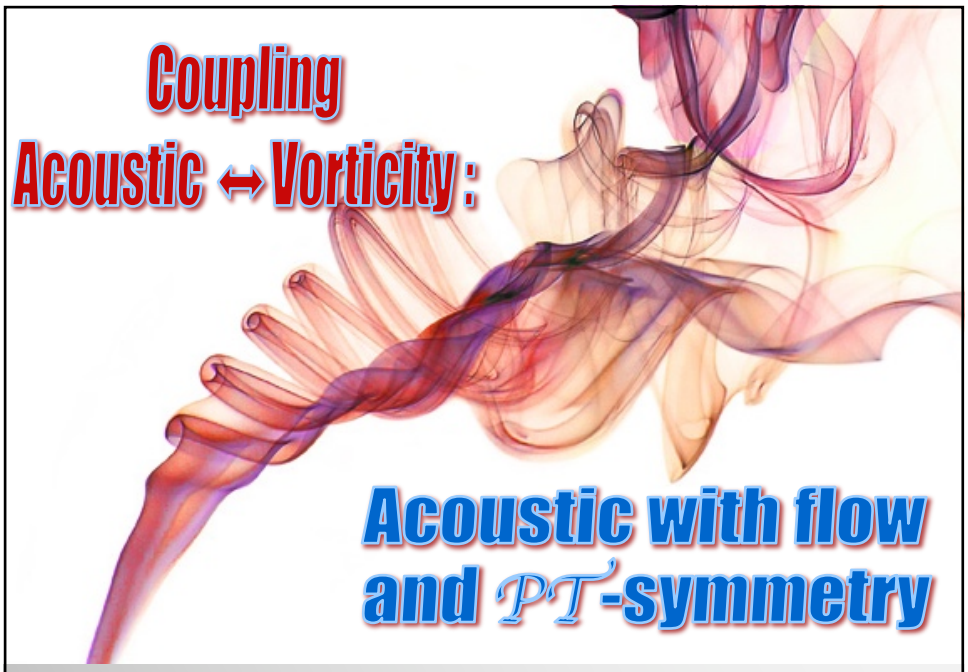
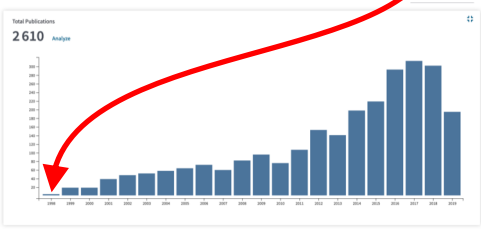


It is a concept coming from **quantum mechanics**.

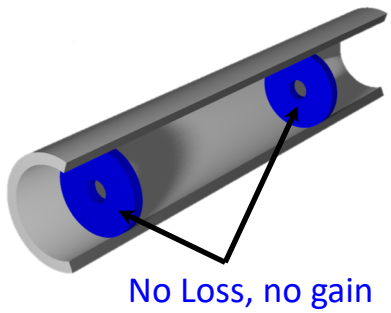
It is commonly believed that the Hamiltonian must be **Hermitian** in order to ensure that the energy spectrum is real and that the probabilities are conserved in time.

Although this axiom is sufficient to guarantee these properties, it is not necessary.

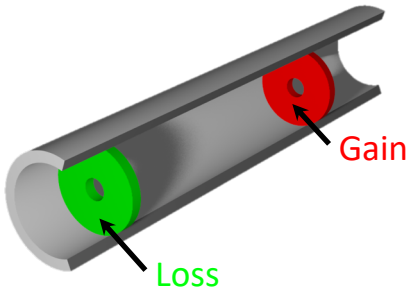
Carl M. Bender *(C. M. Bender and S. Boettcher, Phys. Rev. Lett.* ***80****, 5243, 1998)* demonstrates that there is a more physical alternative axiom, which he refers to as space-time reflection symmetry ($\mathcal{PT}$ symmetry). This symmetry allows for the possibility of **non-Hermitian** and complex Hamiltonians but still leads to a consistent theory of quantum mechanics.



