Institut d'études scientifiques
de
Cargèse

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Homogenization and Inner Resonances
in
Different Physical Contexts

C. Boutin

ENTPE – CeLyA- CNRS 5513

Objectives

This short course aims at

Explaining the asymptotic homogenization method
through the physical principles behind the up-scaling process
for deriving (un)-conventional macro-description in dynamics

This will be done

Considering periodic elastic composites
and in other physical contexts in periodic media
with illustration by experiments on prototypes

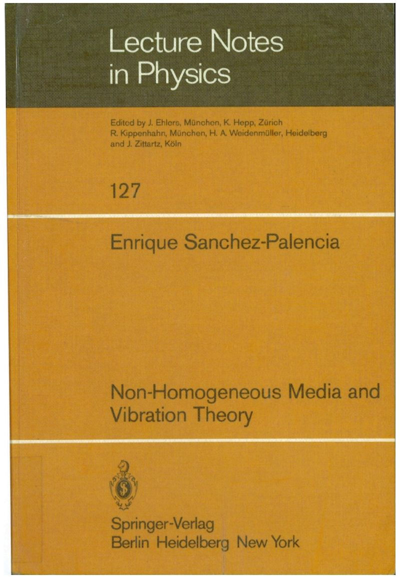
Keywords

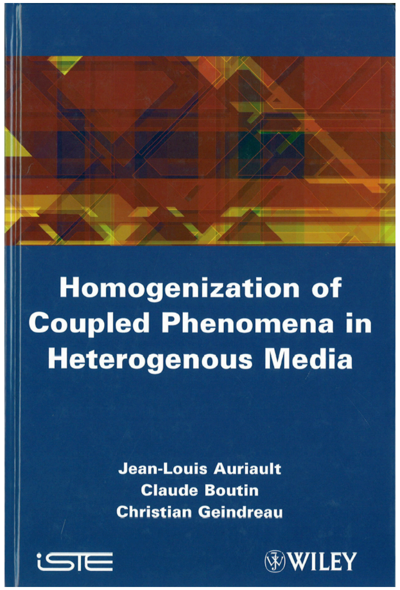
Homogenisation, Composites, Inner resonance, Metamaterials

Content

Content

1 -Introduction to Homogenization





E. Sanchez Palencia, *Nonhomogeneous media and vibration theory*, Lectures notes in Physics, 127, Berlin: Springer-Verlag, 398p. (1980)

From local to global description

Key Issue

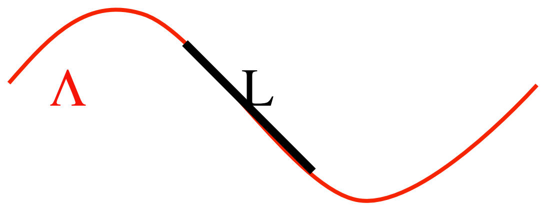
Escape from the detailed description

while

Keep the qualitative and quantitative features

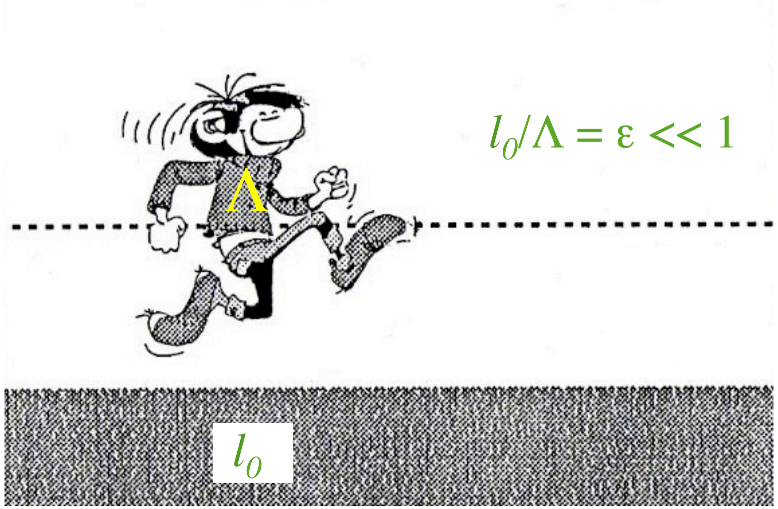
Scale separation is necessary, … and sufficient *(Auriault, 80)*

