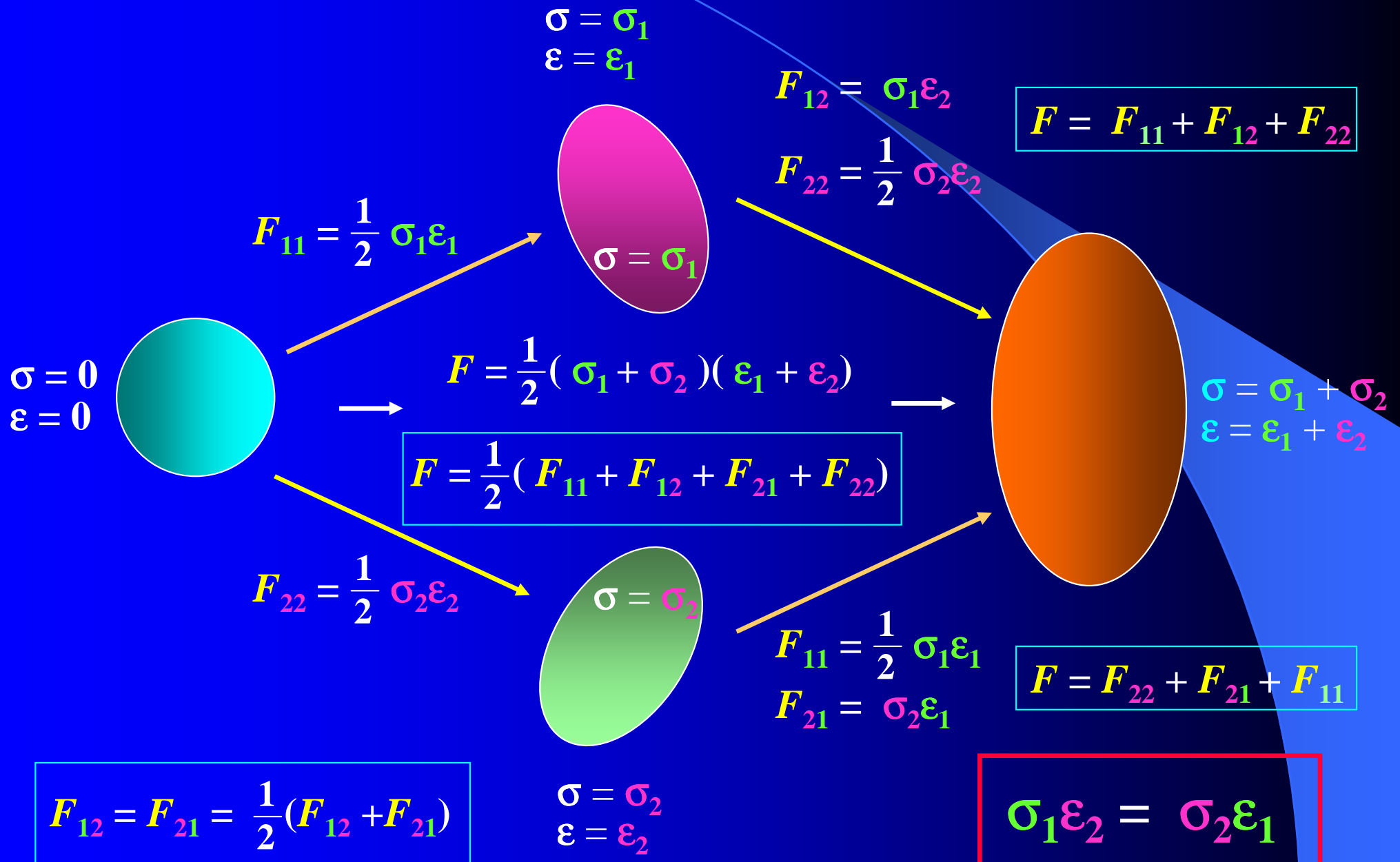
$$\Delta(\text{Tr}(\bar{\sigma})) = 0$$

# IV Théorèmes Énergétiques

- IV-1 Réciprocité des Chargements
- IV-2 Formule de Maxwell - Betti
- IV-3 Théorème de Castigliano
- IV-4 Formule de Ménabréa

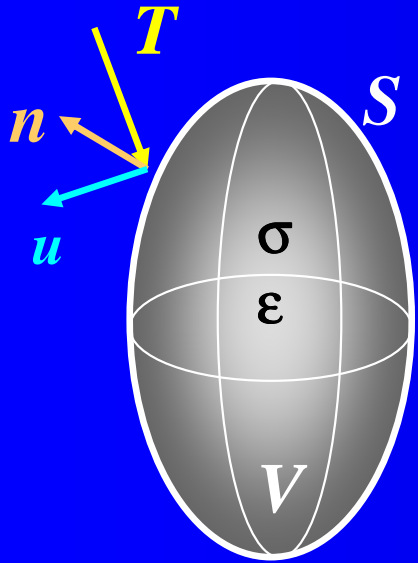


# IV-1 Réciprocité des Chargements



# IV-2 Formule de Maxwell - Betti

Statique et Forces de Volume négligeables  $\vec{\gamma} = \mathbf{0}$  et  $\vec{X} = \mathbf{0}$



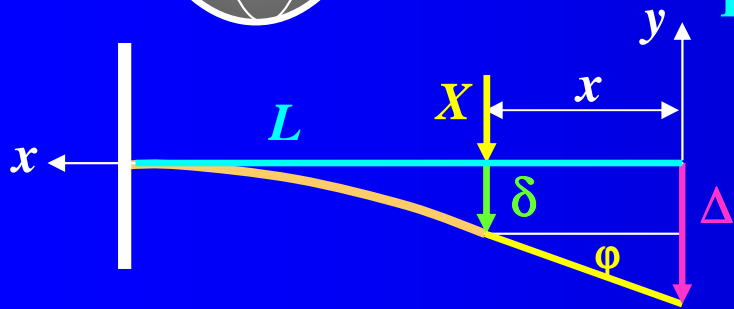
$$W = \frac{1}{2} \int \text{Tr}(\overline{\overline{\sigma \epsilon}}) dV = \frac{1}{2} \int \vec{T} \vec{u} dS$$

Réciprocité  $\vec{T}_1 \cdot \vec{u}_2 = \vec{T}_2 \cdot \vec{u}_1$

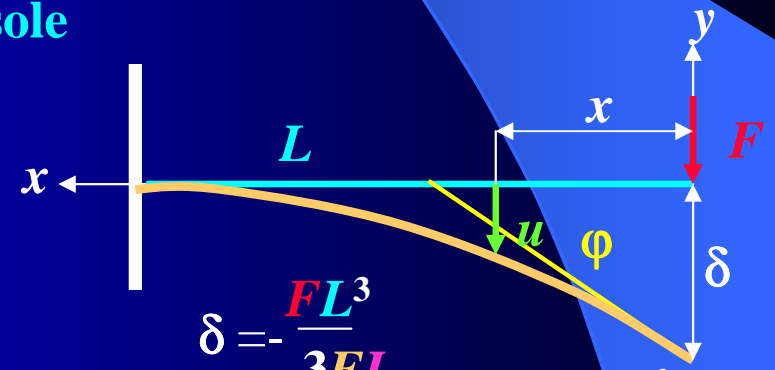
$$\overline{\overline{\epsilon}} = \frac{\partial W}{\partial \overline{\overline{\sigma}}} \Rightarrow \vec{u} = \frac{\partial W}{\partial \vec{T}}$$

$$\overline{\overline{\sigma}} = \frac{\partial W}{\partial \overline{\overline{\epsilon}}} \Rightarrow \vec{T} = \frac{\partial W}{\partial \vec{u}}$$