Hermitian acoustics



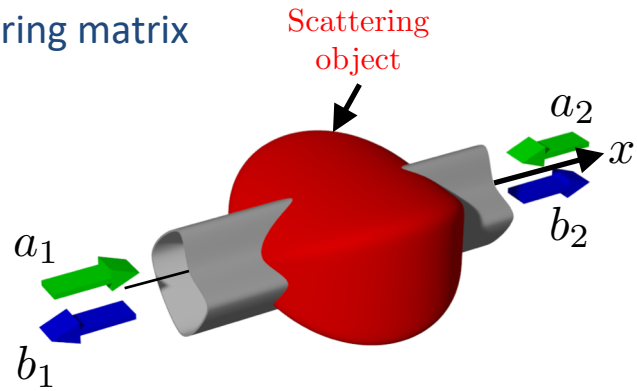
Non-Hermitian acoustics



Particular case : Gain = Loss

can lead to $\mathcal{PT}$-symmetry

Scattering matrix



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underbrace{\begin{bmatrix} r_L & t_R \\ t_L & r_R \end{bmatrix}}_{\text{matrix } S_r} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Scattering matrix

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{bmatrix} r_L & t_R \\ t_L & r_R \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Reciprocity $\quad \mathsf{S_r} = {}^{\mathrm{t}}\mathsf{S_r} \rightarrow t = t_R = t_L$

Time-reversal invariance $\quad \mathsf{S_r}\mathsf{S_r^*} = \mathsf{I_d}$

Energy conservation $\quad {}^{\mathrm{t}}\mathsf{S_r^*}\,\mathsf{S_r} = \mathsf{I_d}$

Hermitian acoustics

Energy conservation
Reciprocity
Time-reversal invariance

(2 of those proprieties imply the third one)

$\rightarrow$ $\mathsf{S_r}$ is a unitary matrix

σ_1^2 and σ_2^2 are the eigenvalues of ${}^{t}\mathsf{S}_r^* \, \mathsf{S}_r$

λ_1 and λ_2 are the eigenvalues of S_r

$$\sigma_1 < |\lambda_1| < |\lambda_2| < \sigma_2$$

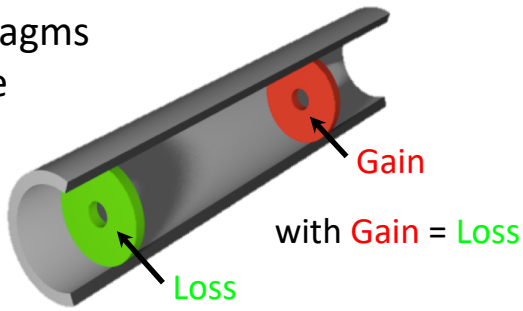
singular values for S_r (real and positive)

For Hermitian acoustics:

$$\sigma_1 = \sigma_2 = 1 \quad \Rightarrow \quad |\lambda_1| = |\lambda_2| = 1$$