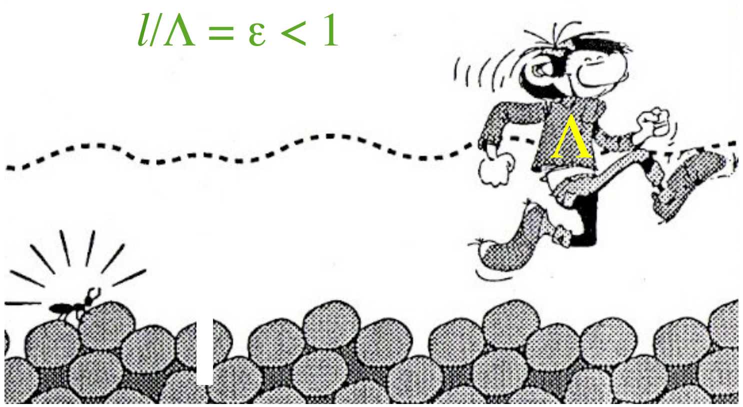
1 - Long wavelength $\Lambda = 2\pi L$

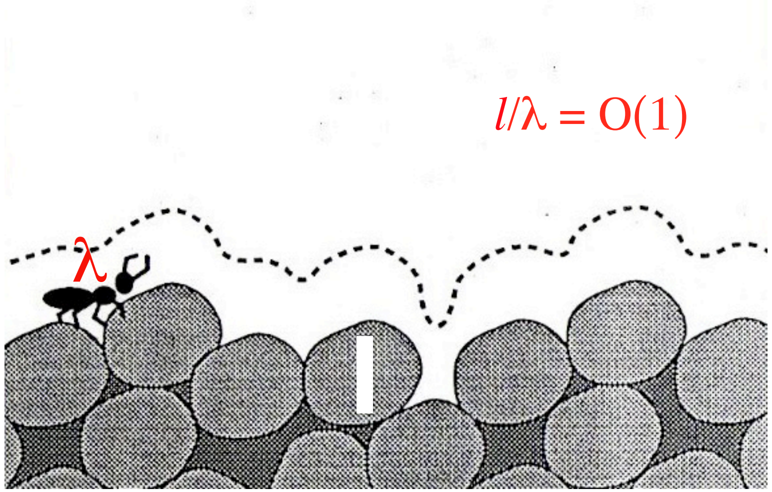


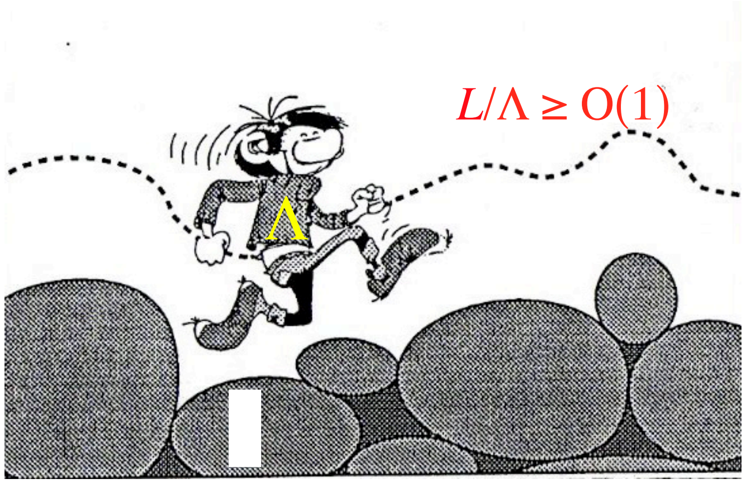
2 - Medium = {ERV} ERV [l] $l/L = \varepsilon << 1$

Intuitive approach









Guide line

Scale separation $l/\Lambda = \varepsilon << 1$ **Macro - Continuum physics**

ERV <==> Particle — not too small (representative)
not too large (infinimum % L)

Global description arises from the local physics

ERV physics Condensed in — Nature of the macro-description
Macro-parameters
Relevant information % L

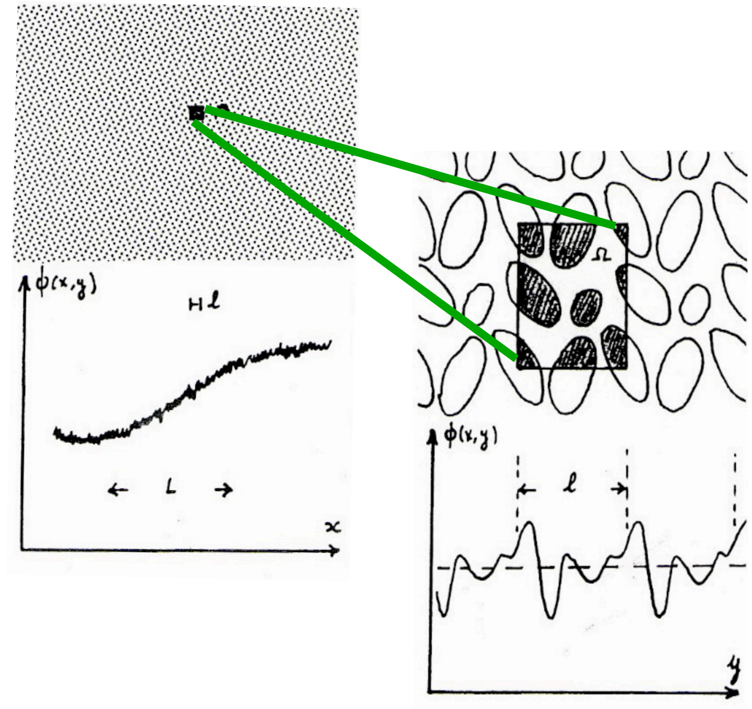
Homogenization method *[Sanchez-Palencia, 80], [Auriault, 80]*

