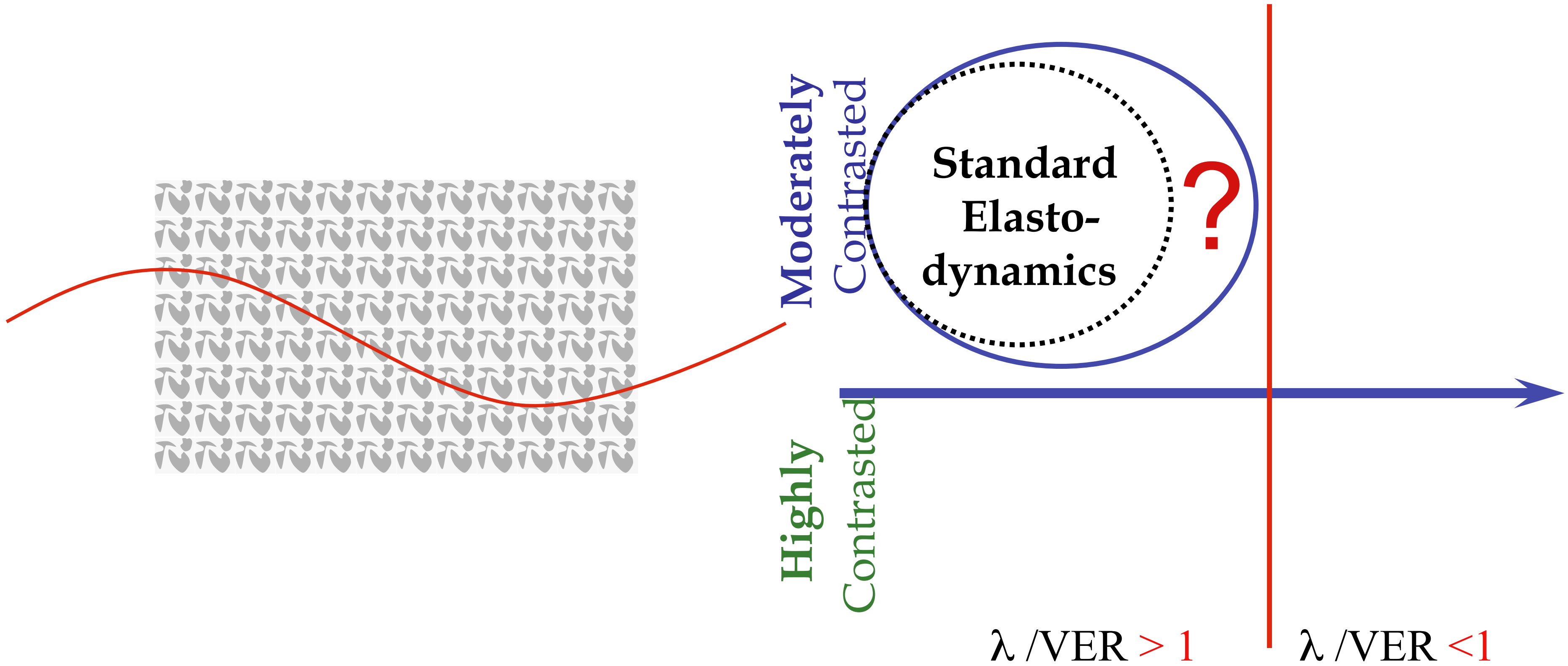
Content

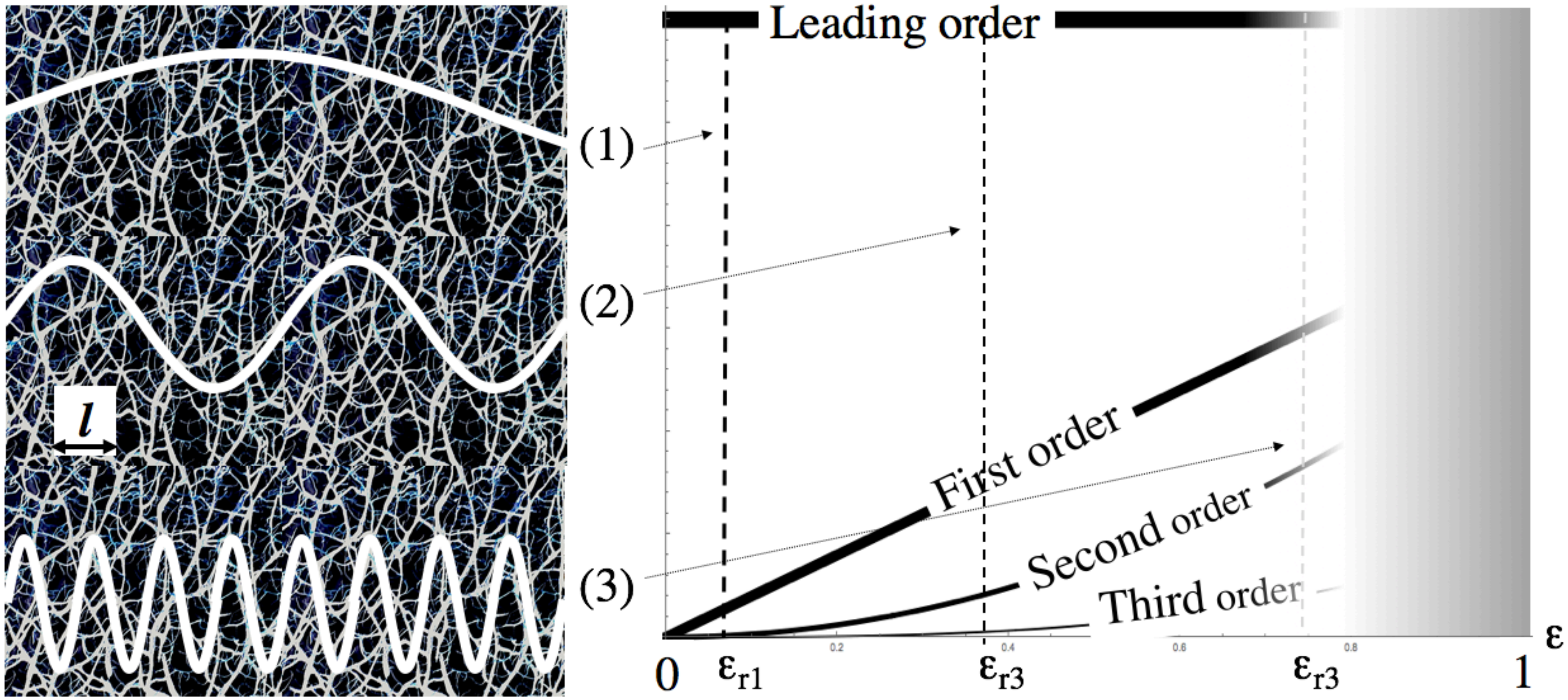
Long Wavelengths

Moderately Contrasted Composites



Elasto-dynamic with correctors

$\varepsilon = 2\pi\, l/\Lambda < 1 \;\backslash\;$ Correctors



Boutin, Auriault, I.J.E.S. 1993

Asymptotic process

Double scale formulation

$$E(u) = \varepsilon^{-2} E^{-2}(u) + \varepsilon^{-1} E^{-1}(u) + E^{0}(u)$$

$$E^{-2}(u) = \mathrm{div}_y[a(y):e_y(u)]$$
$$E^{-1}(u) = \mathrm{div}_y[a(y):e_x(u)] + \mathrm{div}_x[a(y):e_y(u)]$$
$$E^{0}(u) = \mathrm{div}_x[a(y):e_x(u)] + \omega^2 \rho(y)\, u$$

Series of local Ω-periodic problems

$$u = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^2 u^2 + \ldots$$

$$\sigma^i = a(y):[e_x(u^i) + e_y(u^{i+1})]$$

➔

$$E^{-2}(u^0) = 0$$
$$E^{-2}(u^1) + E^{-1}(u^0) = 0$$
$$E^{-2}(u^{i+2}) + E^{-1}(u^{i+1}) + E^{0}(u^i) = 0$$

$$\mathrm{div}_y[\sigma^{-1}] = 0$$
$$\mathrm{div}_y[\sigma^{0}] + \mathrm{div}_x[\sigma^{-1}] = 0$$
$$\mathrm{div}_y[\sigma^{i+1}] + \mathrm{div}_x[\sigma^{i}] + \omega^2 \rho(y)\, u^i = 0$$

