Complementary slides

on

Discrete homogenization

Homogenization of discrete media -1

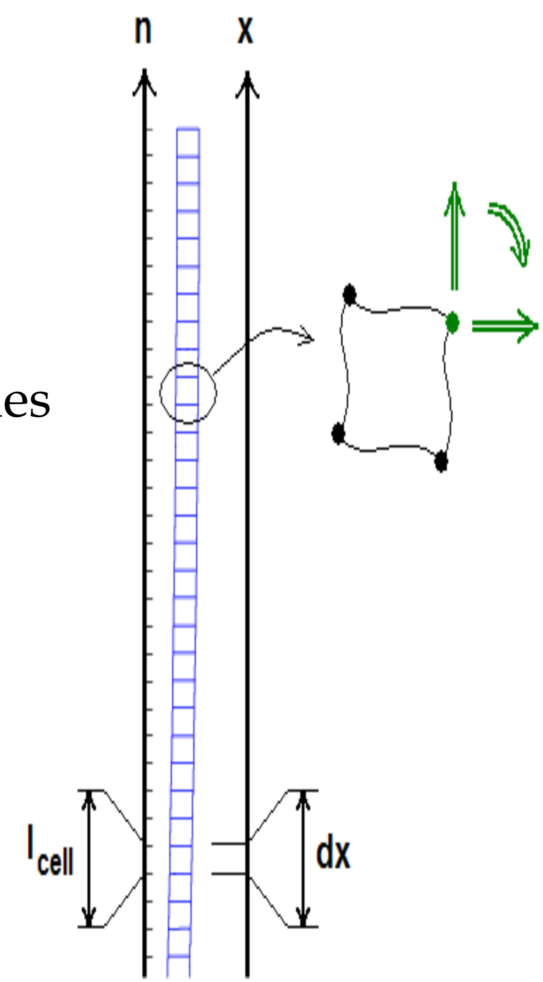
Discrétisation

Variables nodales cinématiques : Inconnues

Intégration de la dynamique de poutre : Forces nodales

Equilibre des Forces nodales

Discrétisation exacte de la dynamique du système



Séparation d'échelles

Taille de cellule l << Longueur d'onde (modale)

? = l_{cell}/L << 1

Homogenization of discrete media - 2

Discretized system

Homogenisation 1D (2D)

–Macro variable x $l = \varepsilon L = dx$

–$U(n) = U(x_n)$,…. Variables continues

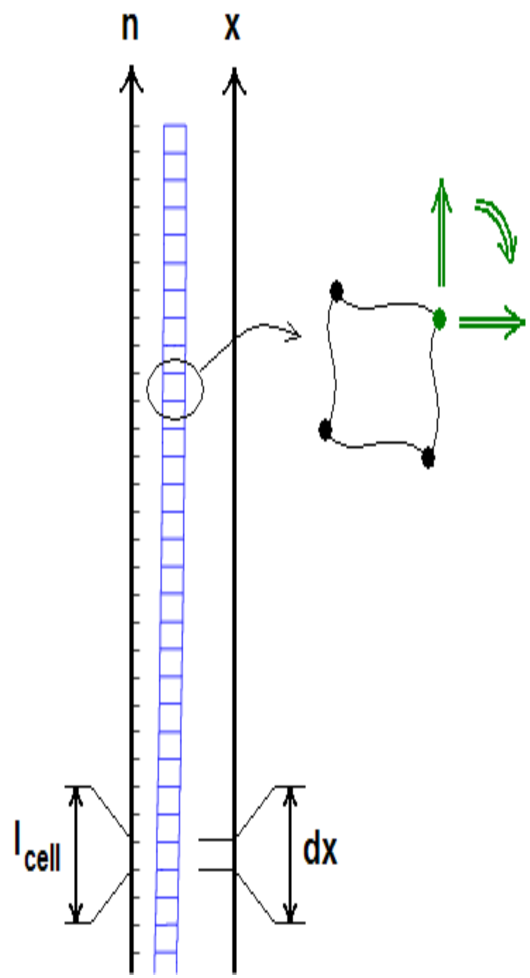
–Variables cinématiques Développées en ε

$$U(x) = \Sigma\, \varepsilon^i U^i(x)$$

–Incréments sur l Séries de Taylor
Macro Dérivée

–Forces nodales (l,…) Développées en $l = \varepsilon L$

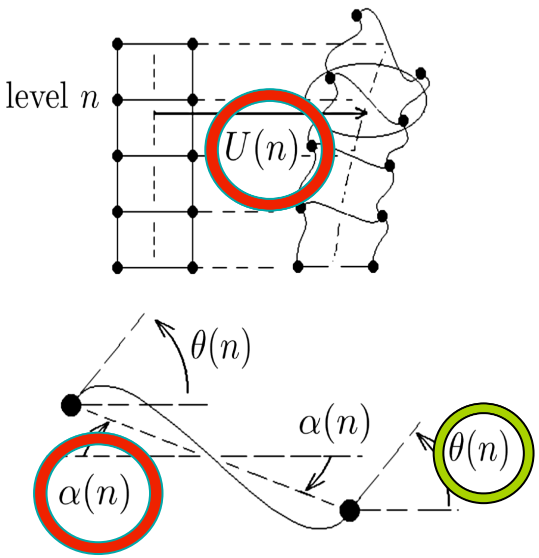
–**Normalisation** (ε^i) de a_m/l_m ; a_p/l_m ; ω/ω_r
–Développements ==> Equilibres nodaux
Résolution par ordre croissant

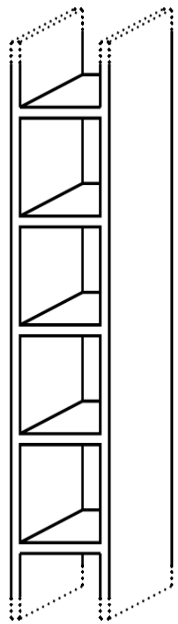


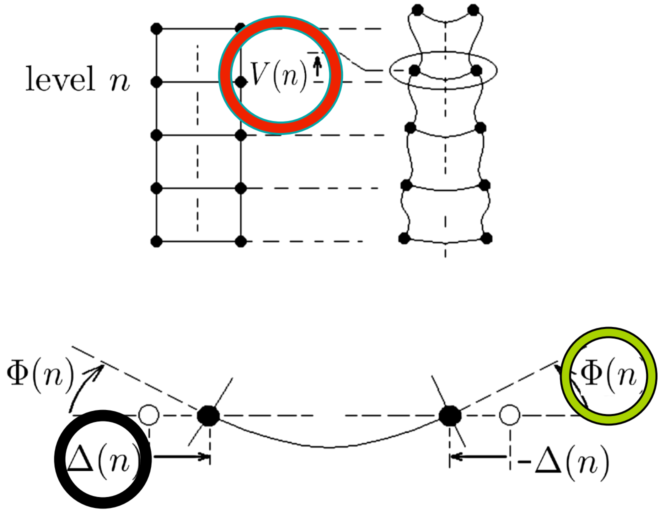
Vibrations Transverses / Longitudinales

Cellules symétriques

Deux jeux de variables cinématiques indépendantes







Transverse		Longitudinal
U ; α	Mouvement de corps rigide de section	V
θ	Deformation de section	Δ ; ϕ

Homogenization of discrete media -2

Expansion of the efforts

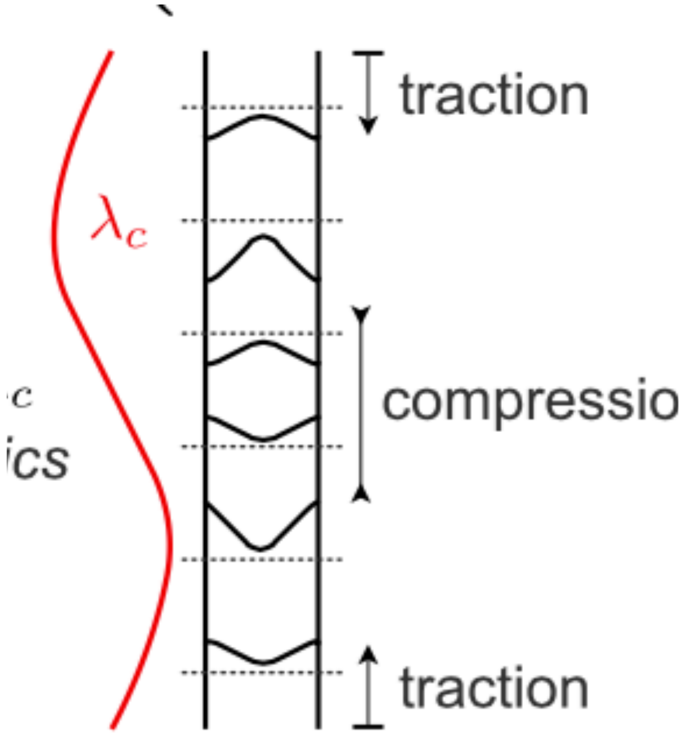
λ_c compression wavelength in the elements
λ_b bending wavelength in the elements