Rigorous mathematical approach of the two requirements

ERV — Ω-Periodic media

Scale separation — $\varepsilon = l/L << 1$
Asymptotic expansions
Two scales method

Homogenization method



Two-scale variables

Macro : x/L Micro : $x/l = x/(L\varepsilon)$

x $y = x/\varepsilon$

Two-scale expansions

➔ $\partial \;\text{-->}\; \partial_x + \varepsilon^{-1}\partial_y$

➔ y-Ω periodic expansions

$$q(x,y) = q^0(x,y) + \varepsilon q^1(x,y) + \varepsilon^2 q^2(x,y) + \dots$$

Resolution

Rescaled Equations $EQ_{x,y}(\text{Expansions}(x,y)) = 0$

$\forall\, \varepsilon \;\text{-->}\; 0$ $\sum \varepsilon^q EQ^q_{x,y}(\text{-}) = 0$

Separate ε power $EQ^q_{x,y}(\text{-}) = 0$

Series of local problems (y-periodic)

Macro description (x)

About convergence

Different approaches

Mathematics

Essential for physics

Enables the use of the homogenized models **for real material with *l* finite**

Practical interest : **much wider** applicability than $\varepsilon \ll 1$; ➔ $\varepsilon < ½$

Up to now systematically prouved with

Scale separation, linear physics, at given scaling

Physically sound : The phenomena tends to stay identical as L is enlarged

Thus

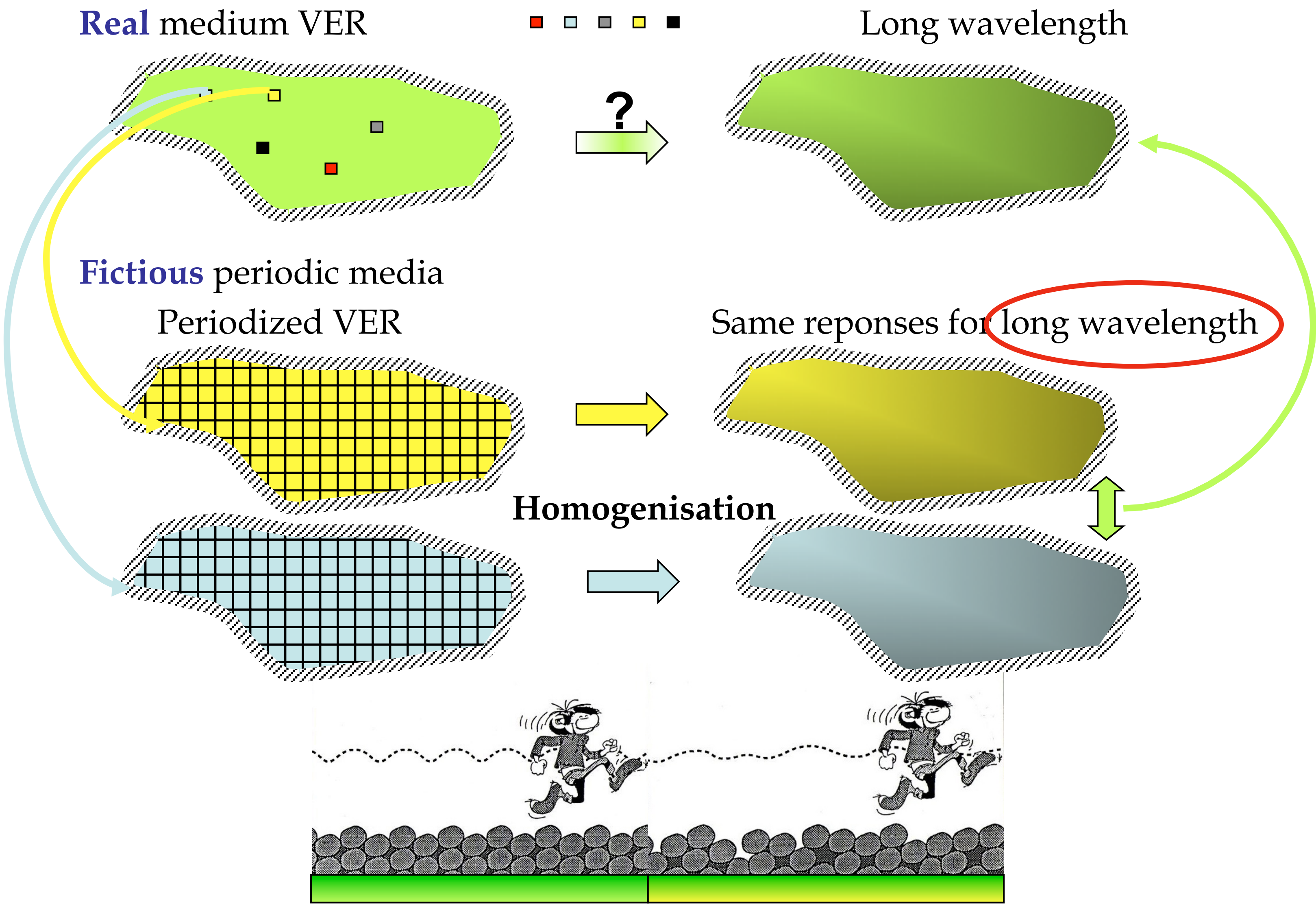
The convergence is of interest, ...