➔ $$\langle \mathrm{div}_x[\sigma^i] \rangle + \omega^2 \langle \rho(y)\, u^i \rangle = 0$$

Recursive resolutions

Each u^i depends upon the previous terms : Multiple gradients

High order Correctors

Recursive resolutions

$$u^0(x,y) = U^0(x)$$

$$u^1(x,y) = U^1(x) + \xi^1(y).e_x(U^0)$$

$$u^2(x,y) = U^2(x) + \xi^1(y).e_x(U^1) + \xi^2(y)..\nabla_x e_x(U^0)$$

i > 2

$$u^i(x,y) = U^i(x) + \xi^1(y).e_x(U^{i-1}) + \xi^2(y)..\nabla_x e_x(U^{i-2}) + \xi^3(y).(.\nabla_x)^2 e_x(U^{i-3}) + \ldots + \xi^i(y).(.\nabla_x)^{i-1} e_x(U^0)$$

Local fields and global balance

➔ ξ^i Real & elastic tensors of rank i+2 $\langle \xi^i \rangle = 0$

($\sigma^i = a(y):[e_x(u^i) + e_y(u^{i+1})]$

➔ $\langle div_x[\sigma^i] \rangle + \omega^2 \langle \rho(y)\, u^i \rangle = 0$

Macro Description with Correctors

Mean cell motion

$$\mathbf{U}(x) = \mathbf{U}^0(x) + \mathbf{U}^1(x) + \mathbf{U}^2(x) + \mathbf{U}^3(x) + \dots \qquad \mathbf{U}^i(x) = \varepsilon^i \underline{U}^i(x)$$

Series of balance equations (x !)

$$\mathrm{div}(C^0 : e(\mathbf{U}^0)) + \langle\rho\rangle\omega^2\, \mathbf{U}^0 = 0$$

$$\mathrm{div}(C^0 : e(\mathbf{U}^1)) + \langle\rho\rangle\omega^2\, \mathbf{U}^1 = -[\,\mathrm{div}(C^1 \ldots \nabla e(\mathbf{U}^0)) + \omega^2 R^1 e(\mathbf{U}^0))] \qquad \varepsilon$$

$$\mathrm{div}(C^0 : e(\mathbf{U}^2)) + \langle\rho\rangle\omega^2\, \mathbf{U}^2 = -[\,\mathrm{div}(C^1 \ldots \nabla e(\mathbf{U}^1)) + \omega^2 R^1 e(\mathbf{U}^1))] - [\mathrm{div}(C^2 \ldots . \nabla\nabla e(\mathbf{U}^0)) + \omega^2 R^2 \nabla e(\mathbf{U}^0))] \qquad \varepsilon^2$$

$$\mathrm{div}(C^0 : e(\mathbf{U}^3)) + \langle\rho\rangle\omega^2\, \mathbf{U}^3 = -[\,\mathrm{div}(C^1 \ldots \nabla e(\mathbf{U}^2)) + \omega^2 R^1 e(\mathbf{U}^2))] - [\mathrm{div}(C^2 \ldots . \nabla\nabla e(\mathbf{U}^1)) + \omega^2 R^2 \nabla e(\mathbf{U}^1))] - [\mathrm{div}(C^3 \ldots .. \nabla\nabla\nabla e(\mathbf{U}^0)) + \omega^2 R^3 \nabla\nabla e(\mathbf{U}^0))] \qquad \varepsilon^3$$

Classic Elastodynamics & Homogenized diffracted sources

$C^i = O(a l^i)$; $R^i = O(r l^i)$

Real & frequency independant

Non local Elasticity, Non local Mass

Correctors « Weak » dynamic local regime

Plane wave propagation

Principle ≅ Huygens $U^0 \rightarrow$ Diffracted Source $\rightarrow U^1 \rightarrow$ Sources $\rightarrow U^2 \rightarrow$ Sources ....

$$\begin{cases} E_M(\mathbf{U}^0) = 0 & \\ E_M(\mathbf{U}^1) = - S^1(\mathbf{U}^0) & \varepsilon \\ E_M(\mathbf{U}^2) = - S^1(\mathbf{U}^1) - S^2(\mathbf{U}^0) & \varepsilon^2 \\ E_M(\mathbf{U}^3) = - S^1(\mathbf{U}^2) - S^2(\mathbf{U}^1) - S^3(\mathbf{U}^0) & \varepsilon^3 \end{cases}$$

Order 0 ➔ $\mathbf{U}^0 = \mathbf{A}\, e^{(-ikx.p)}$ k(p, **A**) : p propagation ; **A** polarization (**B**, **C**)

Order 1 $S^1(\mathbf{U}^0) \approx d\mathbf{U}^0/dx \approx -ikl\, e^{(-ikx.p)}$ $S^1(\mathbf{U}^0) \perp \mathbf{A}$!