➔ Classical homogenization $l << \lambda_b << \lambda_c$

Expansion of N, M ,T ➔ Newtonian macro dynamics

➔ Bending resonance $l \approx \lambda_b << \lambda_c$

Expansion of N, only ➔ Atypical dynamics



Complementary slides

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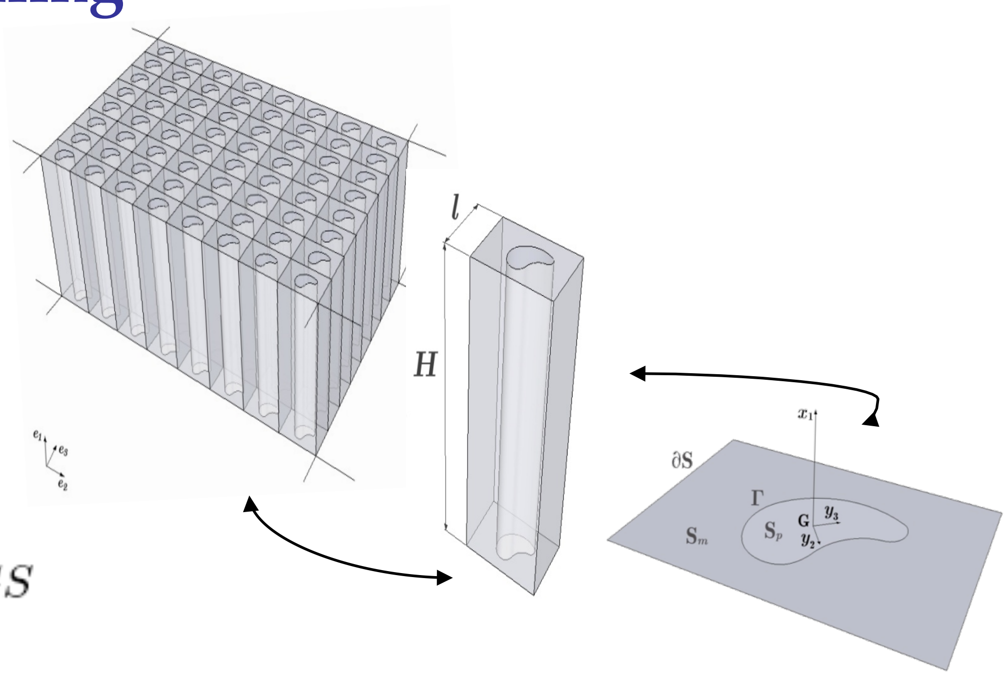
Reinforced media

Scaling



Scale separation

$l/H = \varepsilon << 1$

2D-In plane Periodicity

Scaling of parameters

Soil/Pile contrast $\quad \dfrac{EI}{H^2} \approx GS$

$$G = EI/SH^2 = E(l/H)^2 = \varepsilon^2 E$$

Scaling of motions

Almost free beam $\quad {}^p\sigma_{1\alpha} = \mu_p \left({}^p u_{1,x_\alpha} + {}^p u_{\alpha,x_1} \right)_\alpha \simeq 0$

$${}^p u_{1,x_\alpha} = O\left({}^p u_1 / l \right) \qquad {}^p u_{\alpha,x_1} = O\left({}^p u_\alpha / L \right)$$

$$O\left({}^p u_1 \right) = \varepsilon \, O\left({}^p u_\alpha \right)$$

Inner bending/Second gradient continua

Interpretation of the balance equation

$$\underbrace{-\frac{E_p I_{p\alpha}}{S} U^0_{\alpha,x_1x_1x_1x_1}}_{BENDING} + \underbrace{C^{1\beta}_{1\alpha}\frac{1}{2}U^0_{\beta,x_1x_1}}_{SHEAR} = 0$$

« Matrix » loaded by transverse force of beams

$$[CU_{,x_1}]_{,x_1} = \frac{1}{S}[M_{,x_1}]_{,x_1} \quad , \quad M = EIU_{,x_1x_1}$$

Beam loaded by shear forces in the « matrix »

$$[T]_{,x_1} = S[CU_{,x_1}]_{,x_1} \quad , \quad T - M_{,x_1} = 0 \quad , \quad M = EIU_{,x_1x_1}$$

Influence of boundary conditions

Beams clamped at both extremities

$$U(x_1) = a\left\{x_1 - \frac{\mathcal{L}}{\sinh(H/\mathcal{L})}[\cosh(x_1/\mathcal{L}) - \cosh((x_1 - H)/\mathcal{L}) + \cosh(H/\mathcal{L}) - 1]\right\}$$

Beams clamped on x= H, free on x = 0

$$U(x_1) = a\left\{x_1 - \frac{\mathcal{L}}{\cosh(H/\mathcal{L})}\sinh(x_1/\mathcal{L})\right\}$$

Beams free at both extremities

$$U(x_1) = ax_1$$

General Kinematics

$$\underline{U}(\underline{x}) = \underline{U}^0(\underline{x}) + \varepsilon^2 \underline{U}^2(\underline{x}) + \ldots = \underline{U}^0(\underline{x}) + \underline{\widetilde{U}}^2(\underline{x}) + \ldots$$

$$\langle \underline{\underline{\sigma}} \rangle(\underline{x}) = \langle \underline{\underline{\sigma}}^0 \rangle(\underline{x}) + \varepsilon^2 \langle \underline{\underline{\sigma}}^2 \rangle(\underline{x}) + \ldots = \langle \underline{\underline{\sigma}}^0 \rangle(\underline{x}) + \langle \underline{\underline{\widetilde{\sigma}}}^2 \rangle(\underline{x}) + \ldots$$

$$\underline{\mathrm{div}}_x(\langle \underline{\underline{\sigma}}^0 \rangle) = \underline{0}$$

$$\langle \underline{\underline{\sigma}}^0 \rangle = E_p \frac{|S_p|}{|S|} U^0_{1,x_1} \underline{a}_1 \otimes \underline{a}_1$$

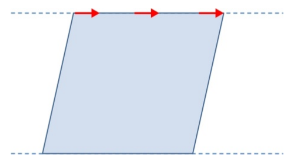
$$\underline{\mathrm{div}}_x(\langle \underline{\underline{\widetilde{\sigma}}}^2 \rangle) = \langle \underline{\widetilde{b}}^2 \rangle$$

$$\langle \underline{\underline{\widetilde{\sigma}}}^2 \rangle = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\mathrm{e}}}_x(\underline{U}^0) + E_p \frac{|S_p|}{|S|} \widetilde{U}^2_{1,x_1} \underline{a}_1 \otimes \underline{a}_1 - \underline{\underline{\mathbb{S}}} - \underline{\underline{\mathbb{S}}}'$$

$$\begin{aligned}\underline{\underline{\mathbb{S}}} = & -E_p \Big\{ \frac{I_{p\alpha}}{2|S|} ([U^0_{1,x_\alpha} + U^0_{\alpha,x_1}]_{,x_1 x_\alpha} + [U^0_{\alpha,x_\alpha}]_{,x_1 x_1}) + \nu_p \frac{I_{p2} + I_{p3}}{2\,|S|} [U^0_{1,x_1}]_{,x_1 x_1} \Big\} \underline{a}_1 \otimes \underline{a}_1 \\ & + E_p \frac{I_{p\alpha}}{|S|} [U^0_{1,x_\alpha} + U^0_{\alpha,x_1}]_{,x_1 x_1} (\underline{a}_1 \otimes \underline{a}_\alpha + \underline{a}_\alpha \otimes \underline{a}_1)\end{aligned}$$

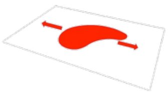
$$\underline{\underline{\mathbb{S}}}' = \mu_p \mathcal{J}_p [\underline{a}_1 \otimes \underline{\mathrm{curl}}_x({}^p\Omega^0_{,x_1} \underline{a}_1) + \underline{\mathrm{curl}}_x({}^p\Omega^0_{,x_1} \underline{a}_1) \otimes \underline{a}_1]$$

Transverse kinematics $U_2(x_1)$

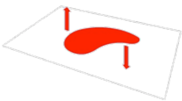


Resolution

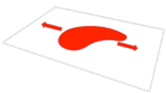
Beam | **Matrix**



$^{P}u^{0}{}_{2} = U_2(x_1)$



$^{P}u^{1}{}_{1} = -y_2\, \partial U_2/\partial x_1$



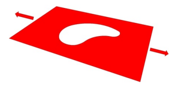
$^{P}e^{1} = -[\boldsymbol{I}_{\nu}]\ y_2 \partial U_2/\partial x_1$

$^{P}\sigma^{1}{}_{11} = -E_p\, y_2 \partial U_2/\partial x_1$

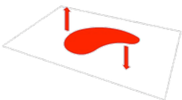
$^{P}u^{2} = \ \ldots.$

$^{m}u^{0}{}_{2} = U_2(x_1)$

$[^{m}\sigma^{1}] = 0$

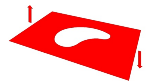


$\partial\, ^{P}\sigma^{1}{}_{11}/\partial x_1 + \mathrm{div}_y([^{P}\sigma^{2}]) = 0$



$\mathrm{div}_y\,([^{m}\sigma^{2}]) = 0$

$^{m}u^{1}{}_{1} = -\,\Psi(y)\partial U_2/\partial x_1$



$$^{p}\underline{\underline{\sigma}} = \begin{bmatrix} ^{p}\sigma^{1}_{11} & ^{p}\sigma^{2}_{12} & ^{p}\sigma^{2}_{13} \\ ^{p}\sigma^{2}_{12} & \ldots & \ldots \\ ^{p}\sigma^{2}_{13} & \ldots & \ldots \end{bmatrix}$$