but pathological situations with no convergence would be even more

if any...

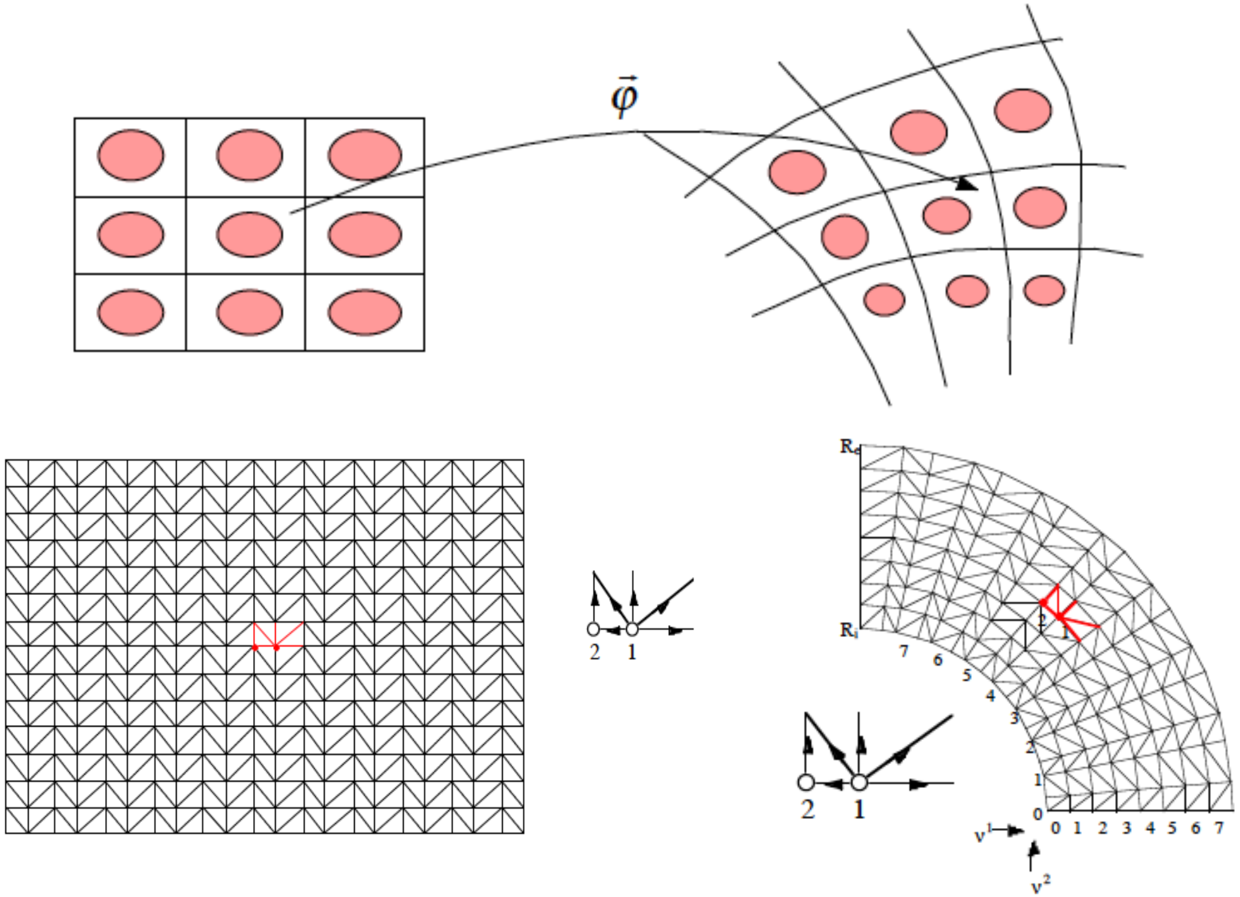
About periodicity - At long wavelength

Periodicity Ω vs statistical invariance (ERV)