Exemple : Poutre Console



$$Xu = F\Delta$$



$$\delta = -\frac{X(L-x)^3}{3EI} \quad \text{tg}(\varphi) = \frac{X(L-x)^2}{2EI}$$

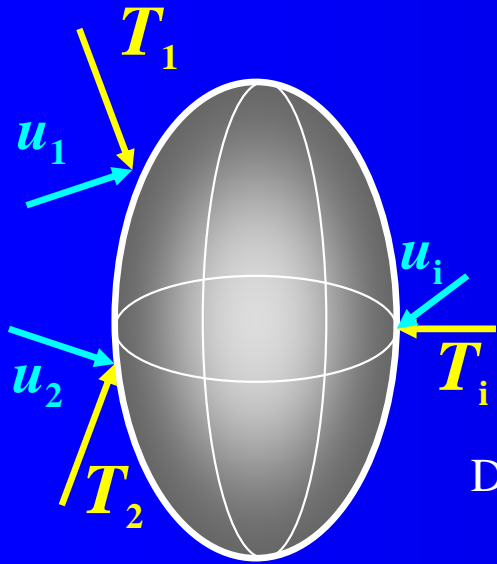
$$\delta = -\frac{FL^3}{3EI} \quad \text{tg}(\varphi) = \frac{FL^2}{2EI}$$

$$\Delta = \delta - \text{tg}(\varphi)x = \frac{X}{6EI} (-x^3 + 3Lx^2 - 2L^3)$$

$$u = \frac{F\Delta}{X} = \frac{F}{6EI} (-x^3 + 3Lx^2 - 2L^3)$$

# IV-3 Théorème de Castigliano

Statique et Forces de Volume négligeables  $\vec{\gamma} = \mathbf{0}$  et  $\vec{X} = \mathbf{0}$



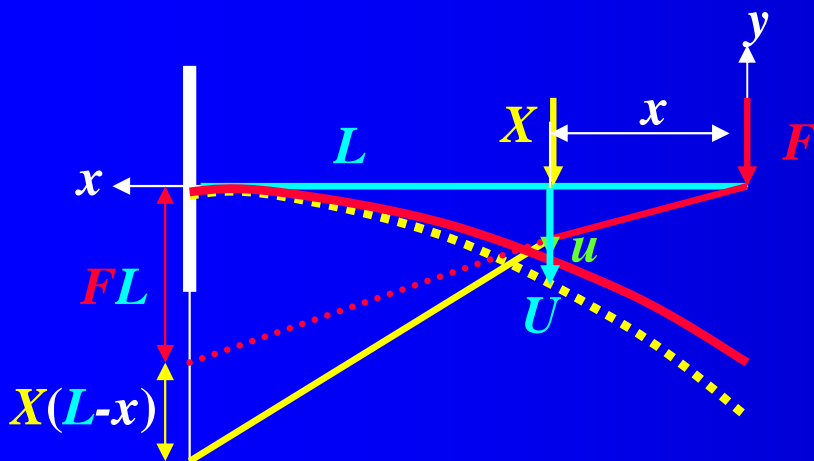
$$W(\vec{T}_1, \vec{T}_2, \dots, \vec{T}_i, \dots) = \frac{1}{2} \sum_i \vec{T}_i \vec{u}_i \quad \vec{u}_i = \frac{\partial W}{\partial \vec{T}_i}$$

Méthode des charges fictives

$$\vec{u}_k(\vec{T}_i) = \frac{\partial W(\vec{T}_1, \dots, \vec{T}_j, \dots, \vec{T}_i, \dots)}{\partial \vec{T}_k} \bigg|_{\vec{T}_j = 0 \quad \forall j \neq i}$$

Déplacement  $\vec{u}_k$  de la charge fictive  $\vec{T}_k$  sous l'action de la **seule** force réelle  $\vec{T}_i$

Exemple : Poutre Console



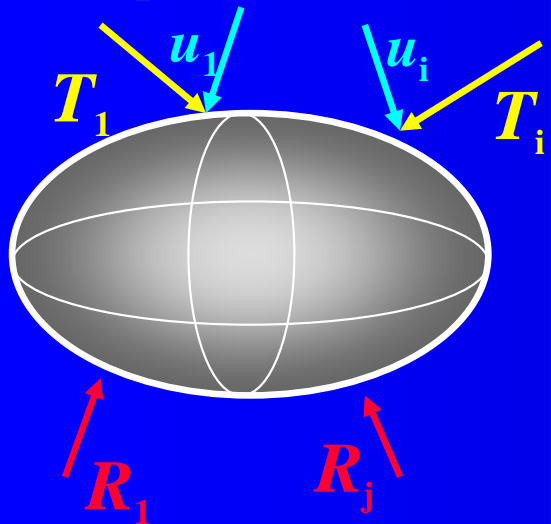
$$W = \frac{1}{2} \int \frac{M^2}{EI} dx = \frac{1}{2EI} \int_0^x F^2 x^2 dx + \frac{1}{2EI} \int_x^L (Fx + X(L-x))^2 dx$$

$$U = \frac{1}{2EI} \frac{\partial W}{\partial X} \bigg|_{X=0} = -2 \int_x^L (Fx + X(L-x))(L-x) dx$$

$$u = \frac{1}{2EI} \frac{\partial W}{\partial X} \bigg|_{X=0} = \frac{F}{6EI} (-x^3 + 3Lx^2 - 2L^3)$$

# IV-4 Formule de Ménébréa

Statique et Forces de Volume négligeables  $\vec{\gamma} = \mathbf{0}$  et  $\vec{X} = \mathbf{0}$



$\vec{T}$  Forces appliquées  
 $\vec{R}$  Réactions d'appui

**Système Hyperstatique**

$\Sigma(\text{forces}) = \mathbf{0}$  et  $\Sigma(\text{moments}) = \mathbf{0}$  ne suffisent pas à exprimer tous les  $\vec{R}$  en fonction des  $\vec{T}$

$$W = f(\vec{T}_1, \dots, \vec{T}_i, \dots, \vec{R}_1, \dots, \vec{R}_j) \quad \vec{u}_i = \frac{\partial W}{\partial \vec{T}_i} \quad \vec{o} = \frac{\partial W}{\partial \vec{R}_j}$$

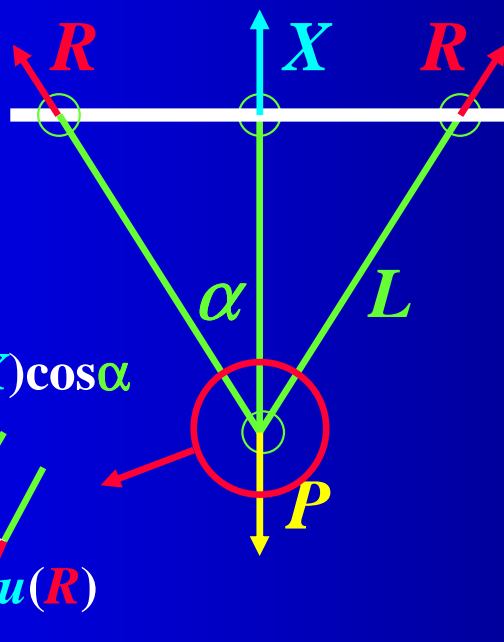
**Exemple : Suspension articulée**

**Symétrie**

$\Sigma(\text{forces horizontales}) = \mathbf{0}$  et  $\Sigma(\text{moments}) = \mathbf{0}$

$$X + 2R \cos \alpha = P$$

$$\sigma = \frac{F}{S} \quad \varepsilon = \frac{\sigma}{E} \quad u = \varepsilon L = \frac{FL}{ES}$$



$$W = \frac{1}{2} \sum_i \vec{F}_i \vec{u}_i$$

$$2W = Xu(X) + 2Ru(R)$$

$$2W = \frac{L}{ES} \left( X^2 \cos \alpha + 2 \frac{(P-X)^2}{4 \cos^2 \alpha} \right)$$

$$u(X) = \frac{L}{ES} X \cos \alpha$$

$$u(R) = \frac{L}{ES} \frac{P-X}{2 \cos \alpha}$$

$$X = \frac{P}{1 + 2 \cos^3 \alpha}$$

$$\frac{\partial W}{\partial X} = 0 \Rightarrow X = \frac{P}{1 + 2 \cos^3 \alpha}$$