➔ $\mathbf{U}^1 = ikl\,(b_1\mathbf{B} + c_1\mathbf{C})\, e^{(-ikx.p)}$

Order 2 $S^1(\mathbf{U}^1) + S^2(\mathbf{U}^0) \approx (-ikl)^2\, e^{(-ikx.p)}$ $S^1(\mathbf{U}^1) + S^2(\mathbf{U}^0) \parallel$ **A**, **B**, **C**

➔ $\mathbf{U}^2 = (-ikl)^2\,(-ikx.p\,a_2\mathbf{A} + b_2\mathbf{B} + c_2\mathbf{C})\, e^{(-ikx.p)}$

Order 3 $S^1(\mathbf{U}^2) + S^2(\mathbf{U}^1) + S^3(\mathbf{U}^0) \approx (-ikl)^3\, e^{(-ikx.p)}$ $S^1(\mathbf{U}^2) + S^2(\mathbf{U}^1) + S^3(\mathbf{U}^0) \parallel$ **A**, **B**, **C**

➔ $\mathbf{U}^3 = (-ikl)^3\,(-ikx.p\,a_3\mathbf{A} + (b_3 - ikx.p\beta)\mathbf{B} + (c_3 - ikx.p\gamma)\mathbf{C})\, e^{(-ikx.p)}$

Rayleigh scattering

≠ Orders

$\mathbf{U}^0 = \mathbf{A}\, e^{(-ikx.p)}$

$\mathbf{U}^1$: depolarized, in quarter of phase

$\mathbf{U}^2$: **A** component linearly amplified, in quarter of phase

$\mathbf{U}^3$: **A** component linearly amplified, in phase opposition
other components linearly amplified

Global field

$\mathbf{U}(x) = \mathbf{U}^0(x) + \mathbf{U}^1(x) + \mathbf{U}^2(x) + \mathbf{U}^3(x) + \ldots \cong \mathbf{A}'\exp(-ik'x.p)$

➔ $\mathbf{A}' \cong \mathbf{A} + (\omega l/c)\,(b_1\mathbf{B} + c_1\mathbf{C}) + \ldots$ $\quad k' \cong k(1 + (\omega l/c)^2\,\kappa + i(\omega l/c)^3\,\kappa')$

Rayleigh scattering

Correction of polarization $\quad O(\omega l/c)$

Dispersion of velocity $\quad O((\omega l/c)^2)$

Apparent attenuation $\quad O((\omega l/c)^3)$

B , C amplified : Mode conversion **A** ➔ **B , C** of ε order : $d > \lambda/\varepsilon^{1/2} = (\omega/cl)^{1/2}$