Relaxed periodicity - At long wavelength

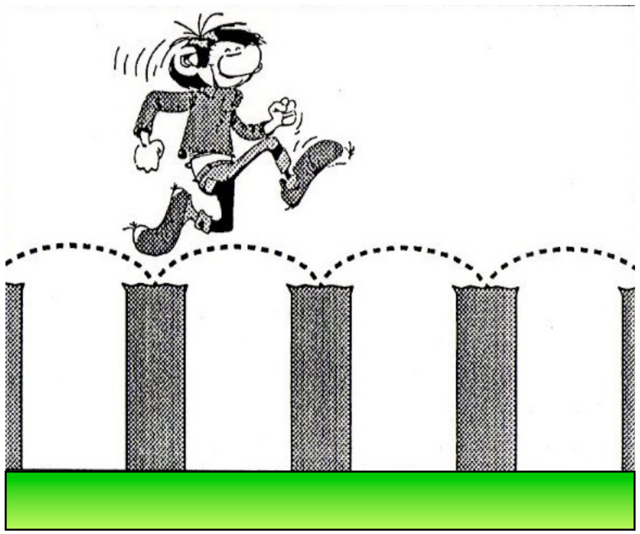
Parametrized periodicity Ω (Caillerie, 2003, 2012)



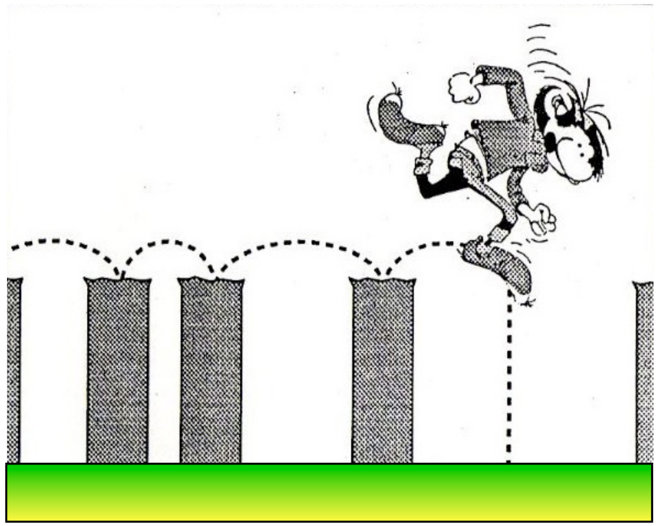
➔ **Large deformations**

About periodicity - Short wavelenght

At Short wavelength : very distinct responses



Periodic



Non Periodic

Global descriptions in dynamics

Classification based on λ /ERV

	λ /ERV > 1	λ /ERV <1
Moderately Contrasted	Large scale dynamics Local statics	Local variations Local dynamics
Highly « Contrasted	Large scale dynamics Local dynamics Co-dynamics » regime Inner resonance	

Diffraction

Floquet-Bloch

Modulation

Floquet-Bloch Waves

$$-\frac{\partial}{\partial y}\left(a(y)\frac{\partial u}{\partial y}\right) = \rho\,\omega^2 u$$

Elastodynamics a Ω_0/l_0−periodic, ρ constant

$$\mathcal{A}(u) = \rho\,\omega^2\,u \qquad \text{with} \qquad \mathcal{A} = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right)$$

Bloch waves Wave number : k

$$\varphi(y) = e^{iky}\phi(y) \qquad \text{with} \qquad \phi(y)\ \Omega\text{-periodic} \qquad 0 \leq k \leq 2\pi/l_0$$

Shifted operator

$$\mathcal{A}_k(\phi) = \rho\,\omega^2\,\phi \qquad \text{with} \qquad \mathcal{A}_k = -\left(\frac{\partial}{\partial y} + ik\right)\left(a(y)\left(\frac{\partial}{\partial y} + ik\right)\right)$$

Spectral resolution

$$0 \leq \omega_1^2(k) \leq \cdots \leq \omega_N^2(k) \leq \ldots$$ Eigenvalues – Dispersion

$\{\phi_N(k,y)\}$ $\{\varphi_N(k,y)\}$ Bloch modes & Bloch waves

Periodic Pseudo-periodic Orthogonal basis

Floquet-Bloch Waves / Homogenization

Longwave length & Low frequency : k → εk ; ω → $\varepsilon\omega$

$$\mathcal{A}_{\varepsilon k}(\phi) = \rho\,\varepsilon^2\omega^2\,\phi \qquad \text{with} \qquad \mathcal{A}_{\varepsilon k} = -\left(\frac{\partial}{\partial y} + i\varepsilon k\right)\left(a(y)\left(\frac{\partial}{\partial y} + i\varepsilon k\right)\right)$$