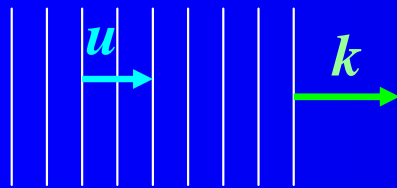
# V La Propagation du Son

- V-1 Les Ondes de Volume
- V-2 Barres, Plaques et Blocs





# V-1 Les Ondes de Volume



$u$  Polarisation (déplacement de la matière)  
 $k$  Vecteur d'onde  $V$  Célérité de l'onde

$$\sigma = E^* \epsilon = E^* \frac{\partial u}{\partial x}$$

$$u = u_0 \sin \omega \left( t - \frac{x}{V} \right)$$

$$2W = \rho \left[ \frac{\partial u}{\partial t} \right]^2 = \rho \omega^2 u_0^2 \cos^2 \omega \left( t - \frac{x}{V} \right)$$

Énergie cinétique

$$V = \sqrt{\frac{E^*}{\rho}}$$

$$2F = \text{Tr}(\sigma \epsilon) = E^* \omega^2 \frac{u_0^2}{V^2} \cos^2 \omega \left( t - \frac{x}{V} \right)$$

Énergie élastique

$$V \sim 10^3 - 10^4 \text{ ms}^{-1}$$

$$\text{Rot } \vec{u}_L = 0$$

$$\text{Div } \vec{u}_L \neq 0$$

$$\frac{dV}{V} \neq 0$$

$$\vec{u} = \vec{u}_L + \vec{u}_T$$

$$\frac{dV}{V} = 0$$

$$\text{Div } \vec{u}_T = 0$$

$$\text{Rot } \vec{u}_T \neq 0$$

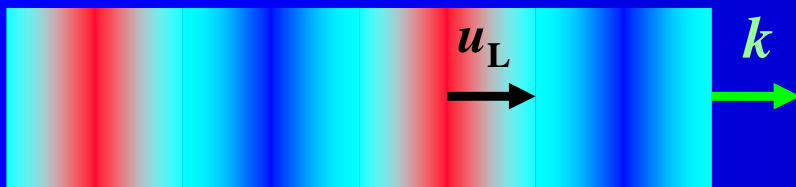
$$(\lambda + \mu) \text{Rot}(\text{Rot } \vec{u}) + (\lambda + 2\mu) \Delta \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

$$(\lambda + \mu) \text{Grad}(\text{Div } \vec{u}) + \mu \Delta \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

$$\vec{u} = \vec{u}_L \parallel \vec{k} \quad V_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$\frac{V_T}{V_L} = \sqrt{\frac{1 - 2\eta}{2(1 - \eta)}} < \sqrt{\frac{1}{2}}$$

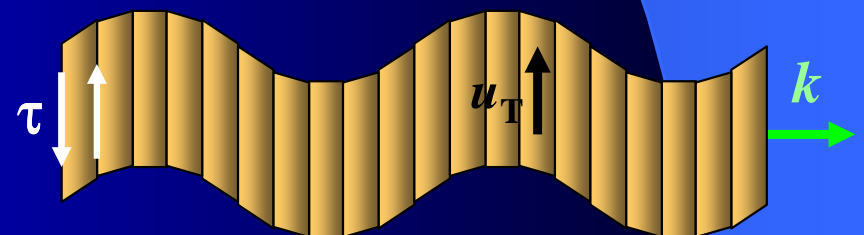
$$\vec{u} = \vec{u}_T \perp \vec{k} \quad V_T = \sqrt{\frac{\mu}{\rho}}$$