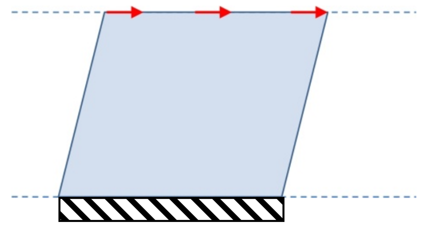
$$^{m}\underline{\underline{\sigma}} = \begin{bmatrix} \ldots & ^{m}\sigma^{2}_{12} & ^{m}\sigma^{2}_{13} \\ ^{m}\sigma^{2}_{12} & \ldots & \ldots \\ ^{m}\sigma^{2}_{13} & \ldots & \ldots \end{bmatrix}$$

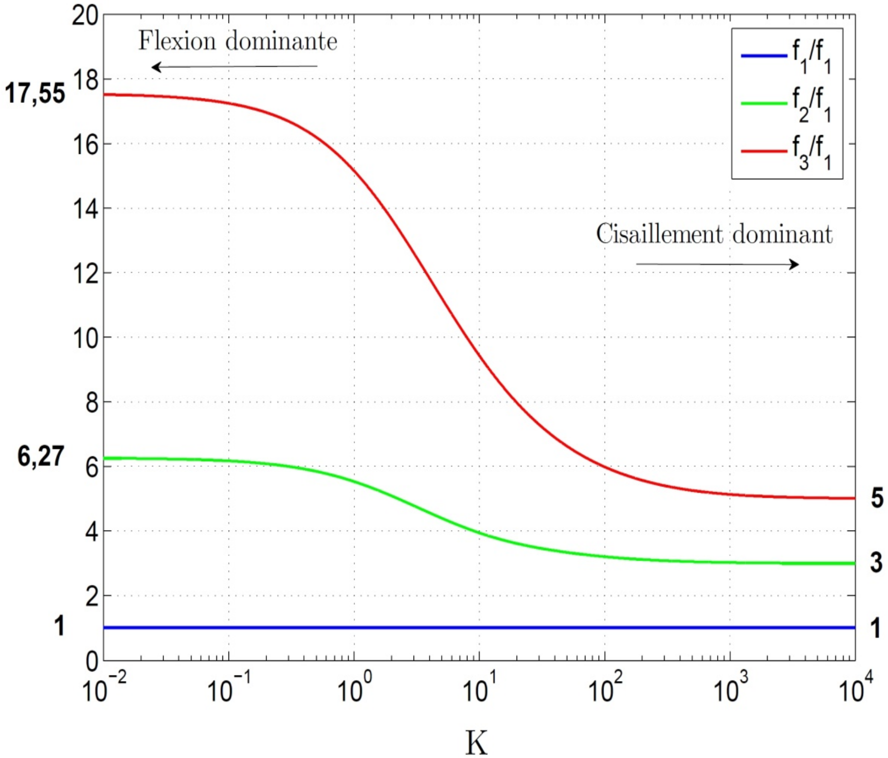
Modes of a Soil-Pile reinforced layer

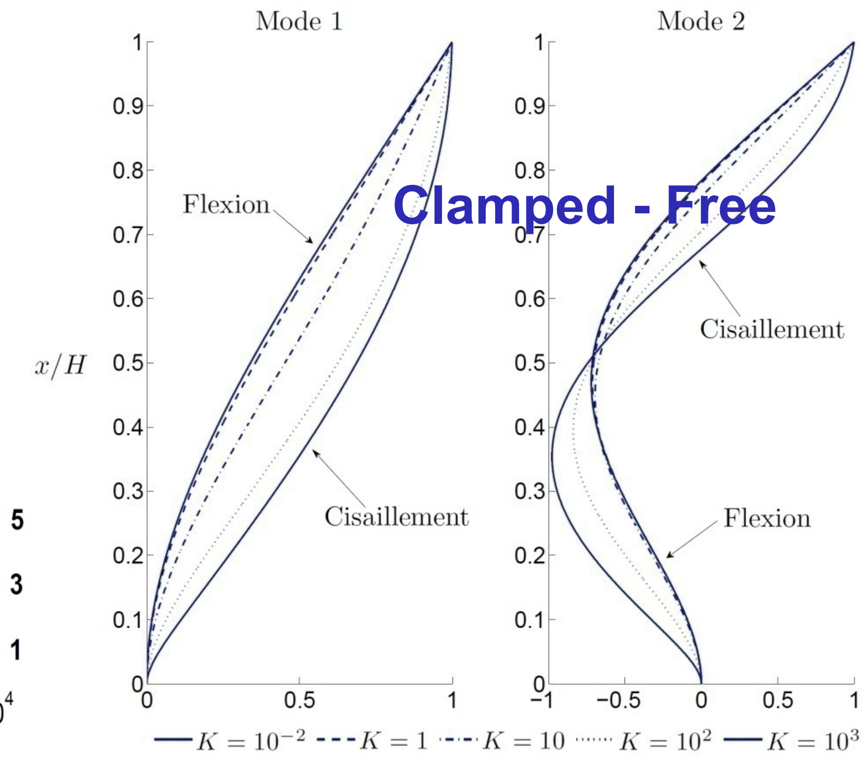
Eigen modes and frequencies depends on

Soil & Pile boundary conditions

Bending/Shear ratio $K = EI/GSH^2$







To sum up

Complement of

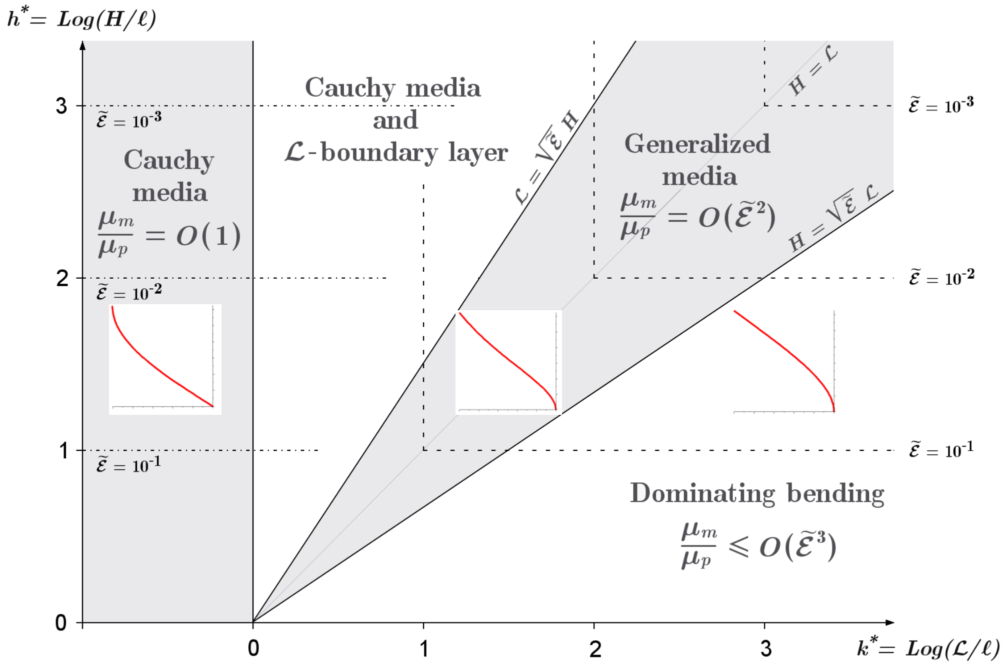
Usual composite — Léné 78, Sanchez-Palencia 1980, Postel 1985
Multiphasic — DeBuhan 1999
Energetic — Pideri & Seppecher 1997

Experimental evidence of second gradient effects

Leading order
Dynamics

Validity for real media

$\mathcal{L} = (E_p I_p / \mu_m S)^{1/2}$



Longitudinal Kinematics $U_1(x_1)$

Macro dynamics & Matrix inner dynamics

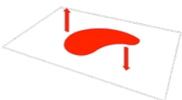
$L = \Lambda/2\pi = O(\sqrt{{}^{P}\mu/\langle\rho\rangle})$

${}^{m}\lambda/2\pi = O(\sqrt{{}^{m}\mu/\langle\rho\rangle}) = O(\varepsilon\sqrt{{}^{P}\mu/\langle\rho\rangle}) = O(\varepsilon\, L) = O(l)$:

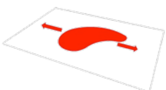
Resolution

Beam

Matrix



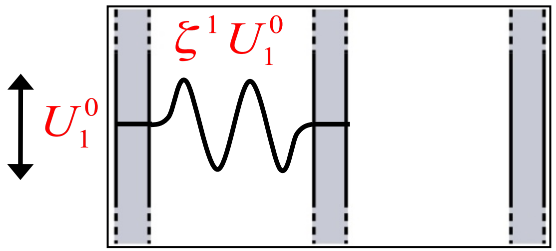
${}^{P}u^{0}{}_{1} = U_1(x_1)$

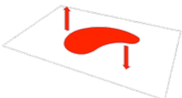


${}^{P}e^{1} = -[\boldsymbol{I}_v]\ \partial U_1/\partial x_1$

${}^{P}\sigma^{1}{}_{11} = -E_p\, \partial U_1/\partial x_1$

${}^{P}u^{2} = \ \ldots.$

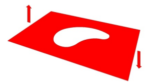




$\partial\, {}^{P}\sigma^{1}{}_{11}/\partial x_1 + \mathrm{div}_y([{}^{P}\sigma^{2}]) = -{}^{P}\rho\omega^2 U_1$