Two scale formulation x = εy

$$u = e^{i\varepsilon k y}\phi_{\varepsilon k}(y) = e^{ikx}\phi_{\varepsilon k}(y)$$

$$\frac{\partial}{\partial y}e^{i\varepsilon k y} = i\varepsilon k e^{i\varepsilon k y} = i\varepsilon k e^{ikx} = \varepsilon\frac{\partial}{\partial x}e^{ikx}$$

$$\mathcal{A}_{\varepsilon k}(ue^{-ikx}) = \rho\,\varepsilon^2\omega^2\,(ue^{-ikx}) \qquad \text{with} \qquad \mathcal{A}_{\varepsilon k} = -\left(\frac{\partial}{\partial y} + \varepsilon\frac{\partial}{\partial x}\right)\left(a(y)\left(\frac{\partial}{\partial y} + \varepsilon\frac{\partial}{\partial x}\right)\right)$$

Expansion + Rigorous analysis (Turbe & Wilcox 1984)

$$ue^{-ikx} = \phi_{\varepsilon k}(y) = \phi_o(x) + \varepsilon\phi_0'(x,y) + \ldots$$

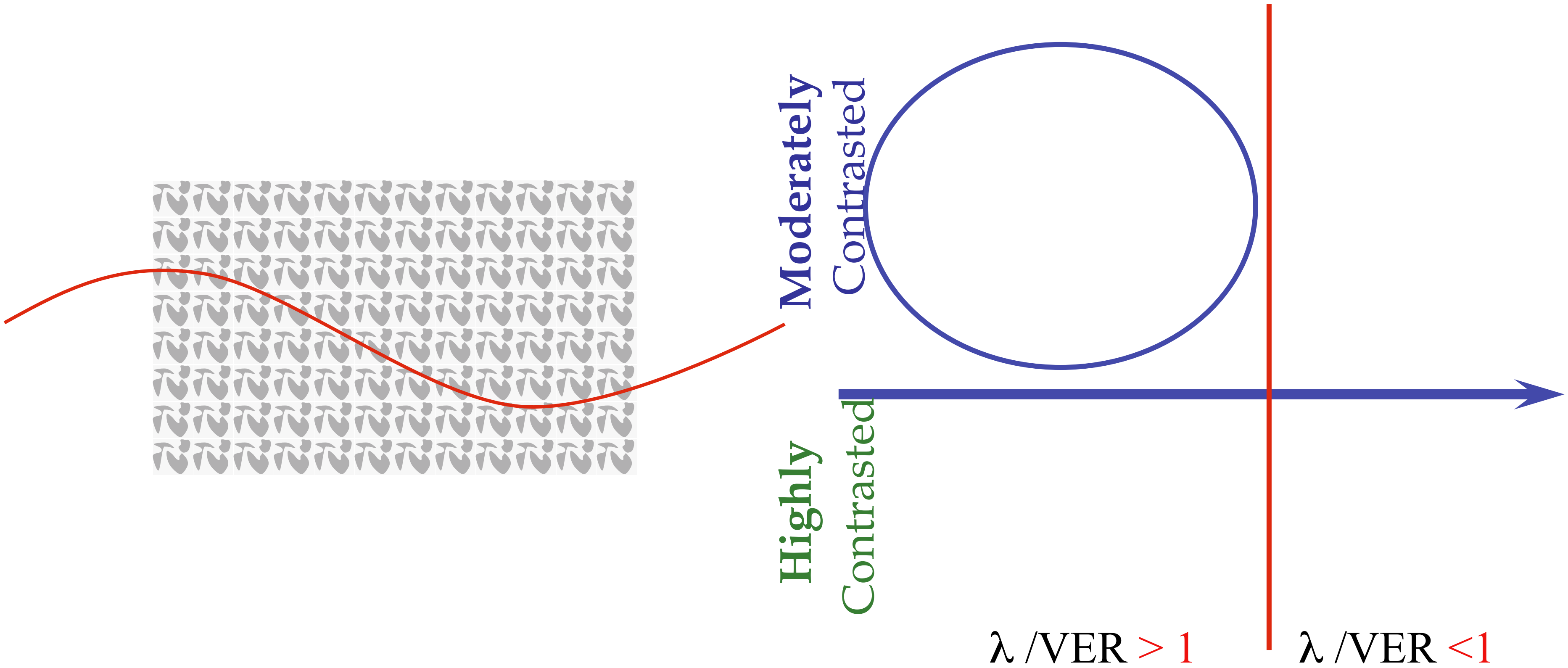
At the limit ε → 0 Homogenization of $\mathcal{A}(u) = e^{ikx}\mathcal{A}_{\varepsilon k}(ue^{-ikx}) = \rho\,\varepsilon^2\omega^2 u$