Longitudinales : Compression

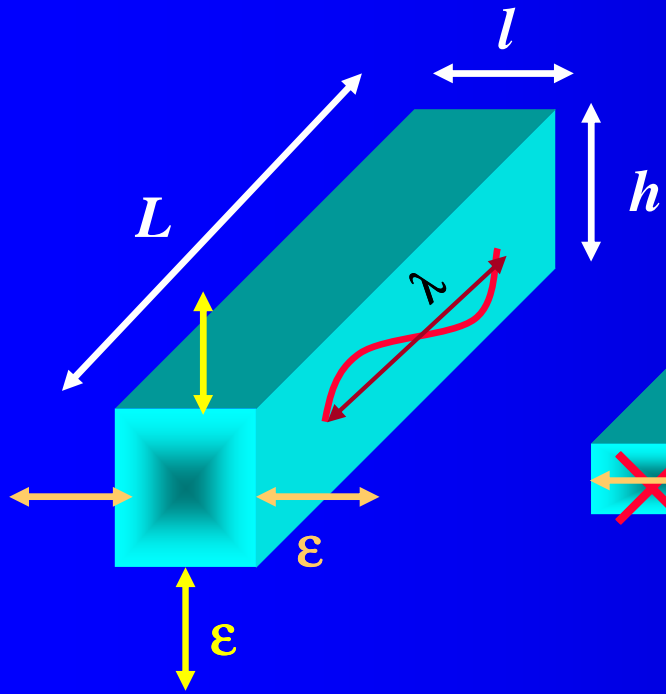
$$\eta = \frac{1}{3}$$

$$\frac{V_T}{V_L} = \frac{1}{2}$$



Transversales : Cisaillement

# V-2 Barres, Plaques et Blocs

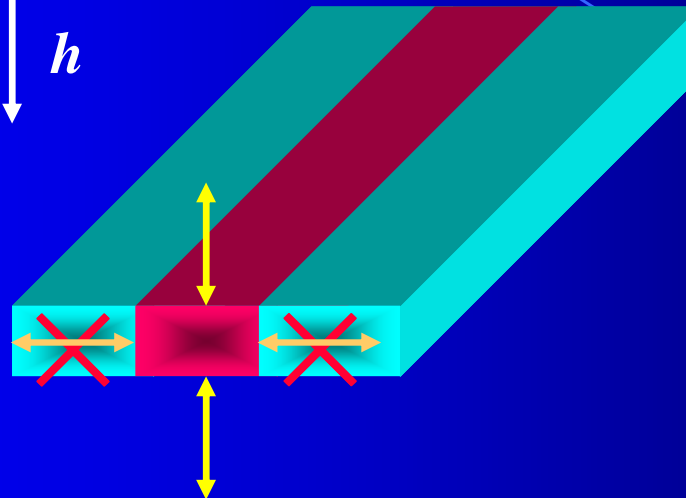


**Barre**

$l \sim h \ll \lambda \ll L$   
 $\varepsilon$  et  $\varepsilon$  libres

$$E^* = E$$

$$V_L = \sqrt{\frac{E}{\rho}}$$

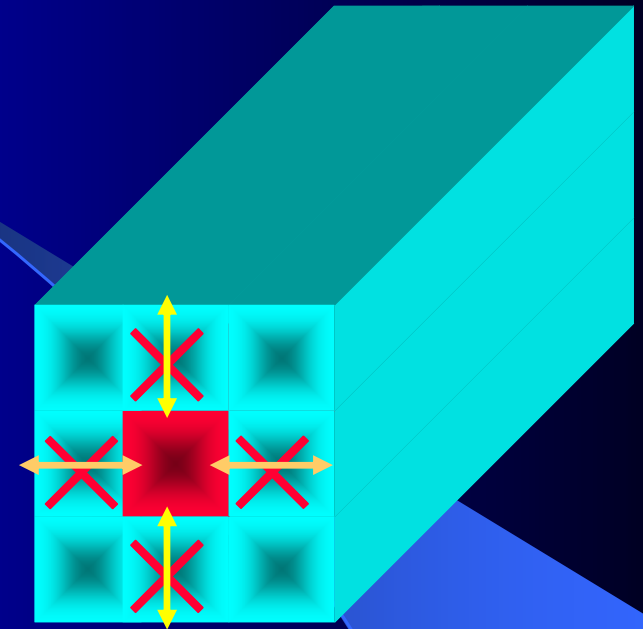


**Plaque**

$h \ll \lambda \ll l \ll L$   
 $\varepsilon = 0$  et  $\varepsilon$  libre

$$E^* = \frac{E}{1-\eta^2}$$

$$V_L = \sqrt{\frac{E}{\rho(1-\eta^2)}}$$



**Bloc**

$\lambda \ll l \sim h \ll L$   
 $\varepsilon = 0$  et  $\varepsilon = 0$

$$E^* = \frac{E(1-\eta)}{(1+\eta)(1-2\eta)} = \lambda + 2\mu$$

$$V_L = \sqrt{\frac{E(1-\eta)}{\rho(1+\eta)(1-2\eta)}}$$

# VI Les Problèmes Plans

- VI-1 Déformations Planes
- VI-2 Contraintes Planes
- VI-3 Techniques Expérimentales
- VI-4 Réseaux Caractéristiques



# VI-1 Déformations Planes

Problème **Mathématique**

$$\begin{cases} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{cases} \Rightarrow \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

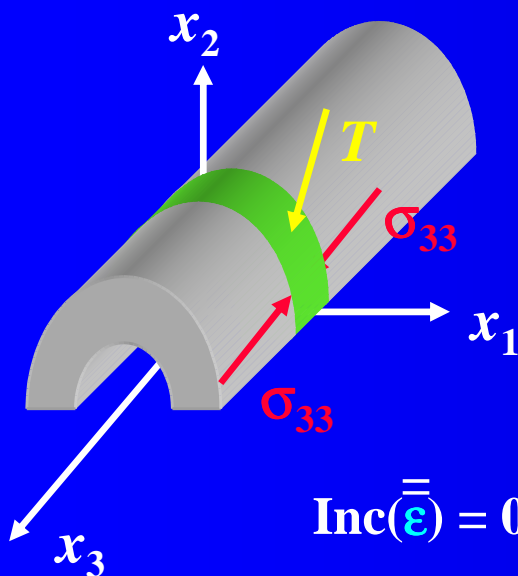
Problème **Mécanique**

$Ox_3$  Direction principale et **Chargement**

Si **Poids propre**  $\vec{X} = \rho \vec{g} \Rightarrow Ox_3$  **horizontal**

$$\begin{cases} T_1(x_1, x_2) \\ T_2(x_1, x_2) \\ 0 \end{cases} \begin{cases} X_1(x_1, x_2) \\ X_2(x_1, x_2) \\ 0 \end{cases}$$

## Déformation Plane



$$\begin{cases} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3 = \text{Cte} \end{cases} \Leftarrow \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{vmatrix}$$

$$E^* \bar{\bar{\epsilon}} = -\eta^* \text{Tr}(\bar{\bar{\sigma}}) \bar{\bar{\delta}} + (1 + \eta^*) \bar{\bar{\sigma}}$$

$$\begin{aligned} \epsilon_{33} &= 0 \\ \sigma_{33} &= \eta(\sigma_{11} + \sigma_{22}) \end{aligned}$$

$$E^* = \frac{E}{1 - \eta^2} \quad \eta^* = \frac{\eta}{1 - \eta}$$

$$\text{Inc}(\bar{\bar{\epsilon}}) = 0 \Rightarrow \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad \vec{X} = 0 \Rightarrow \Delta(\sigma_{11} + \sigma_{22}) = 0$$

# VI-2 Contraintes Planes

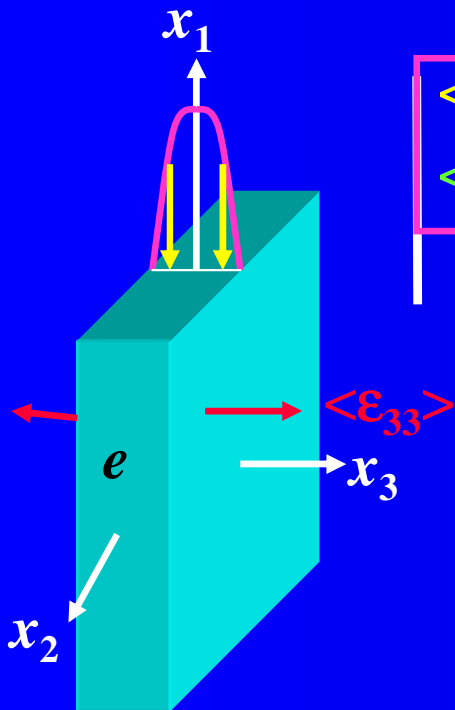
Contrainte Plane 
$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{vmatrix} \Rightarrow \begin{cases} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{cases}$$

$\Rightarrow \epsilon_{33} = 0 \Rightarrow$  Contrainte et Déformation Planes  $\Rightarrow \sigma_{11} + \sigma_{22} = 0 \Rightarrow$  Cisaillement pur

Contrainte Quasi Plane

$Ox_3$  Direction principale et **Chargement**  
Symétrique sur l'épaisseur du feuillet

$$\begin{cases} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ 0 \end{cases} \quad \vec{X} = 0$$



$$\begin{vmatrix} \langle \sigma_{11} \rangle & \langle \sigma_{12} \rangle & 0 \\ \langle \sigma_2 \rangle & \langle \sigma_{22} \rangle & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} \langle \epsilon_{11} \rangle & \langle \epsilon_{12} \rangle & 0 \\ \langle \epsilon_2 \rangle & \langle \epsilon_{22} \rangle & 0 \\ 0 & 0 & \langle \epsilon_{33} \rangle \end{vmatrix}$$

$$\langle \sigma \rangle = \frac{1}{e} \int_0^e \sigma(x_1, x_2, x_3) dx_3$$

$$\langle \epsilon \rangle = \frac{1}{e} \int_0^e \epsilon(x_1, x_2, x_3) dx_3$$

$$E \langle \bar{\bar{\epsilon}} \rangle = -\eta \text{Tr}(\langle \bar{\bar{\sigma}} \rangle) \bar{\bar{\delta}} + (1 + \eta) \langle \bar{\bar{\sigma}} \rangle$$

$$\langle \sigma_{33} \rangle = 0$$

$$E \langle \epsilon_{33} \rangle = -\eta (\langle \sigma_{11} \rangle + \langle \sigma_{22} \rangle)$$

$$\Delta(\langle \sigma_{11} \rangle + \langle \sigma_{22} \rangle) = 0$$

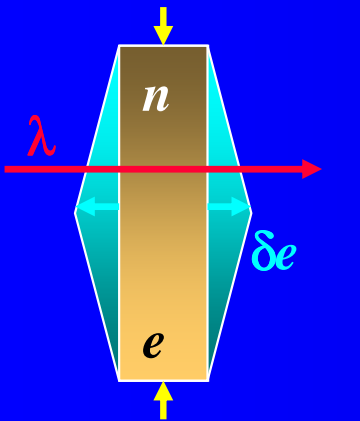
# VI-3 Techniques Expérimentales

Contraintes et Déformations Planes  $\vec{X} = 0 \Rightarrow \Delta(\sigma_{11} + \sigma_{22}) = 0$  Indépendant de  $E$  et  $\eta$

$$\begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} = \frac{1}{2} (\sigma_1 + \sigma_2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \frac{1}{2} (\sigma_1 - \sigma_2) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