Introduction to $\mathcal{PT}$–symmetry

2 diaphragms
in a tube

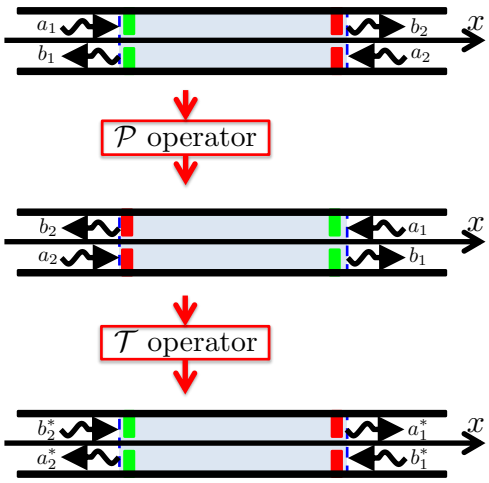


with Gain = Loss

$\mathcal{P}$ operator (Parity operator) $\quad \vec{x} \to -\vec{x}$

$\mathcal{T}$ operator (Time operator) $\quad t \to -t$

Scattering with $\mathcal{PT}$–symmetry



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underbrace{\begin{bmatrix} r_L & t_R \\ t_L & r_R \end{bmatrix}}_{\text{matrix } \mathsf{S_r}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} a_2^* \\ a_1^* \end{pmatrix} = \begin{bmatrix} r_L & t_R \\ t_L & r_R \end{bmatrix} \begin{pmatrix} b_2^* \\ b_1^* \end{pmatrix}$$

Introduction to $\mathcal{PT}$–symmetry

Unusual conservation relation

$$(\mathsf{S_r}\sigma)\,(\mathsf{S_r^*}\sigma) = \mathsf{I}$$

or

$$\mathsf{S_t}\mathsf{S_t^*} = \mathsf{I}$$

$$\mathsf{S_r} = \begin{bmatrix} r_L & t \\ t & r_R \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Pauli matrix

$$\mathsf{S_t} = \mathsf{S}\sigma = \begin{bmatrix} t & r_L \\ r_R & t \end{bmatrix}$$

Different from the energy conservation relations:

S is a unitary matrix

$$\mathsf{S_r}^t\mathsf{S_r^*} = \mathsf{I}$$

$$\mathsf{S_t}^t\mathsf{S_t^*} = \mathsf{I}$$

Introduction to $\mathcal{PT}$–symmetry

$$\mathsf{S}_t\mathsf{S}_t^* = \mathsf{I} \quad \Leftrightarrow \quad \begin{cases} r_R^* r_L + t_R t_R^* = 1 \\ r_L^* r_R + t_L t_L^* = 1 \\ r_R^* t_L + r_R t_R^* = 0 \\ r_L^* t_R + r_L t_L^* = 0 \end{cases}$$

$$\boxed{r_R^* r_L + |t|^2 = 1}$$

Generalization of the more familiar conservation relation $|r|^2 + |t|^2 = 1$, which applies for energy conservation

Introduction to $\mathcal{PT}$–symmetry

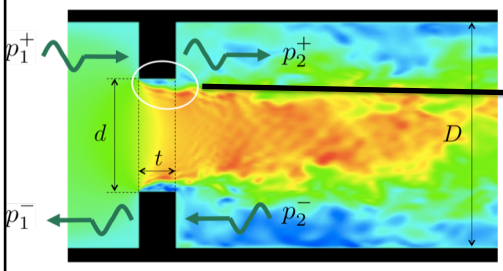
$$\left.\begin{array}{r} s_1,\ s_2 \text{ eigenvalues of } \mathsf{S} \\ \Psi_1,\ \Psi_2 \text{ eigenvectors of } \mathsf{S} \end{array}\right\} \qquad \begin{cases} \mathsf{S}\Psi_1 = s_1\Psi_1 \\ \mathsf{S}\Psi_2 = s_2\Psi_2 \end{cases}$$

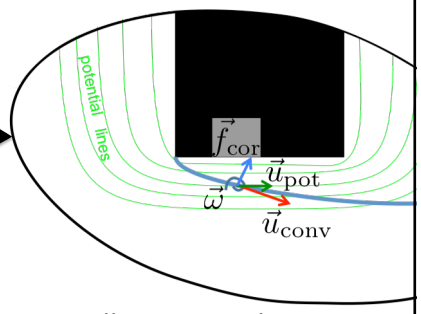
$\mathcal{PT}$–symmetry implies: $|s_1||s_2| = 1$

$$\begin{cases} \text{if} \quad |s_1| = |s_2| = 1 \longrightarrow \text{exact } \mathcal{PT} \text{ symmetric phase} \\ \text{if} \quad |s_1| = 1/|s_2| \neq 1 \longrightarrow \text{broken } \mathcal{PT} \text{ symmetric phase} \end{cases}$$