$\mathrm{div}_y\,([{}^{m}\sigma^{2}]) = -{}^{m}\rho\omega^{2}\, {}^{m}u^{1}{}_{1}$

${}^{m}u^{1}{}_{1} = \zeta(y, \omega) U_1$



Complementary slides

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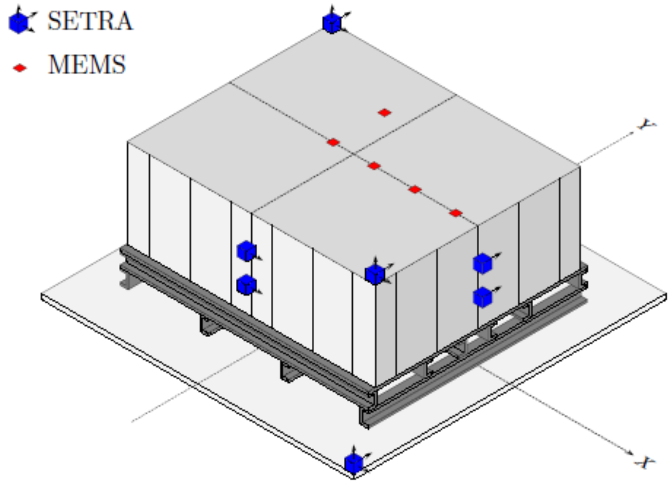
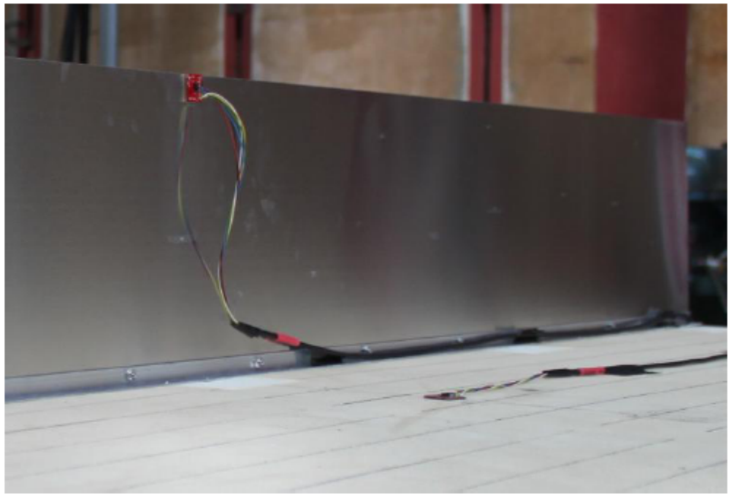
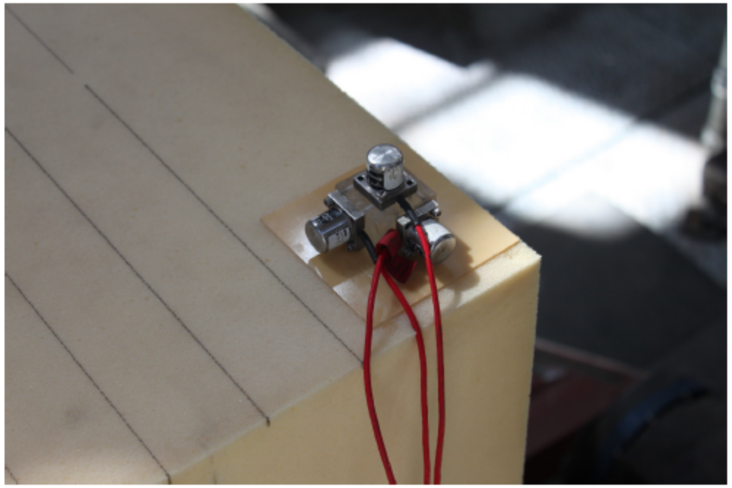
Resonant surfaces