Sphérique :  $\sigma_1 + \sigma_2$   
Déviateur :  $\sigma_1 - \sigma_2$

## Interférométrie



$$\delta\varphi = \frac{2\pi}{\lambda} n e \epsilon_{33}$$

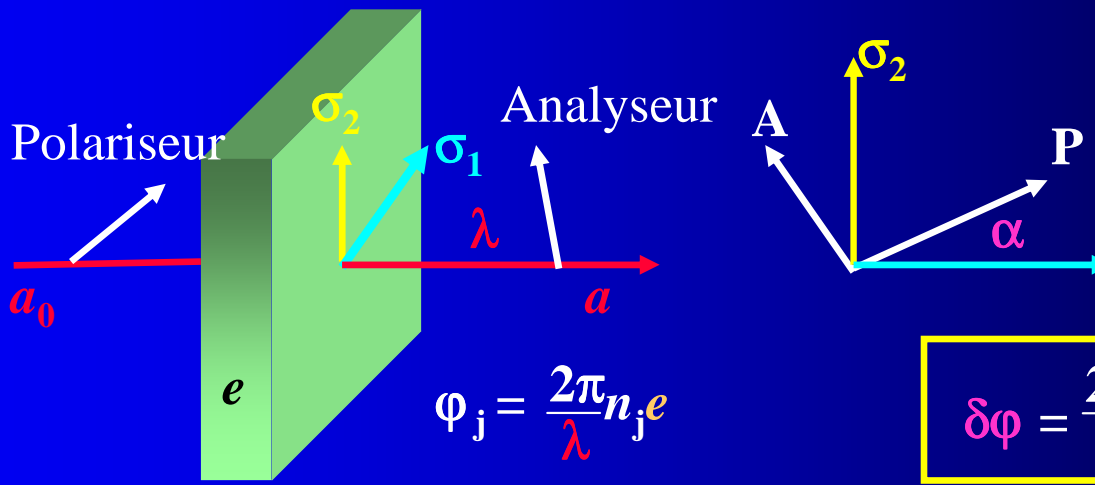
$$E \epsilon_{33} = -\eta (\sigma_{11} + \sigma_{22})$$

$$\sigma_{11} + \sigma_{22} = k \frac{\lambda}{n e} \frac{E}{\eta}$$

Sphérique

## Photoélasticimétrie

$C$  Constante photoélastique : 1 Brewster =  $10^{-12} \text{ m}^2 \text{ N}^{-1}$



$$\varphi_j = \frac{2\pi}{\lambda} n_j e$$

Neumann  $n_1 - n = C \sigma_1$   
 $n_2 - n = C \sigma_2$

$$\delta\varphi = \frac{2\pi}{\lambda} C e (\sigma_1 - \sigma_2)$$

$$I = I_0 \sin^2 2\alpha \sin^2 \frac{\delta\varphi}{2}$$

$$\begin{vmatrix} a_0 \cos \alpha \\ a_0 \sin \alpha \end{vmatrix} \begin{vmatrix} a_0 \cos \alpha \exp(-i\varphi_1) \\ a_0 \sin \alpha \exp(-i\varphi_2) \end{vmatrix}$$

$$a = a_0 \sin \alpha \cos \alpha (\exp(-i\varphi_2) - \exp(-i\varphi_1))$$

$$a = i a_0 \exp(-i \frac{\varphi_1 + \varphi_2}{2}) \sin 2\alpha \sin \frac{\delta\varphi}{2}$$

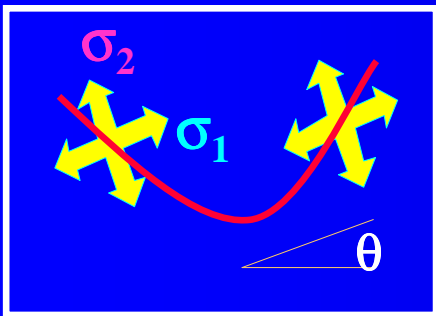
Axes principaux Déviateur

$$\alpha = k \frac{\pi}{2}$$

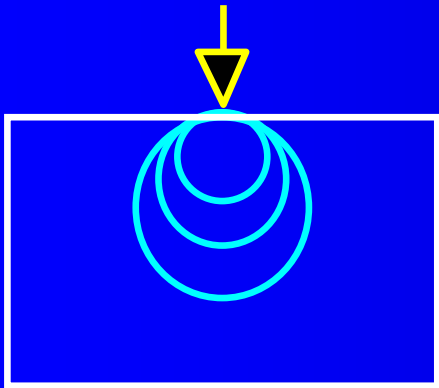
$$\sigma_1 - \sigma_2 = k \frac{\lambda}{C e}$$

# VI-4 Réseaux Caractéristiques

**Isoclines**  
Axes principaux

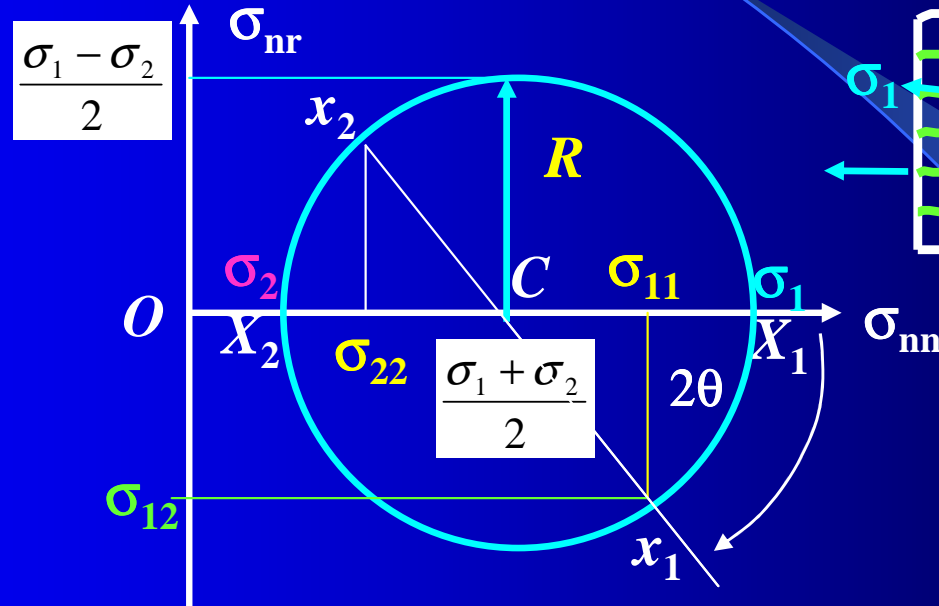


**Isopaches**  
Sphérique



$$\sigma_{11}(x_1, x_2) + \sigma_{22}(x_1, x_2) = \text{Cte}$$

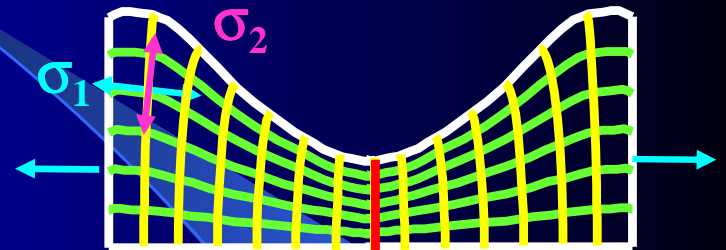
$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \quad \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix}$$