How to make acoustic gain with flow

Thick diaphragm:





Important parameter: Strouhal number

$$S_t = \frac{f\,t}{U_d}$$

Powell-Howe analogy

$$\mathcal{P} \simeq -\rho_0 \int_V \langle \vec{f}_{\rm cor} \cdot \vec{u}_{\rm pot} \rangle \mathrm{dx}^3$$

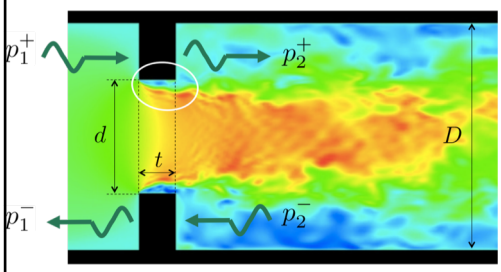
$$\vec{f}_{\rm cor} = \vec{\Omega} \times \vec{u}_{\rm conv}$$

$\mathcal{P} > 0$: gain

$\mathcal{P} < 0$: loss

How to make acoustic gain with flow

Thick diaphragm:



$$t \ll \frac{c_0}{f}$$

velocity conservation

$$u_1 = u_2 \Rightarrow p_1^+ - p_1^- = p_2^+ - p_2^-$$

Resistance $\begin{cases} \text{if } R > 0 \Rightarrow \text{Loss} \\ \text{if } R < 0 \Rightarrow \text{Gain} \end{cases}$

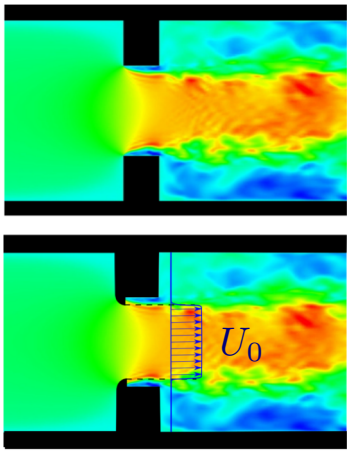
$$p_1 = p_2 + Z u_2$$

Impedance

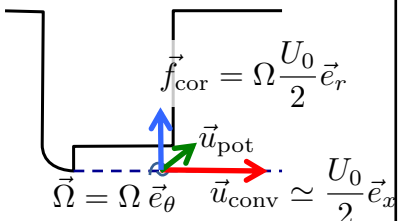
$$Z = R + \mathrm{jX}$$

Reactance

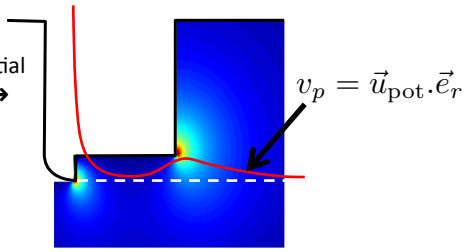
How to made acoustic gain with flow



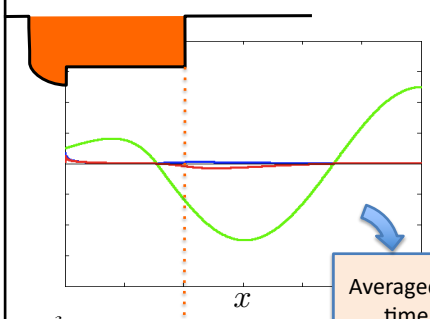
$$\mathcal{P} \simeq -\rho_0 \int_V \langle \vec{f}_{\rm cor} \cdot \vec{u}_{\rm pot} \rangle \mathrm{dx}^3$$