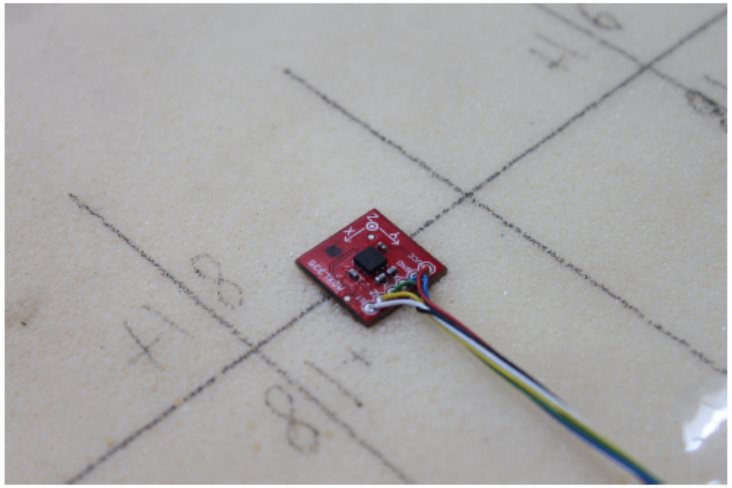
Instrumentation





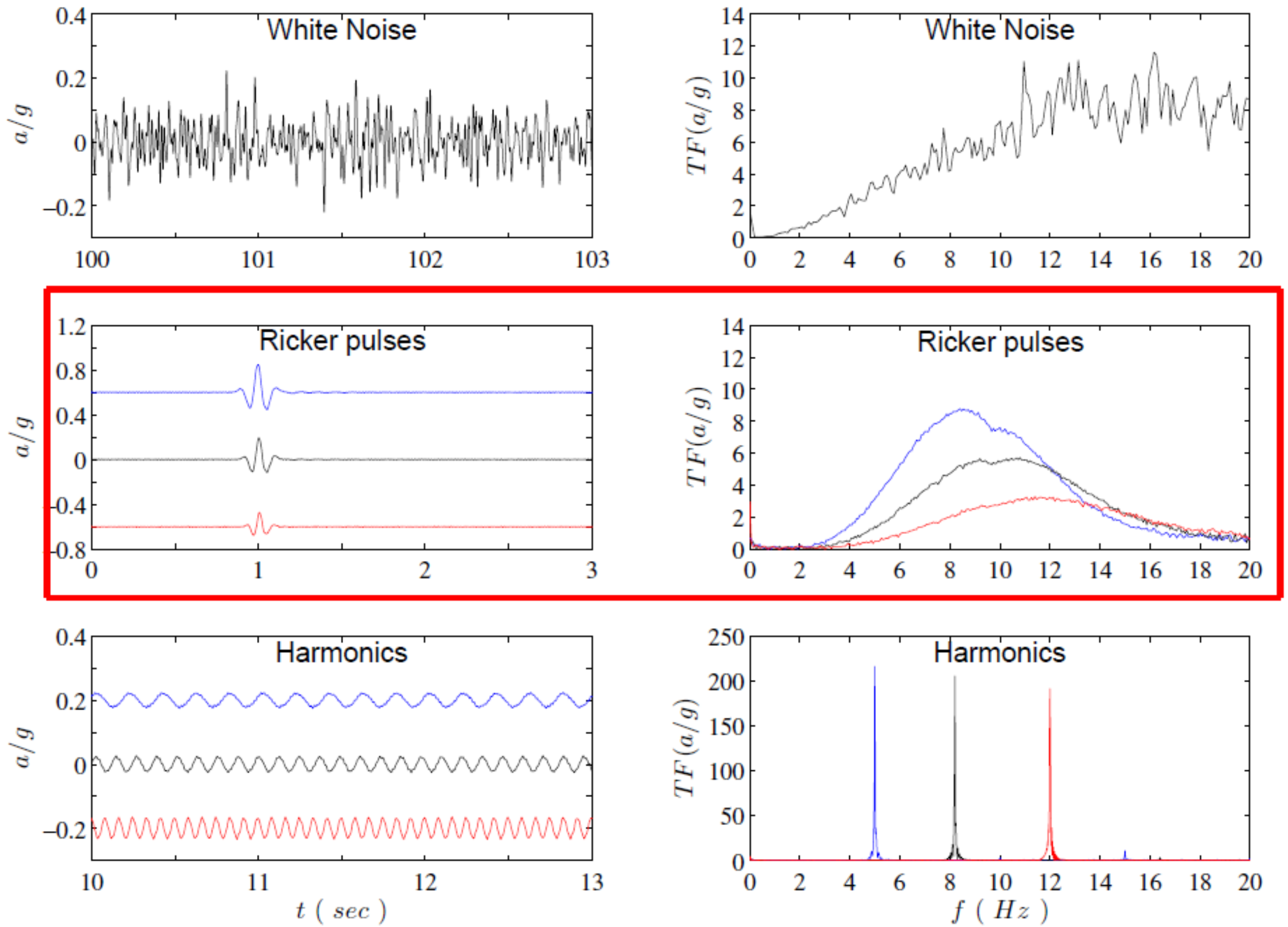


SETRA : 1D, 30 grams each, $8\ cm$-wide base plate



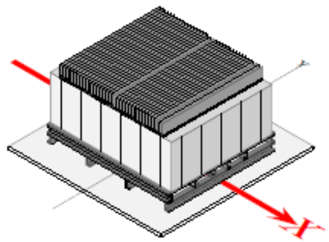
MEMS : 3D, 2 grams, $2\ cm$ wide

Shakings



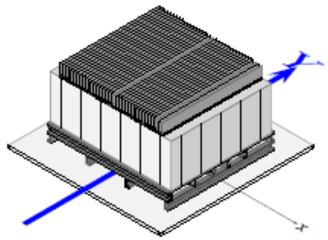
in X

resonant direction



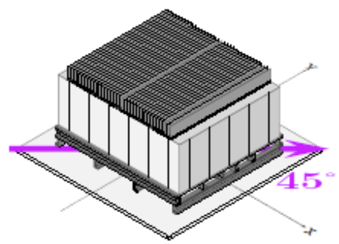
in Y

inert direction

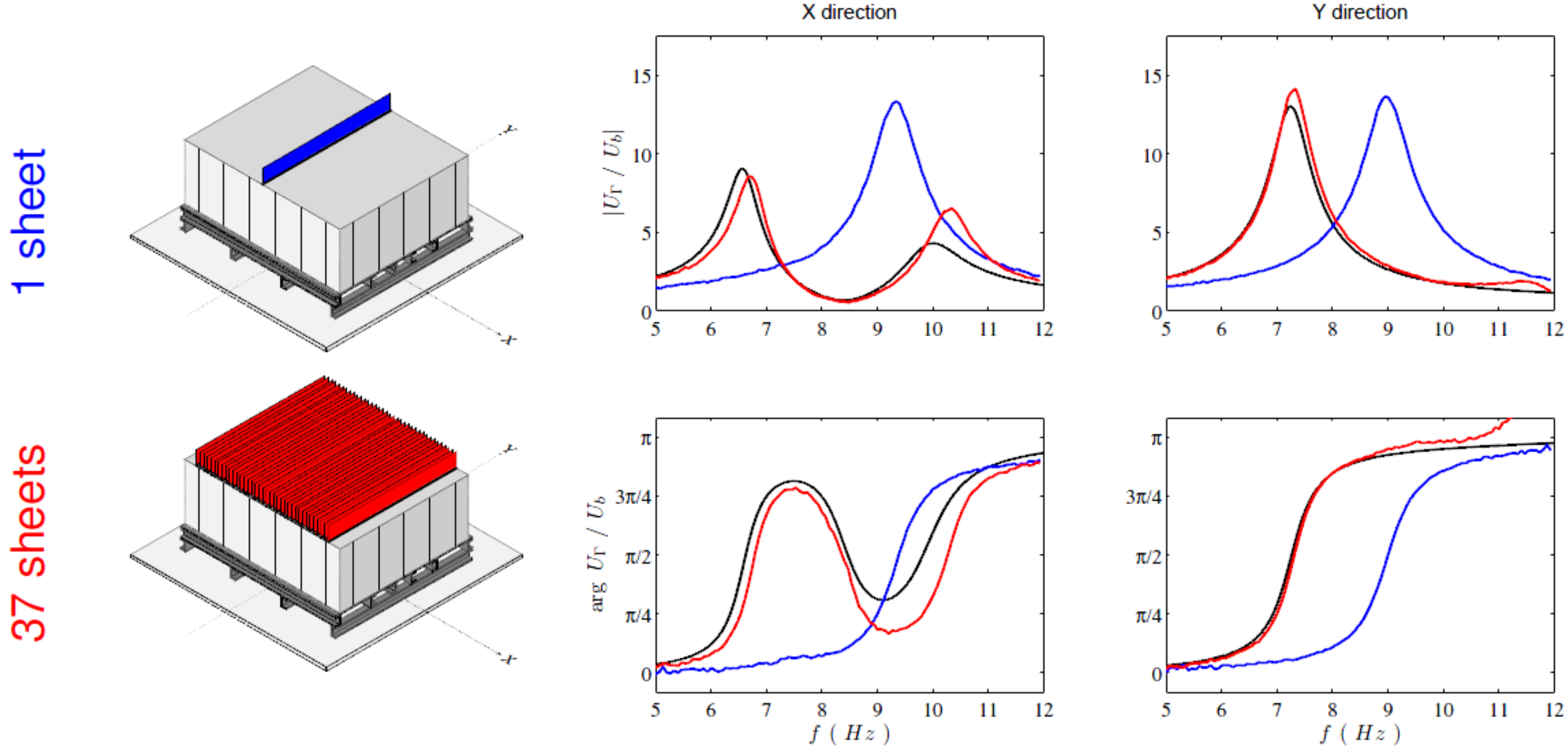


at 45°

superposition of X and Y



Drastic changes in spectrum Surface/Table



- 1 resonator : usual layer's resonance
- 37 resonators : in X resonant direction : drastic change in layer's resonance
- 37 resonators : in Y inert direction : usual resonance peak
- City impedance analysis : qualitatively and quantitatively accurate

Complementary slides

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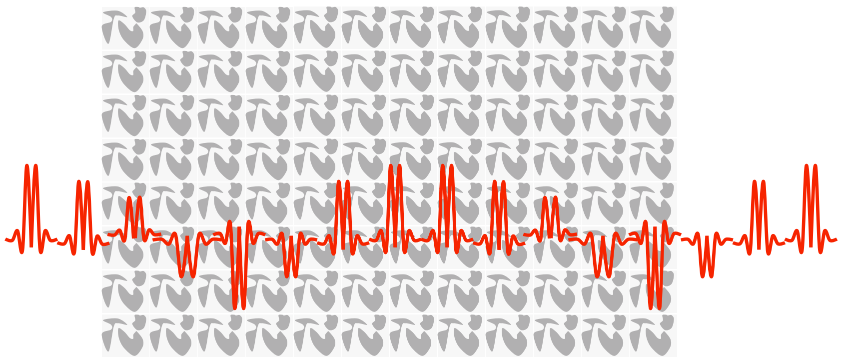
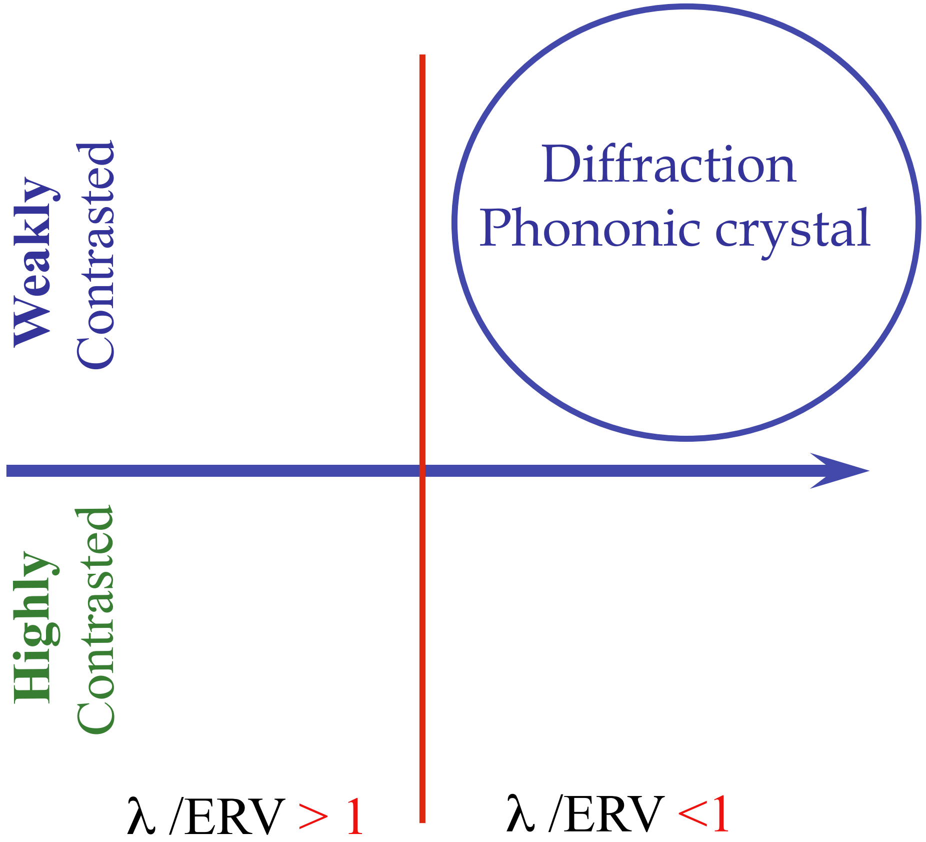
Modulation approach at high frequency

Short Wavelengths

Weakly Contrasted Composites





Bloch waves

No scale separation for U

Then ?

∃? Macrodescription ➔ ***Daya et al., 2002 ; Craster et al., 2010 ; Boutin et al.,2012***

Augmented Paradigm of Scale Separation

Continuum modeling for the modulated vibration modes of large repetitive structures

El Mostafa Daya [a], Bouazza Braikat [b], Noureddine Damil [b], Michel Potier-Ferry [a]

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[b] Laboratoire de calcul scientifique en mécanique, Faculté des sciences Ben M'Sik, Université Hassan II – Mohammedia, Casablanca, Maroc

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Note presented by Évariste Sanchez-Palencia.