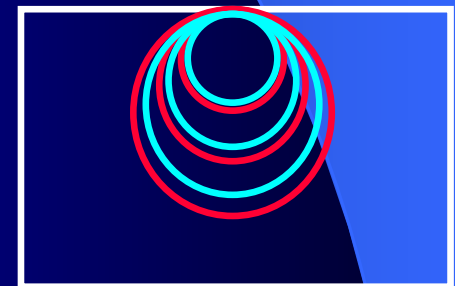
$$\text{tg}2\theta = \frac{2\sigma_{12}(x_1, x_2)}{\sigma_{11}(x_1, x_2) - \sigma_{22}(x_1, x_2)}$$

$$\text{tg}2\theta = \frac{dx_1}{dx_2}$$

**Isostatiques**  
Trajectoire  
Potentielle de fissure



**Isochromes**  
Déviateur : Plasticité  
Cisaillement Max



$$\sigma_1(x_1, x_2) - \sigma_2(x_1, x_2) = \text{Cte}$$

Tous les réseaux respectent les **symétries** de la pièce et de son chargement

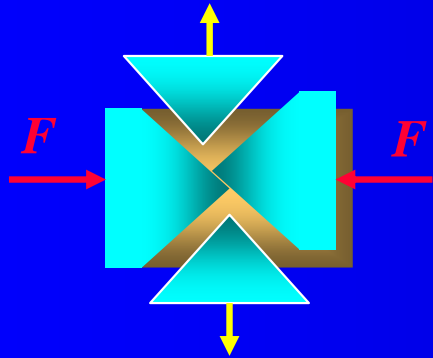
# VII Le Flambement

- VII-1 Charge Critique d'Euler
- VII-2 Optimisation Géométrique
- VII-3 Optimisation Matériau



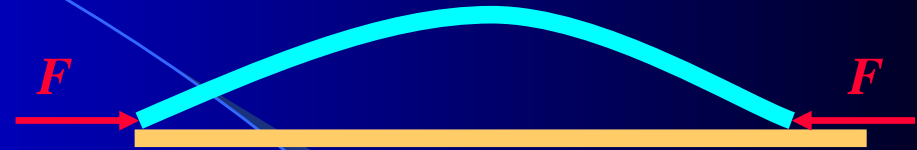


# VII-1 Charge Critique d'Euler

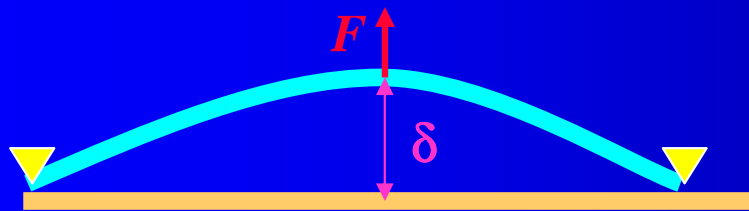


En compression

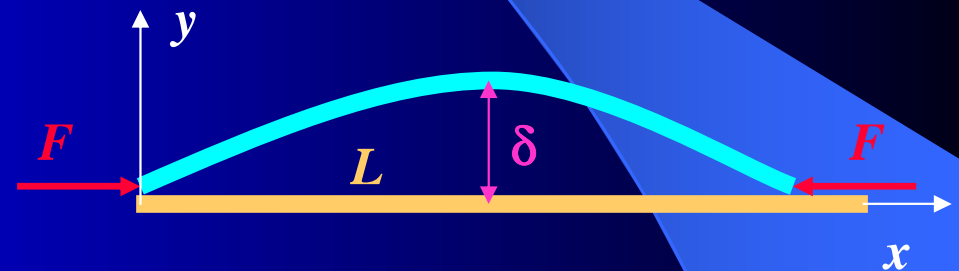
La structure **massive** cède par cisaillement



La structure **élancée** cède par flambement



**Flexion** sans seuil  $\delta = f(F)$



**Flambement** avec seuil  $\delta = f(F - F_C)$

Charge **Critique**  $F_C$

$$EI \frac{d^2y}{dx^2} = M$$

$$y \sim \delta$$

$$x \sim L$$

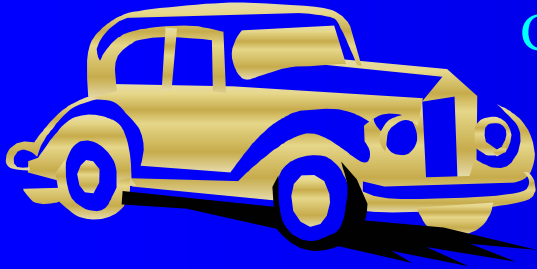
$$M \sim F\delta$$

$$\Rightarrow EI \frac{\delta}{L^2} = F\delta$$

$\Rightarrow$  au seuil

$$F_C = k \frac{EI}{L^2}$$

# VII-2 Optimisation Géométrique



Compression : Fortes charges sur Courtes Distances

$$F_C \sim \frac{EI}{L^2}$$

Augmenter  $I$  à masse  $Cte$

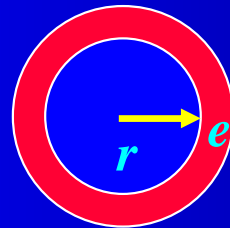


$$L = Cte \quad S = Cte$$

$$S \sim R^2 \sim re$$

$$I \sim R^4 \quad I^* \sim r^3 e$$

$$\frac{F^*}{F} = \frac{I^*}{I} = \frac{r}{e} = \alpha^2$$



$$r = \alpha R$$

$$e = R/\alpha$$

Profils et Structures tubulaires

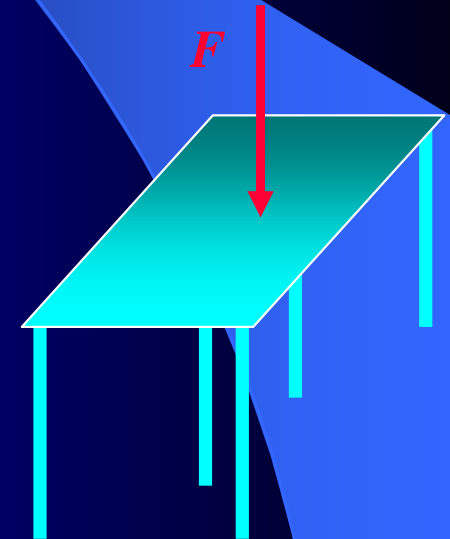
Table à  $N$  pieds (de masse  $Cte$ )

$$S = Cte \sim NR^2$$

$$I \sim R^4 \sim (S/N)^2$$

$$F_C \sim N \frac{EI}{L^2}$$

$$F_C = \frac{1}{N} \frac{ES^2}{L^2}$$



# VII-3 Optimisation Matériau

Recherche du matériau le plus performant en **compression** :  
à géométrie  $L$  et à force  $F$  imposées

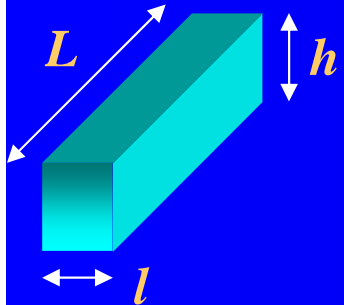
$$m = \rho LS \quad F \sim \frac{EI}{L^2}$$