Potential flow →



How to made acoustic gain with flow



Saturated

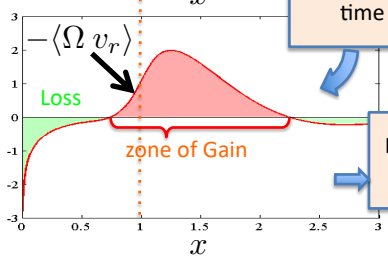
$$\Omega = \Omega_0 \mathrm{e}^{\omega x / U_\Gamma} \sin(\omega(x/U_\Gamma - t))$$

$$v_r = v_r(x) \sin(\omega t)$$

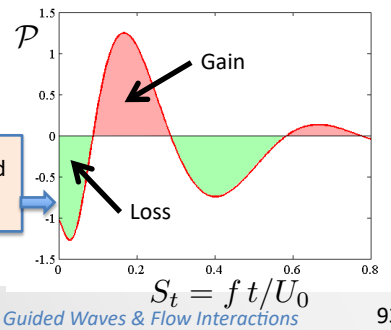
$$-\Omega\, v_r$$

$$\mathcal{P} \simeq -\rho_0 \int_V \langle \vec{f}_{\rm cor} \cdot \vec{u}_{\rm pot} \rangle \mathrm{dx}^3$$

Averaged in time



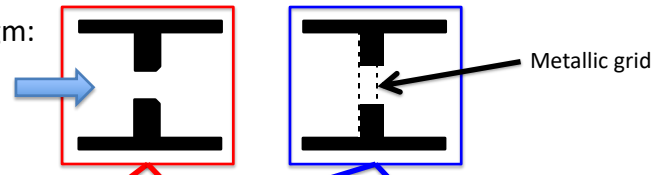
Integrated in space

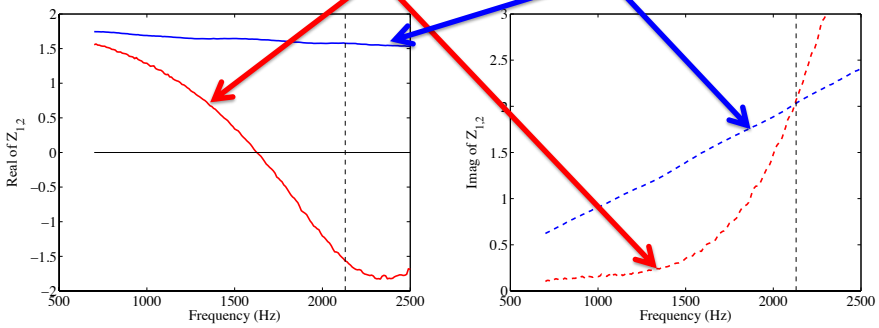


How to made acoustic gain with flow

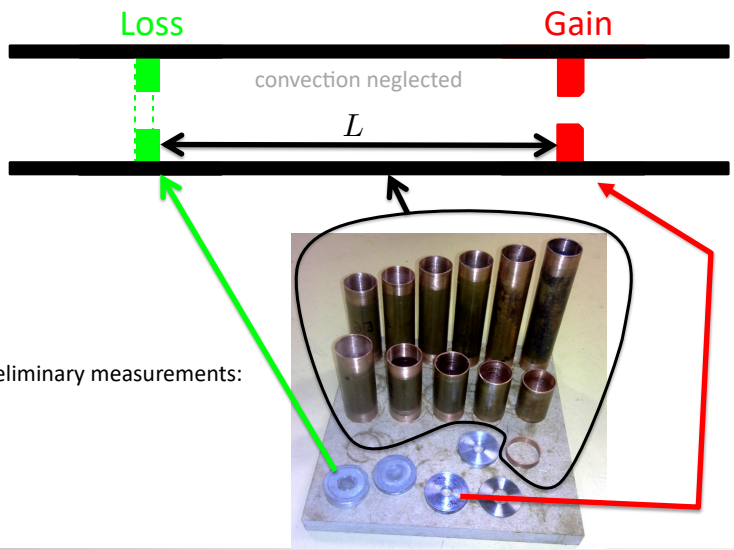
Thick diaphragm:

M=0.02



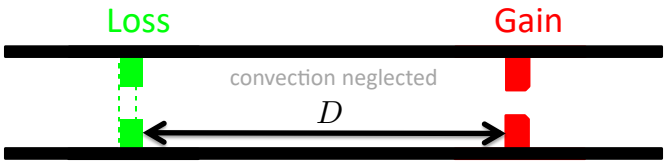


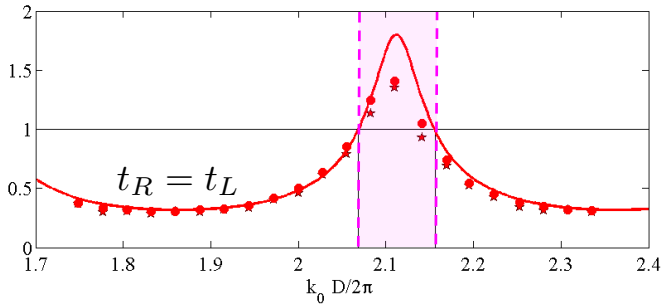
An acoustic $\mathcal{PT}$ symmetric system with flow



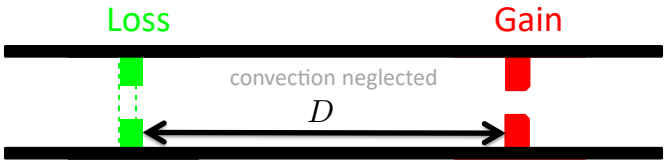
Preliminary measurements:

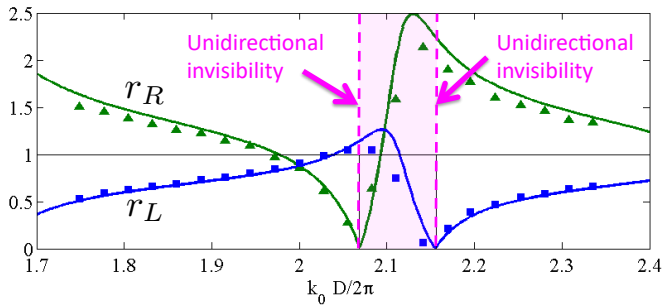
An acoustic $\mathcal{PT}$ symmetric system with flow



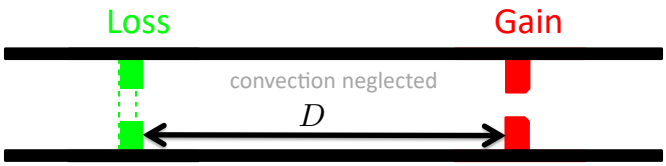


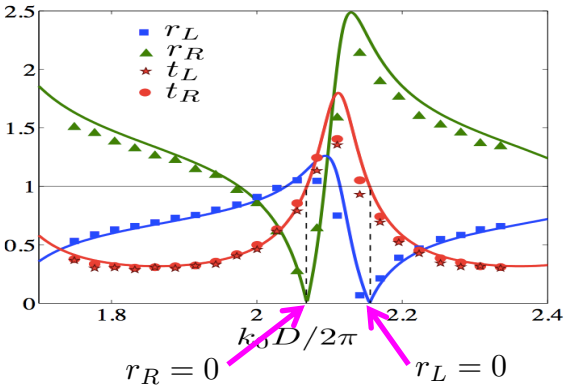
An acoustic $\mathcal{PT}$ symmetric system with flow





An acoustic $\mathcal{PT}$ symmetric system with flow

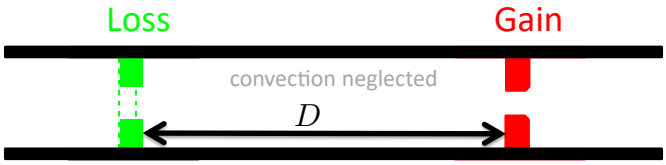


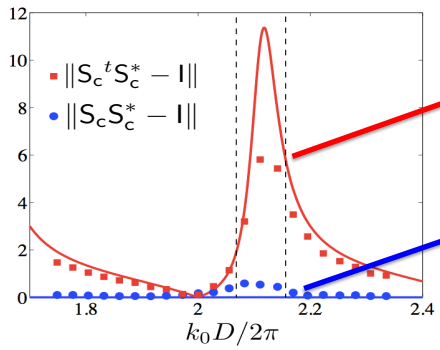


Unidirectional "invisibility":
Zero reflection only form one side
$(r_L = 0 \text{ or } r_R = 0)$