Abstract By homogenization theory, one can predict the vibrations of long repetitive structures in the low frequency range. Beyond this range, many modes have a modulated shape. Based on a multiple scale analysis, a continuum model is presented, that is able to account for this class of modes. This model involves a real coefficient that can be computed from the finite element resolution of problems defined on a few basic cells. An application in 2D elasticity is presented. ***To cite this article: E.M. Daya et al., C. R. Mecanique 330 (2002) 333–338.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / solids and structures

Modèle continu pour les modes de vibrations modulés des longues structures répétitives

Résumé Grâce à la théorie de l'homogénéisation, on peut prédire les basses fréquences de vibrations des structures longues et répétitives. Pour des fréquences moyennes, beaucoup de modes ont une forme modulée. Nous présentons ici un modèle continu qui permet de prendre en compte cette classe de modes, grâce à la méthode des échelles multiples. Ce modèle dépend d'un paramètre réel qu'on peut calculer en résolvant par éléments finis des problèmes définis sur quelques cellules de base. Une application est présentée dans le cas de l'élasticité 2D. ***Pour citer cet article : E.M. Daya et al., C. R. Mecanique 330 (2002) 333–338.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

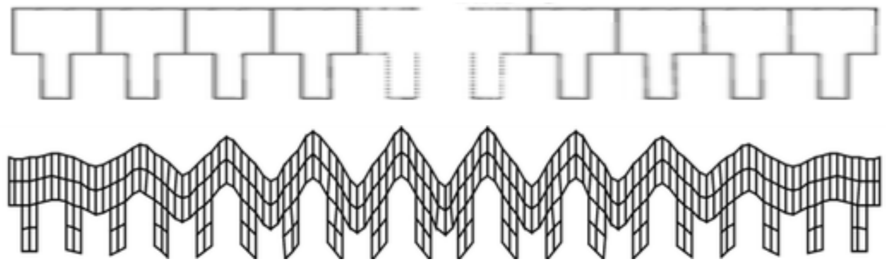
mécanique des solides numérique / solides et structures

1. Introduction

Large structures exhibiting a repetitive form are used in many domains, as aerospace industry. Generally, the eigenmodes of these structures can appear as overall modes or modulated ones. For instance, let us consider a structure as the one pictured in Fig. 1. If the displacement is locked at one or several points of each basic cell, only modulated modes exist, sometimes together with a few localized modes [1]. On the contrary if all the basic cells have stress free boundaries except the first and the last one, the smallest eigenfrequencies correspond to overall modes, also called beam modes. In these two cases, most of the eigenfrequencies are closely located in well separated bands, see Fig. 2. Typical shapes for the modulated modes are presented in Fig. 3: they appear as slow modulations of a periodic mode. The latter property

E-mail address: daya@lpmm.univ-metz.fr (E.M. Daya).

S1631-0721(02)01464-X/FLA



Moustaghfir, Daya, Braikat, Damil, Potier-Ferry, IJSS 2007

Craster, Kaplunov, Pichugin, PRSA 2010

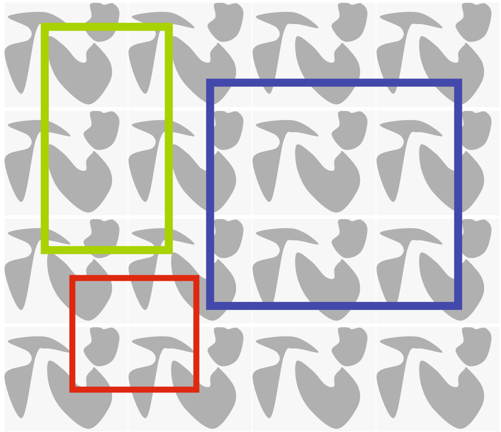
Boutin, Rallu, Hans, JMPS 2012

Macro Description ?

Local dynamics (ρ constant)

$$\underline{\mathcal{G}}\,(\underline{u}) + \rho\,\omega^2\,\underline{u} = \underline{0} \qquad ; \qquad \underline{\mathcal{G}}\,(\underline{u}) = \underline{\mathrm{div}}(\underline{\underline{\underline{\underline{a}}}} : \underline{\underline{e}}\,(\underline{u}))$$

$$\underline{\underline{\underline{\underline{a}}}}\,;\ \rho\ \Omega_0 - \text{periodic}$$



Ω−Eigen modes at eigen frequencies

$$\begin{cases} \underline{\mathcal{G}}\,(\underline{\Phi}^J) + \rho\omega_J^2\,\underline{\Phi}^J = \underline{0} \quad , \quad \text{on} \quad \Omega \\ \underline{\Phi}^J\ \ \Omega - \text{periodic} \end{cases}$$

$\Phi^J(y)$ of constant MODAL amplitude

Scale separation

Modal amplitude varying at large scale

$U^0(x,y) = A(x)\ \Phi^J(y) \qquad \omega \approx \omega_J$

Two-scale Formalism ≈ Homogenization

Modifications

Frequency range $\omega \approx O(\omega_K)$

$$\omega = \omega_K + \sum_{i=1}^{\infty} \varepsilon^i \, \omega^{(i)}$$

Ω-periodic

Multicell

Classic steps

$\varepsilon = l/L << 1$

$x = \varepsilon y \; ; \; y$

$$\underline{u}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \varepsilon^i \, \underline{u}^{(i)} \, (\underline{x}, \underline{y})$$

$$\underline{\mathcal{G}} \to \underline{\mathcal{G}}_{y^2} + \varepsilon \, \underline{\mathcal{G}}_{yx} + \varepsilon^2 \, \underline{\mathcal{G}}_{x^2}$$

$$\underline{\mathcal{G}}_{y^2}(\underline{u}) = \underline{\mathrm{div}}_y \left(\underline{\underline{\sigma}}_y \, (\underline{u}) \right)$$

$$\underline{\mathcal{G}}_{yx}(\underline{u}) = \underline{\mathrm{div}}_x \left(\underline{\underline{\sigma}}_y \, (\underline{u}) \right) + \underline{\mathrm{div}}_y \left(\underline{\underline{\sigma}}_x \, (\underline{u}) \right)$$

$$\underline{\mathcal{G}}_{x^2}(\underline{u}) = \underline{\mathrm{div}}_x \left(\underline{\underline{\sigma}}_x \, (\underline{u}) \right)$$

Series of local problems in Ω

Leading Order

Local dynamics at ω_K

$$\begin{cases} \underline{\mathcal{G}}_{y^2}(\underline{u}^{(0)}) + \rho\omega_K^2\, \underline{u}^{(0)} = \underline{0} \quad , \quad \text{on} \quad \Omega \\ \underline{u}^{(0)}\ \Omega - \text{periodic} \end{cases}$$

Simple mode

$$\underline{u}^{(0)}(\underline{x},\underline{y}) = A^{(0)}(\underline{x})\underline{\Phi}^K(\underline{y})$$

Double mode

$$\underline{u}^{(0)}(\underline{x},\underline{y}) = \overset{1}{A}{}^{(0)}(\underline{x})\,\overset{1}{\underline{\Phi}}{}^K(\underline{y}) + \overset{2}{A}{}^{(0)}(\underline{x})\,\overset{2}{\underline{\Phi}}{}^K(\underline{y})$$

Multiple mode

...

First Order Problem - Simple Mode

y-Eigen value problem + Source

$$\underline{u}^{(0)}(\underline{x},\underline{y}) = A^{(0)}(\underline{x})\underline{\Phi}^K(\underline{y})$$

$$\begin{cases} \boxed{\underline{\mathcal{G}}_{y^2}(\underline{u}^{(1)}) + \rho\omega_K^2\,\underline{u}^{(1)}} + \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) = \underline{0} \quad , \quad \text{on} \quad \Omega \\ \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) = \underline{\mathcal{G}}_{yx}(\underline{u}^{(0)}) + 2\rho\,\omega_K\,\omega^{(1)}\,\underline{u}^{(0)} \\ \underline{u}^{(1)} \;\; \Omega - \text{periodic} \end{cases}$$

Fredholm alternative $G(\Phi^K) - \lambda_K\Phi^K = 0$; $G(u) - \lambda_K u + S = 0$ ➔ $\langle S\Phi^K \rangle = 0$

$$\left\langle \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}).\underline{\Phi}^K \right\rangle = 0$$

Periodicity

$$\left\langle \underline{\mathcal{G}}_{yx}(\underline{u}^{(0)}).\underline{\Phi}^K \right\rangle = \frac{1}{\langle\sqrt{\rho}\rangle^2}\underline{\underline{\text{grad}}}_x(A^{(0)}(\underline{x})).\underline{\mathbb{V}}\left(\underline{\Phi}^K,\underline{\Phi}^K\right) = 0$$

$$\underline{\mathbb{V}}\left(\underline{\Phi}^K,\underline{\Phi}^J\right) = \langle\sqrt{\rho}\rangle^2\left\langle \underline{\underline{\sigma}}_y\left(\frac{\underline{\Phi}^J}{\sqrt{\rho}}\right).\frac{\underline{\Phi}^K}{\sqrt{\rho}} - \underline{\underline{\sigma}}_y\left(\frac{\underline{\Phi}^K}{\sqrt{\rho}}\right).\frac{\underline{\Phi}^J}{\sqrt{\rho}}\right\rangle = -\underline{\mathbb{V}}\left(\underline{\Phi}^J,\underline{\Phi}^K\right)$$

Frequency corrector ➔ ${}^1\omega = 0$

Resolution

Source

$$\underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) = \underline{\mathcal{G}}_{yx}(\underline{u}^{(0)})$$