**Barres** :  $h \sim l \ll L$        $S = f(I)$        $m = \rho L f\left(\frac{FL^2}{E}\right)$

$$S \sim h^2 \quad I \sim h^4$$

$$f = \sqrt{\quad}$$

$$m = \frac{\rho}{\sqrt{E}} L \sqrt{F}$$

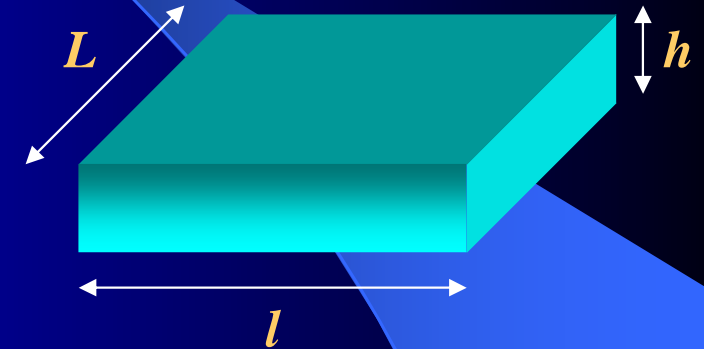


$$S \sim Lh \quad I \sim Lh^3$$

$$f = L^{2/3} \sqrt[3]{\quad}$$

$$m = \frac{\rho}{\sqrt[3]{E}} L^{7/3} \sqrt[3]{F}$$

**Panneaux** :  $h \ll l \sim L$



Performances de quelques Matériaux en compression

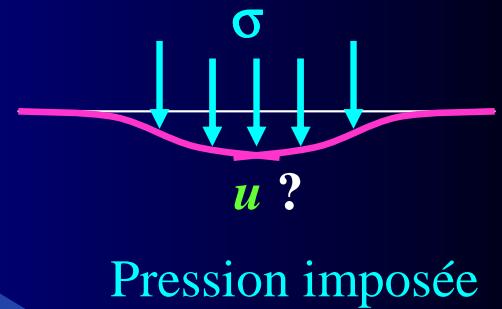
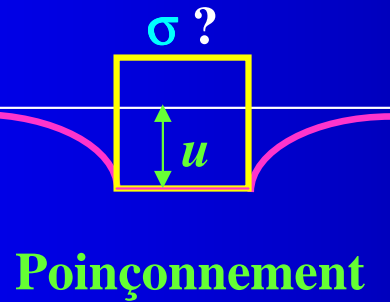
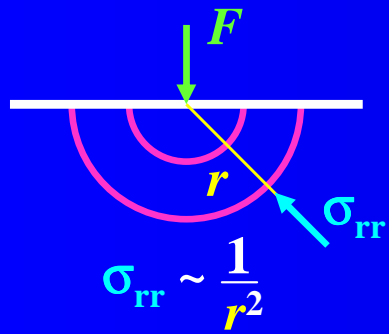
Matériaux	$E$ (Gpa)	$\rho$ (kg.m-3)	$\frac{\sqrt{E}}{\rho}$	$\frac{\sqrt[3]{E}}{\rho}$
Acier	210	7800	59	7,7
Aluminium	73	2800	99	15,0
Béton	15	2500	48	10,0
Composites fibre de carbone	200	2000	223	29,0
Os	18	2000	67	21,1
Bois tendre	10	350	290	61,4
Bois dur	12	650	170	35,5

# VIII Le Contact de Hertz

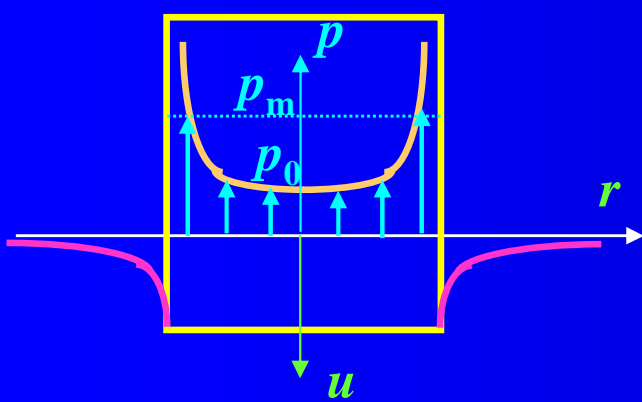
- VIII-1 Le Contact : Phénomène Local
- VIII-2 Le Contact de Hertz
- VIII-3 Plastification et Rupture



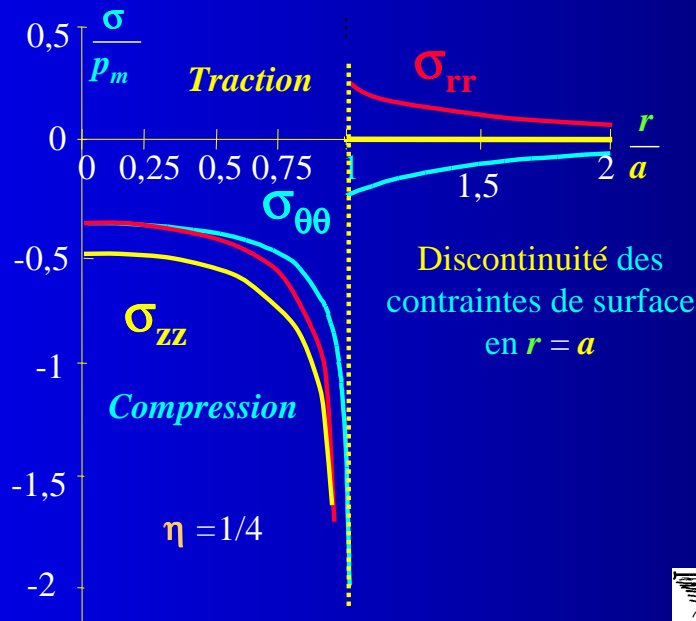
# VIII-1 Le Contact : Phénomène Local



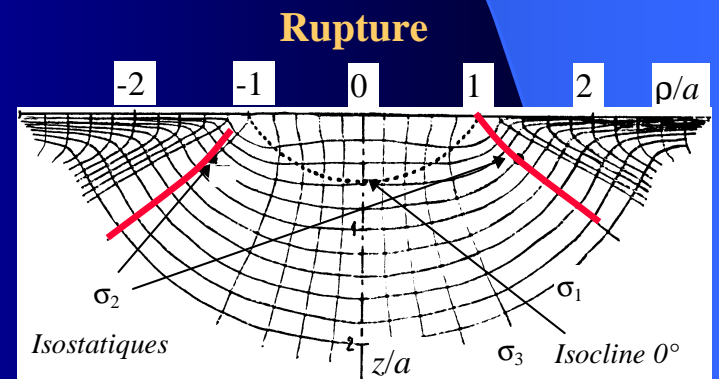
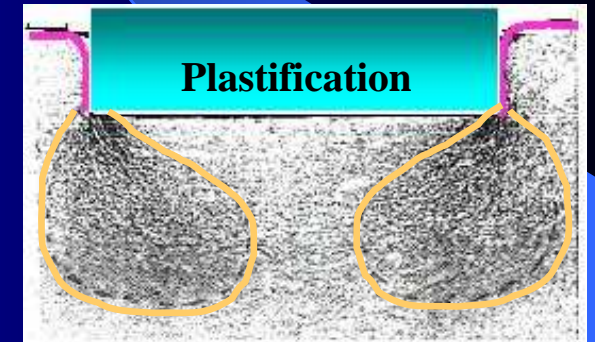
**Poinçonnement**  
*a*