$$r_R^* r_L + |t|^2 = 1$$

An acoustic $\mathcal{PT}$ symmetric system with flow

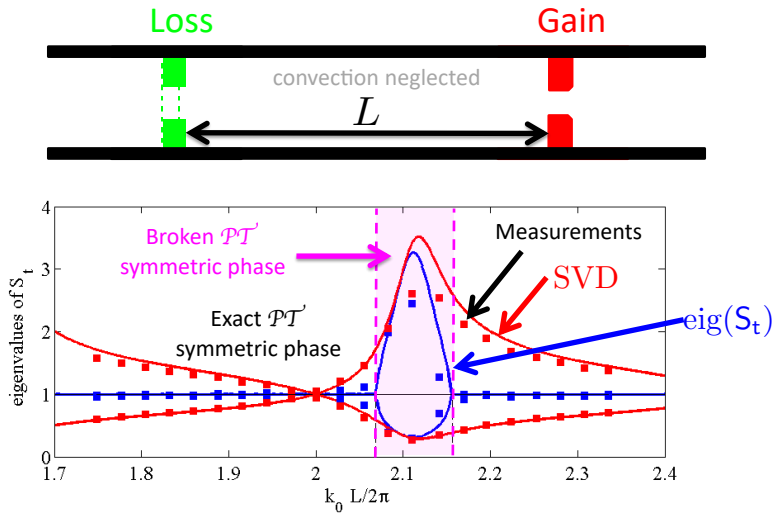




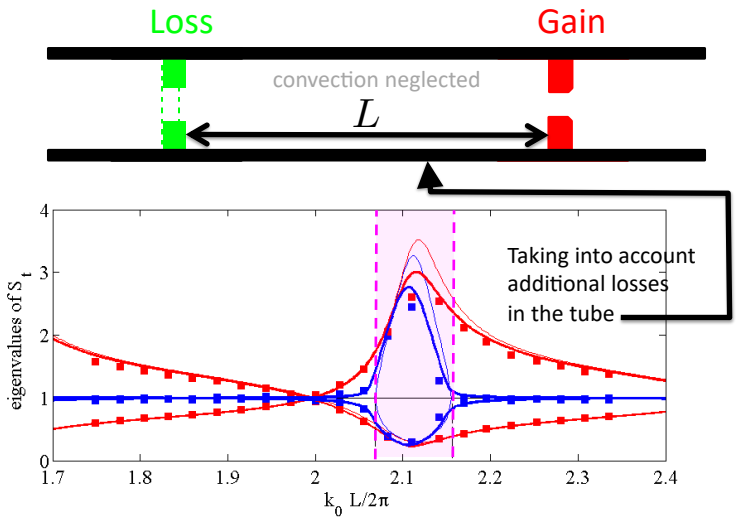
Non-unitary S (energy not conserved)