Forcing in **grad$_x$(A)**

Modes : Orthonormal basis

$$\underline{u}^{(1)}\left(\underline{x},\underline{y}\right) = A^{(1)}(\underline{x})\underline{\Phi}^{K}(\underline{y}) - \sum_{J\neq K}\left[\frac{\underline{\text{grad}}_x\left(A^{(0)}(\underline{x})\right).\underline{\mathbb{V}}\left(\underline{\Phi}^{J},\underline{\Phi}^{K}\right)}{\langle\sqrt{\rho}\rangle^{2}(\omega_K^2-\omega_J^2)}\right]\underline{\Phi}^{J}(\underline{y})$$

Exact expression

Second Order Problem - Simple mode

y- Eigen value problem + Sources

$$\begin{cases} \underline{\mathcal{G}}_{y^2}(\underline{u}^{(2)}) + \omega_K^2\, \underline{u}^{(2)} + \underline{\mathcal{S}}^{(1)}(\underline{u}^{(1)}) + \underline{\mathcal{S}}^{(2)}(\underline{u}^{(0)}) = \underline{0} \quad , \quad \text{on} \quad \Omega \\ \underline{\mathcal{S}}^{(1)}(\underline{u}^{(1)}) = \underline{\mathcal{G}}_{yx}(\underline{u}^{(1)}) \\ \underline{\mathcal{S}}^{(2)}(\underline{u}^{(0)}) = \underline{\mathcal{G}}_{x^2}(\underline{u}^{(0)}) + 2\,\rho\omega_K\,\omega^{(2)}\,\underline{u}^{(0)} \\ \underline{u}^{(2)} \;\; \Omega - \text{periodic} \end{cases}$$

Fredholm alternative

$$\left\langle [\underline{\mathcal{S}}^{(1)}(\underline{u}^{(1)}) + \underline{\mathcal{S}}^{(2)}(\underline{u}^{(0)})].\underline{\Phi}^K \right\rangle = 0$$

….. Governing equation for A !

$$\mathrm{div}_x \left(\underline{\underline{Q}}^K . \underline{\mathrm{grad}}_x \left(A^{(0)}\,(\underline{x}) \right) \right) + 2\,\langle\sqrt{\rho}\rangle^2 \omega_K\, \omega^{(2)}\, A^{(0)}\,(\underline{x}) = 0$$

Macro-modulation of Simple Mode

Governing equation of Macro-modulation

$$\mathrm{div}\left(\underline{\underline{Q}}^K.\underline{\mathrm{grad}}(A)\right) + \langle\sqrt{\rho}\rangle^2\ (\omega^2 - \omega_K^2)A = o(\varepsilon^2)$$

$$\omega^2 - \omega_K^2 = {}^2\omega(\varepsilon^2 2\omega_K)$$

Effective Elasto-Inertial 2nd Rank tensor

$$\underline{\underline{Q}}^K = \underline{\underline{\mathbb{T}}}\left(\underline{\Phi}^K, \underline{\Phi}^K\right) + \sum_{J\neq K} \frac{\underline{\mathbb{V}}\left(\underline{\Phi}^J, \underline{\Phi}^K\right) \otimes \underline{\mathbb{V}}\left(\underline{\Phi}^J, \underline{\Phi}^K\right)}{\langle\sqrt{\rho}\rangle^2(\omega_K^2 - \omega_J^2)}$$

$$\underline{\underline{\mathbb{T}}}\left(\underline{\Phi}^K, \underline{\Phi}^J\right) = \langle\sqrt{\rho}\rangle^2 \left\langle \frac{\underline{\Phi}^J}{\sqrt{\rho}} . \underline{\underline{\underline{\underline{a}}}} . \frac{\underline{\Phi}^K}{\sqrt{\rho}} \right\rangle$$

Symmetry : 3 ≠ Principal values > 0 or < 0

Effective « Differential Inertia » > 0 or < 0 ➔ ≈ Guided waves

Features of simple mode modulation

Modulation in a principal direction

$$\mathcal{C}^2 T_\alpha^K A'' + (\omega^2 - \omega_K^2) A = 0$$

$$A(x) = A_+ \exp(+i\kappa_\alpha \mathrm{x}) + A_- \exp(-i\kappa_\alpha \mathrm{x})$$

Modulation number

$$\kappa_\alpha(\omega) = \frac{1}{\mathcal{C}} \sqrt{\frac{\omega^2 - \omega_K^2}{T_\alpha^K}} = \frac{\omega_K}{\mathcal{C}} \sqrt{\frac{2}{T_\alpha^K} (\frac{\omega}{\omega_K} - 1)}$$

Real / Imaginary

Modulation length

$$\Lambda_\alpha(\omega) = \lambda_K \sqrt{\frac{T_\alpha^K}{2} \frac{\omega_K}{\omega - \omega_K}}$$

High dispersion

First Order Problem - Double Mode

Double mode

$$\underline{u}^{(0)}(\underline{x},\underline{y}) = \overset{1}{A}{}^{(0)}(\underline{x})\,\underline{\overset{1}{\Phi}}{}^{K}(\underline{y}) + \overset{2}{A}{}^{(0)}(\underline{x})\,\underline{\overset{2}{\Phi}}{}^{K}(\underline{y})$$

Same y- Eigen value problem + Source

$$\begin{cases} \underline{\mathcal{G}}_{y^2}(\underline{u}^{(1)}) + \rho\omega_K^2\,\underline{u}^{(1)} + \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) = \underline{0} \quad , \quad \text{on} \quad \Omega \\ \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) = \underline{\mathcal{G}}_{yx}(\underline{u}^{(0)}) + 2\rho\,\omega_K\,\omega^{(1)}\,\underline{u}^{(0)} \\ \underline{u}^{(1)}\ \Omega - \text{periodic} \end{cases}$$

Fredholm alternative x 2

$$\langle \underline{\mathcal{S}}^{(1)}(\underline{v}^{(0)}).\underline{\overset{1}{\Phi}}{}^{K}\rangle = 0$$

$$\langle \underline{\mathcal{S}}^{(1)}(\underline{v}^{(0)}).\underline{\overset{2}{\Phi}}{}^{K}\rangle = 0$$

Macro-modulation of Double Mode

Coupled equations

$$\begin{cases} \underline{\text{grad}}_x(\overset{2}{A}{}^{(0)}(\underline{x})).\underline{\mathbb{V}}\left(\overset{1}{\underline{\Phi}}{}^K, \overset{2}{\underline{\Phi}}{}^K\right) + 2\,\langle\sqrt{\rho}\rangle^2 \omega_K\, \omega^{(1)}\, \overset{1}{A}{}^{(0)}(\underline{x}) = 0 \\ \underline{\text{grad}}_x(\overset{1}{A}{}^{(0)}(\underline{x})).\underline{\mathbb{V}}\left(\overset{2}{\underline{\Phi}}{}^K, \overset{1}{\underline{\Phi}}{}^K\right) + 2\,\langle\sqrt{\rho}\rangle^2 \omega_K\, \omega^{(1)}\, \overset{2}{A}{}^{(0)}(\underline{x}) = 0 \end{cases}$$

Governing equation of Macro-modulation

$$\boxed{\text{div}\left(\underline{\underline{\mathcal{R}}}^K.\underline{\text{grad}}(\overset{1}{A})\right) + 4\,\langle\sqrt{\rho}\rangle^2\,(\omega - \omega_K)^2\,\overset{1}{A} = o(\varepsilon^2)}$$

Effective Elasto-Inertial 2nd Rank tensor

$$\underline{\underline{\mathcal{R}}}^K = \frac{\underline{\mathbb{V}}\left(\overset{1}{\underline{\Phi}}{}^K, \overset{2}{\underline{\Phi}}{}^K\right) \otimes \underline{\mathbb{V}}\left(\overset{1}{\underline{\Phi}}{}^K, \overset{2}{\underline{\Phi}}{}^K\right)}{\langle\sqrt{\rho}\rangle^2\,\omega_K^2}$$

➔ Principal values $\geq 0, 0, 0$!

Effective « Differential Inertia » > 0

Feature of double mode modulation

Modulation in the principal direction

$$\mathcal{C}^2 R^K A'' + \ 4(\omega - \omega_K)^2 A = 0$$

$$A(x) = A_+ \exp(+i\kappa_\alpha \mathrm{x}) + A_- \exp(-i\kappa_\alpha \mathrm{x})$$

Modulation number

$$\kappa(\omega) = \frac{1}{\mathcal{C}}\sqrt{\frac{4(\omega - \omega_K)^2}{R^K}} = \frac{2}{\mathcal{C}}\frac{|\omega - \omega_K|}{\sqrt{R^K}}$$

Real

Modulation length

$$\Lambda_K(\omega) = \lambda_K \sqrt{\frac{R^K}{4}} \frac{\omega_K}{|\omega - \omega_K|} \gg \lambda_K$$

High dispersion

In quasi-statics ?

Modulation in the principal direction

$$\mathcal{C}^2 R^K A'' + \ 4(\omega - \omega_K)^2 A = 0$$

$$A(x) = A_+ \exp(+i\kappa_\alpha \mathrm{x}) + A_- \exp(-i\kappa_\alpha \mathrm{x})$$

Modulation number

$$\kappa(\omega) = \frac{1}{\mathcal{C}}\sqrt{\frac{4(\omega - \omega_K)^2}{R^K}} = \frac{2}{\mathcal{C}}\frac{|\omega - \omega_K|}{\sqrt{R^K}}$$

Real

Modulation length

$$\Lambda_K(\omega) = \lambda_K \sqrt{\frac{R^K}{4}}\frac{\omega_K}{|\omega - \omega_K|} \gg \lambda_K$$

High dispersion

In quasi-statics ?