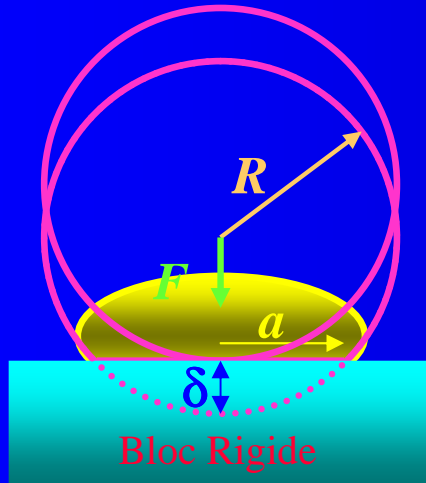
$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}$$



$$p_m = 2p_0$$



# VIII-2 Le Contact de Hertz



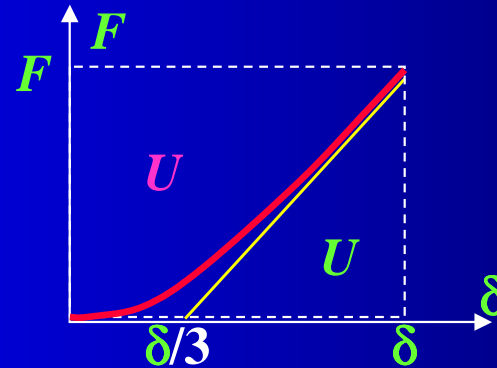
$$\varepsilon = \frac{\sigma}{E} \quad \begin{array}{l} \text{Linéarité} \\ \text{Locale} \end{array}$$



$$\begin{array}{l} F \sim \frac{Ea^3}{R} \\ F \sim ER^{1/2} \delta^{3/2} \end{array} \quad \begin{array}{l} \text{Non Linéarité} \\ \text{Globale} \end{array}$$

$$\sigma \sim \frac{F}{a^2} \quad \frac{1}{R} = \frac{d^2y}{dx^2} \sim \frac{\delta}{a^2}$$

~~$\varepsilon \sim \frac{\delta}{R}$~~  Déformation localisée  $\Rightarrow \varepsilon \sim \frac{\delta}{a}$



Énergie élastique

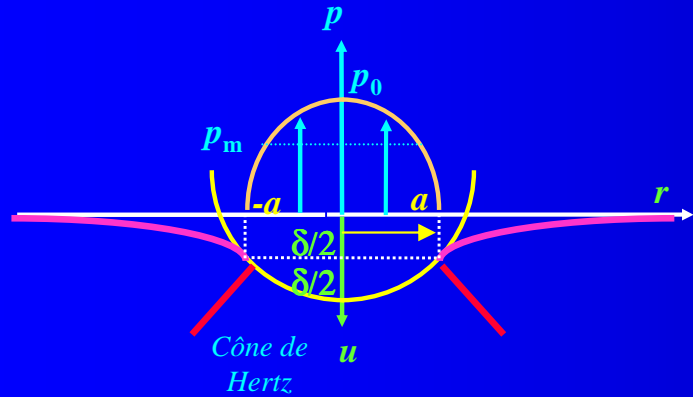
$$U = \frac{1}{2} \int_V \sigma \varepsilon dV \sim \frac{F \delta}{a^2 a} V$$

$$U = \frac{2}{5} F \delta \Rightarrow V \sim a^3$$

Énergie élastique  $U = \int_0^\delta F d\delta = \frac{2}{5} F \delta = \frac{2}{5} ER^{1/2} \delta^{5/2} = \frac{2}{5} E^{-2/3} R^{-1/3} F^{5/3}$   $F = \frac{dU}{d\delta}$   ~~$\delta = \frac{dU}{dF}$~~

Énergie complémentaire  $U = F\delta - U = \frac{3}{5} ER^{1/2} \delta^{5/2} = \frac{3}{5} E^{-2/3} R^{-1/3} F^{5/3}$   $\delta = \frac{dU}{dF}$   ~~$F = \frac{dU}{d\delta}$~~

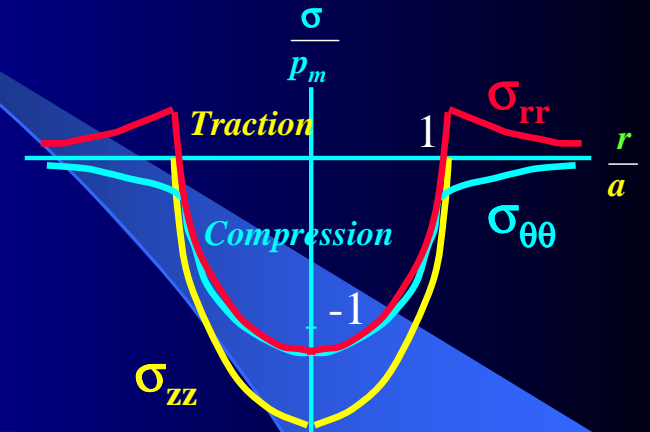
# VIII-3 Plastification et Rupture



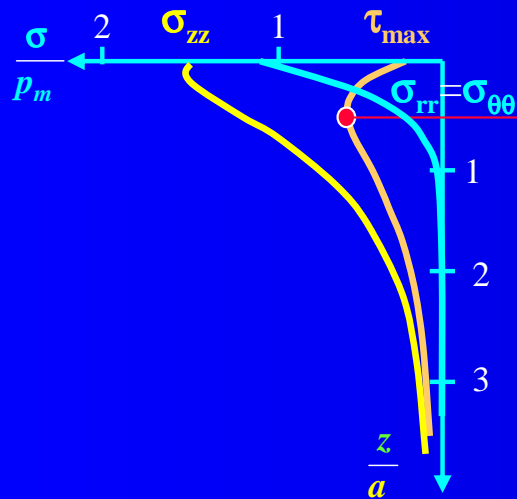
$$p(r) = -p_0 \sqrt{1 - \frac{r^2}{a^2}}$$

$$p_m = \frac{2}{3} p_0$$

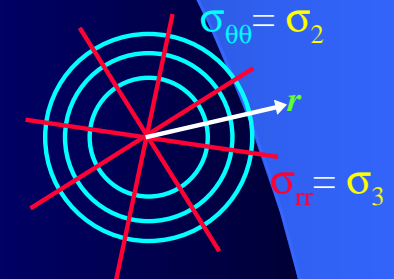
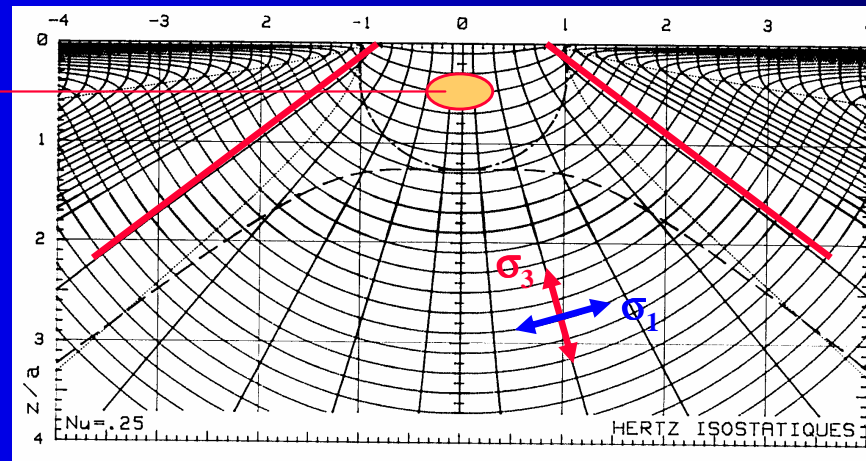
Contraintes en surface



Contraintes le long de l'axe z



Point de Plastification de Hertz



Cône de fissuration de Hertz