Small phase shift ➔ Variation at large scale

Content

Long Wavelengths

Moderately Contrasted Composites



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Scale separation

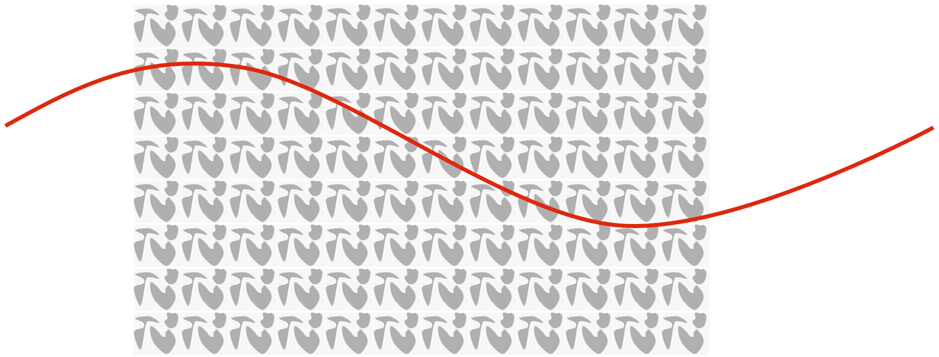
Long wavelength Λ

$\Lambda >> l$ ERV [*l*]

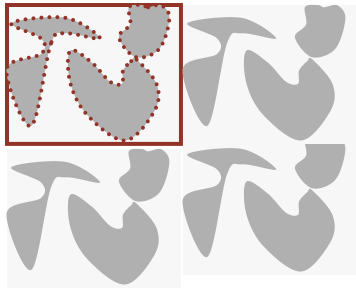
$\varepsilon = 2\pi\, l/\Lambda << 1$ Macro dynamics

➔ Quasi static local regime

Scale separation for U^0



Homogenization



Elasto-dynamics – Harmonic regime ω

$\sigma = a(y):e\,(u)$ $\qquad$ $\mathrm{div}[\sigma] + \omega^2\rho u = 0$

$E(u) + \omega^2\rho(y)\,u = 0$ $\qquad$ $E(u) = \mathrm{div}(a:e\,(u))$ $\qquad$ $a(y)\,;\,\rho(y)$ Ω-periodic

Two space variables $x\,,\ y = \varepsilon^{-1}x$

$$E \rightarrow E_{xy}(u) = \varepsilon^{-2} E_y^{-2}(u) + \varepsilon^{-1}E^{-1}(u) + E^0(u)$$

$$E_y^{-2}(u) = \mathrm{div}_y[a(y):e_y(u)]$$

$$E^{-1}(u) = \mathrm{div}_y[a(y):e_x(u)] + \mathrm{div}_x[a(y):e_y(u)]$$

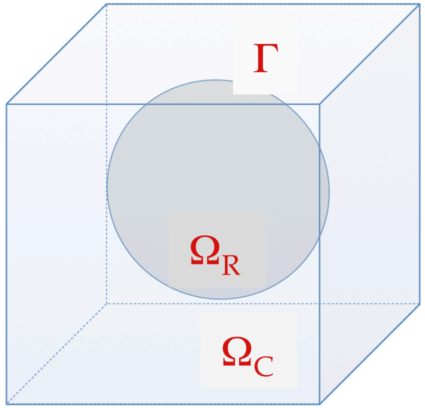
$$E^0(u) = \mathrm{div}_x[a(y):e_x(u)] + \omega^2\rho(y)\,u$$

Macro dynamics

Homogenization - 1

Governing equations (x,y)

$$\left\{\begin{array}{rcll} \mathrm{div}(\mathbf{a}_c : \mathbf{e}(\mathbf{u}_c)) &=& -\omega^2 \rho_c \mathbf{u}_c & \text{in } \Omega_c \\ \mathrm{div}(\mathbf{a}_r : \mathbf{e}(\mathbf{u}_r)) &=& -\omega^2 \rho_r \mathbf{u}_r & \text{in } \Omega_r \\ (\mathbf{a}_c : \mathbf{e}(\mathbf{u}_c) - \mathbf{a}_r : \mathbf{e}(\mathbf{u}_r)).\mathbf{n} &=& 0 & \text{over } \Gamma \\ \mathbf{u}_r - \mathbf{u}_c &=& 0 & \text{over } \Gamma \end{array}\right.$$



$$\mathbf{u} = \mathbf{u}^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 \mathbf{u}^{(2)}(\mathbf{x},\mathbf{y}) + \cdots \quad , \quad \mathbf{u}^{(i)}(\mathbf{x},\mathbf{y})\ \Omega - \text{periodic in } \mathbf{y}$$