Scaling $\omega^2 = (\varepsilon\omega^{(1)})^2$

Leading order

Elastostatic

Triple static eigenmode $u^{(0)} = U_1^{(0)}(x)e_1 + U_2^{(0)}(x)e_2 + U_3^{(0)}(x)e_3$

First order

Fredholm alternative trivially satisfied

$u^{(1)}$ as in classical homogenization

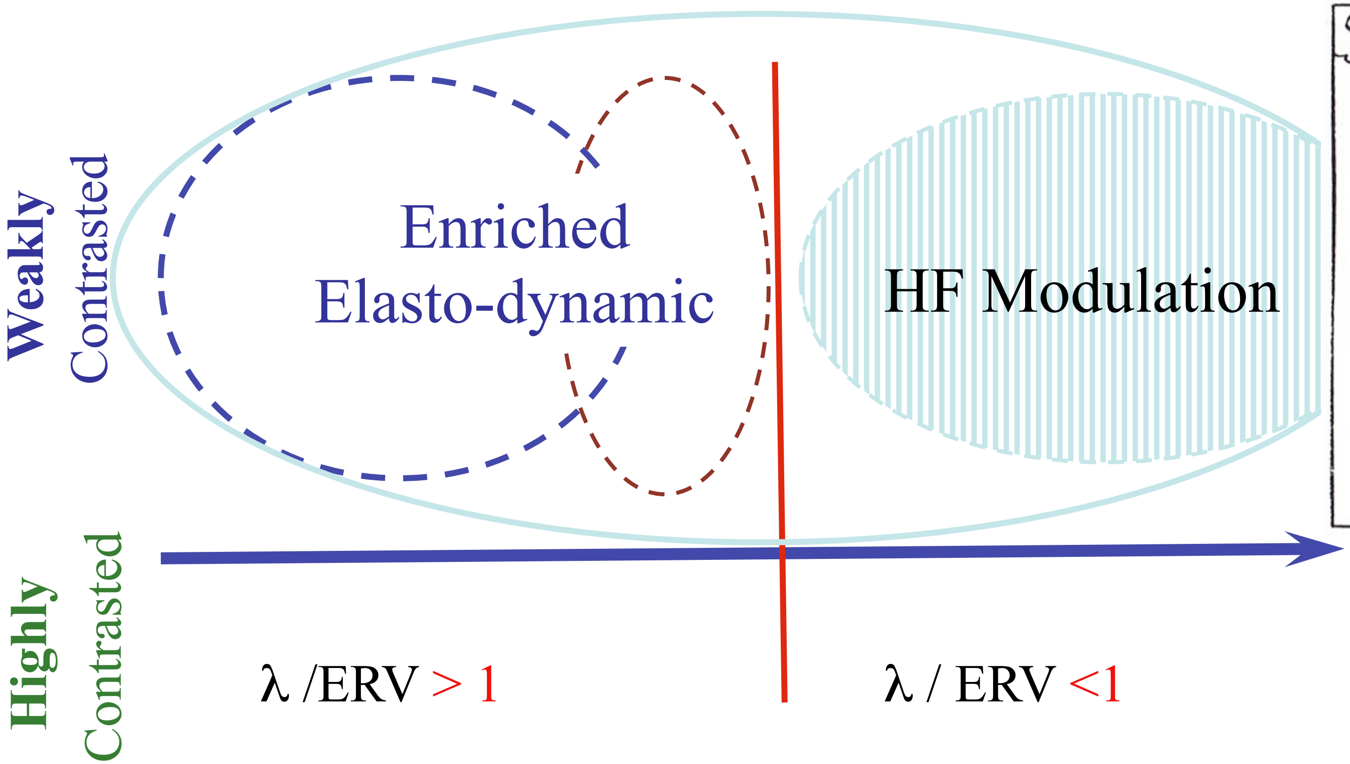
Second order

Fredholm alternative $(e_1; e_2; e_3)$: 3 balance equations

➔ Classical homogenized elasto-dynamics

Effective parameters expressed on the modal basis

Modulation versus Enriched Elasto-dynamic

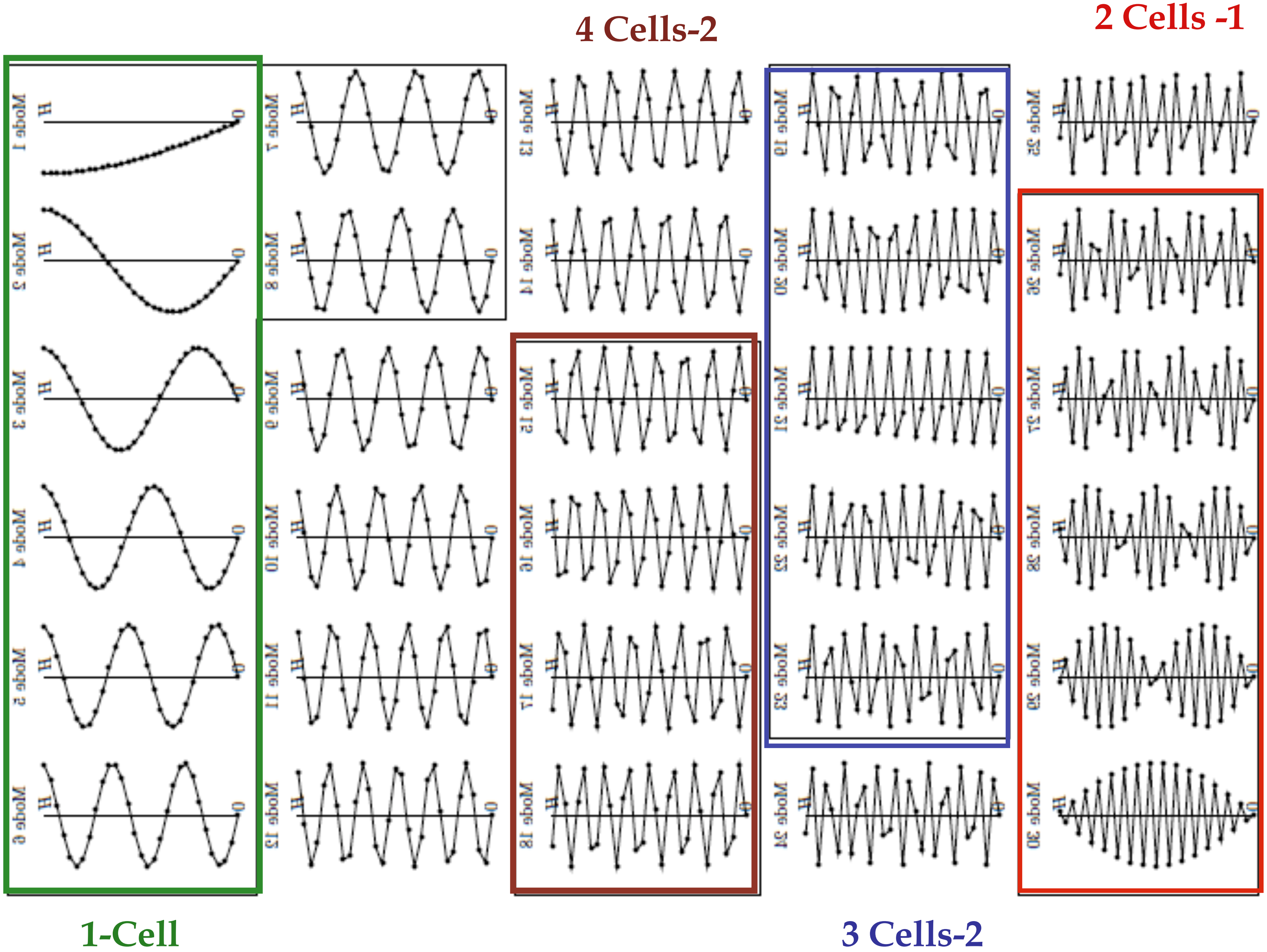


Single modelling — Multiple modelling : Mode & Multi-cell

3 Variables: **Motion amplitudes** — 1-N variables Mode(s) amplitude (s)

Stress concept } Inertia concept } **Enriched**	Stress concept } Inertia concept } Fundamental change

Illustration - 1D



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HOMOGENIZATION IN MECHANICS
A SURVEY OF SOLVED AND OPEN PROBLEMS

To conclude this survey, we mention *two widely open problems.*

The first one is the buckling of periodic structures made of elastic bars. Local buckling may appear for deformations with a local period differen from that of the structure (Fig. 13.1).

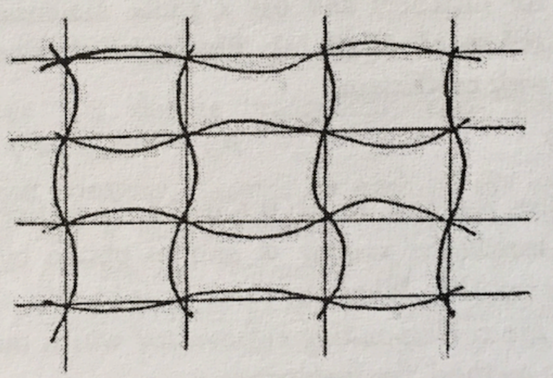

Figure 13.1

More than 30 years after …