$\mathcal{PT}$ symmetric property for S

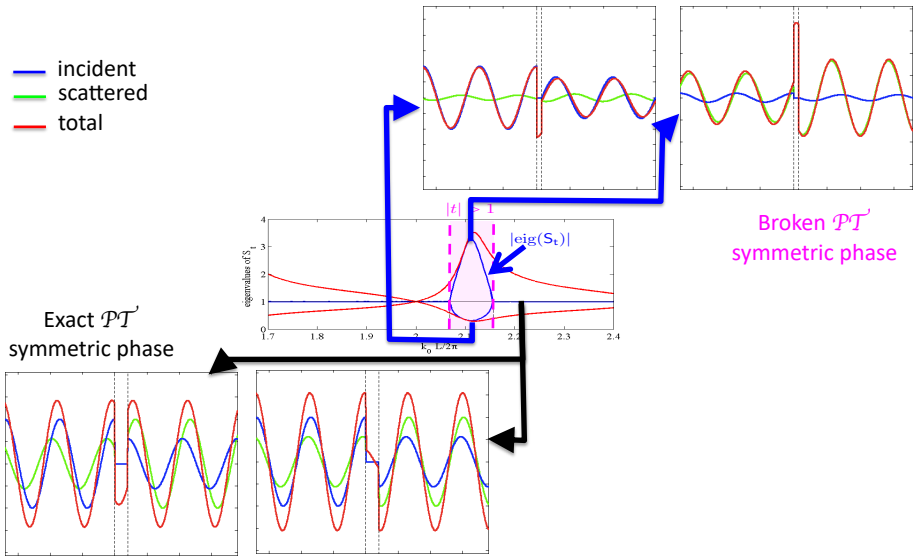
An acoustic $\mathcal{PT}$ symmetric system with flow



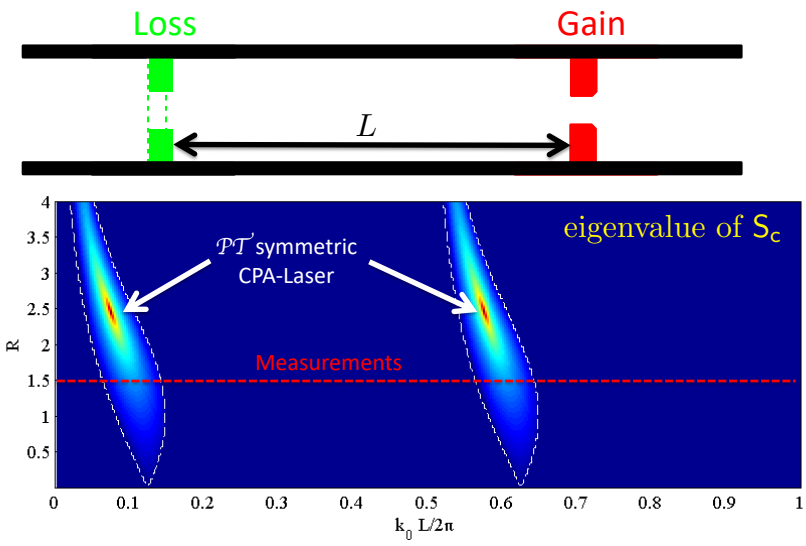
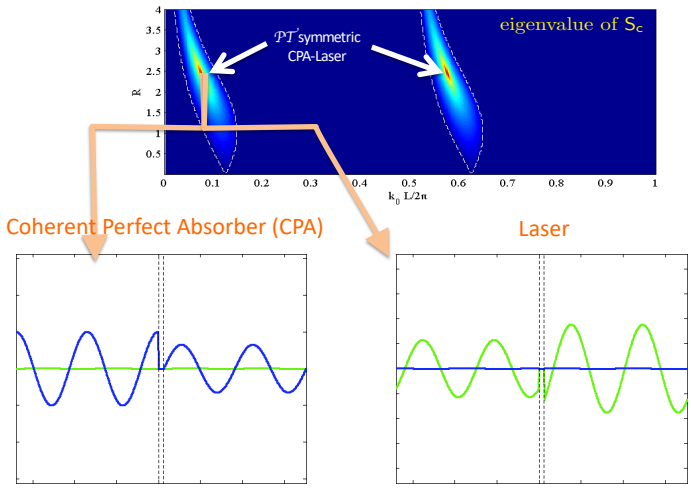
An acoustic $\mathcal{PT}$ symmetric system with flow



An acoustic $\mathcal{PT}$ symmetric system with flow



An acoustic $\mathcal{PT}$ symmetric system with flow



An acoustic $\mathcal{PT}$ symmetric system with flow