Dominant order – periodic in y

$$\left\{\begin{array}{rcll} \mathrm{div}_y(\mathbf{a} : \mathbf{e}_y(\mathbf{u}^0)) &=& 0 & \text{in } \Omega \quad (\varepsilon^{-2}) \\ \left[\mathbf{a} : \mathbf{e}_y(\mathbf{u}^0)\right].\mathbf{n} &=& 0 & \text{over } \Gamma \quad \varepsilon^{-1} \\ \left[\mathbf{u}^0\right] &=& 0 & \text{over } \Gamma \end{array}\right.$$

& Ω -periodicity

➔ **Local quasi-statics** $\quad \mathbf{u}^0(\mathbf{x},\mathbf{y}) = \mathbf{U}^0(\mathbf{x})$

Homogenization - 2

Local fields at next order

$$\left\{\begin{array}{rcll} \mathrm{div}_y(\mathbf{a}:(\mathbf{e}_y(\mathbf{u^1})+\mathbf{e_x}(\mathbf{U^0})) & = & 0 & \text{in } \Omega \quad (\varepsilon^{-1}) \\ \left[\mathbf{a}:(\mathbf{e}_y(\mathbf{u^1})+\mathbf{e_x}(\mathbf{U^0}))\right].\mathbf{n} & = & 0 & \text{over } \Gamma \quad (\varepsilon^{0}) \\ \left[\mathbf{u^1}\right] & = & 0 & \text{over } \Gamma \quad (\varepsilon^{1}) \end{array}\right.$$

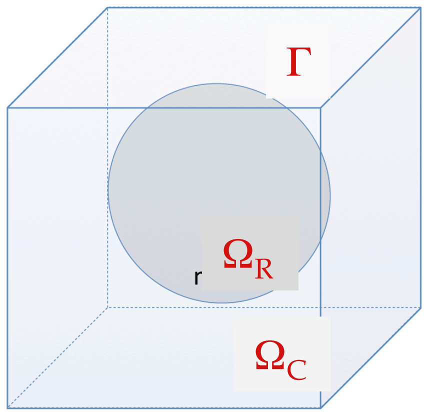
& Ω -periodicity

➔ **Local quasi-statics**

Forcing by $e_x(U^0)$

Lax-Milgram

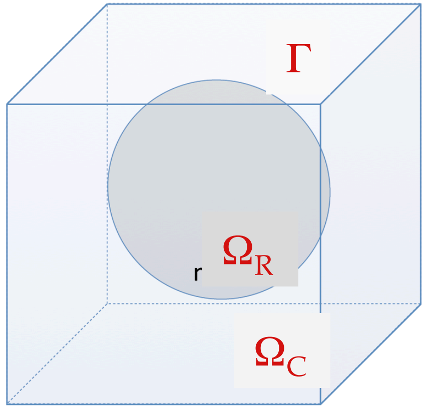
By linearity



$$\mathbf{u^1}(\mathbf{x},\mathbf{y}) = \boldsymbol{\xi}(\mathbf{y}).\mathbf{e_x}(\mathbf{U^0}) + \mathbf{U^1}(\mathbf{x})$$

$<\xi> = 0$

Homogenization - 3



Macroscopic description at leading order

$$\left\{\begin{array}{rcll} \mathrm{div}_y(\boldsymbol{\sigma}^1) + \mathrm{div}_x(\mathbf{a} : (\mathbf{e}_y(\mathbf{u}^1) + \mathbf{e}_x(\mathbf{U}^0)) & = & -\omega^2 \rho \mathbf{U}^0 & \text{in } \Omega_c \\ \left[\boldsymbol{\sigma}^1\right].\mathbf{n} & = & 0 & \text{over } \Gamma \end{array}\right.$$

& Ω -periodicity

Mean balance $\qquad \langle . \rangle = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega . \, d\Omega$

Divergence theorem + periodicity $\qquad \langle \mathrm{div}_y(\boldsymbol{\sigma}^1) \rangle = 0$

➔ **Macro conventional elasto-dynamics (x!)**

$$\mathrm{div}_x(\mathbf{C}^0 : \mathbf{e}_x(\mathbf{U}^0)) \quad = \quad -\omega^2 \langle \rho \rangle \mathbf{U}^0$$

$\mathbf{C}^0 = \langle \mathbf{a} : (\mathbf{e}_y(\boldsymbol{\xi}) + \mathbf{a} \rangle$ Elasto-static stiffness

<ρ> Mean density

Learning : Standard Elastodynamics