! Correctors ➔ No change of direction of propagation

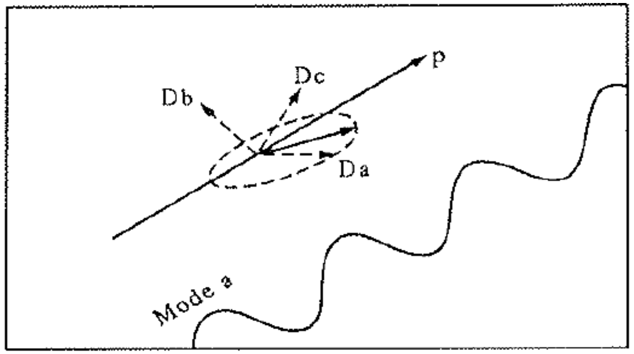
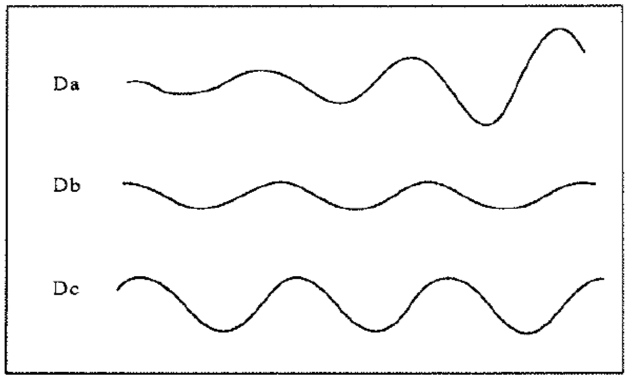
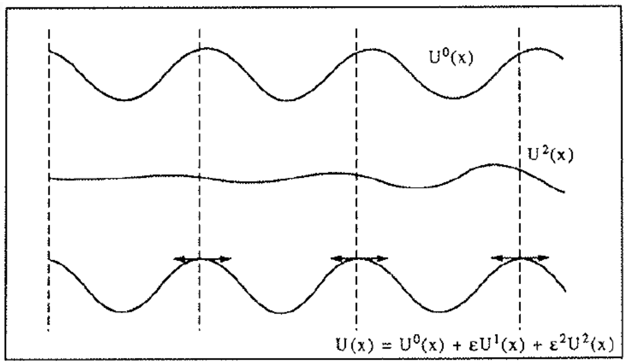


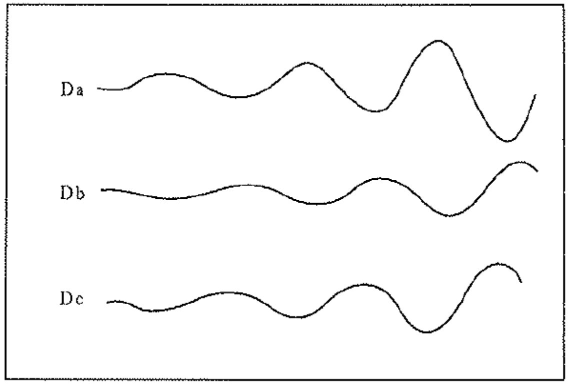
Fig. 3. Correction of polarization at the first order.



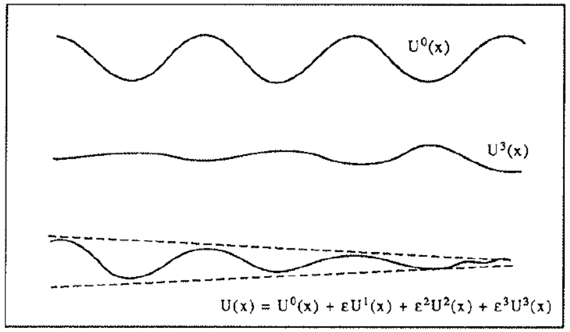
Wave field at the second order: the vibration polarized in direction Da is amplified.



The effect of dispersion results of the summation of the wave U^0 and the wave U^2 (in phase quadrature and amplified).



Wave field at the third order: the vibration for each polarization direction is amplified.



The effect of attenuation results of the summation of the wave U^0 and the wave U^3 (in phase opposition and amplified).

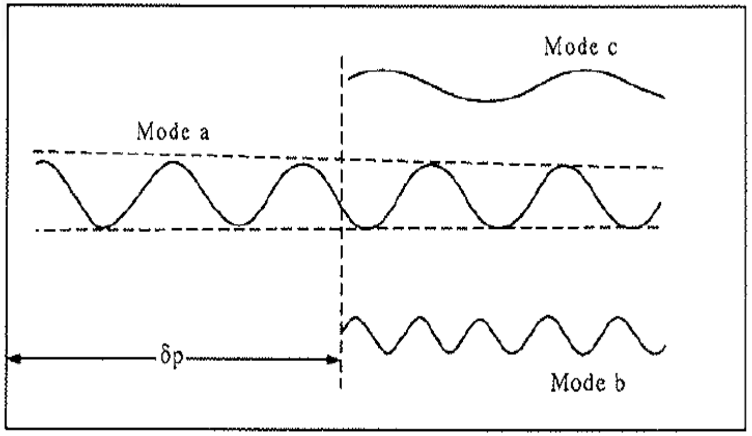
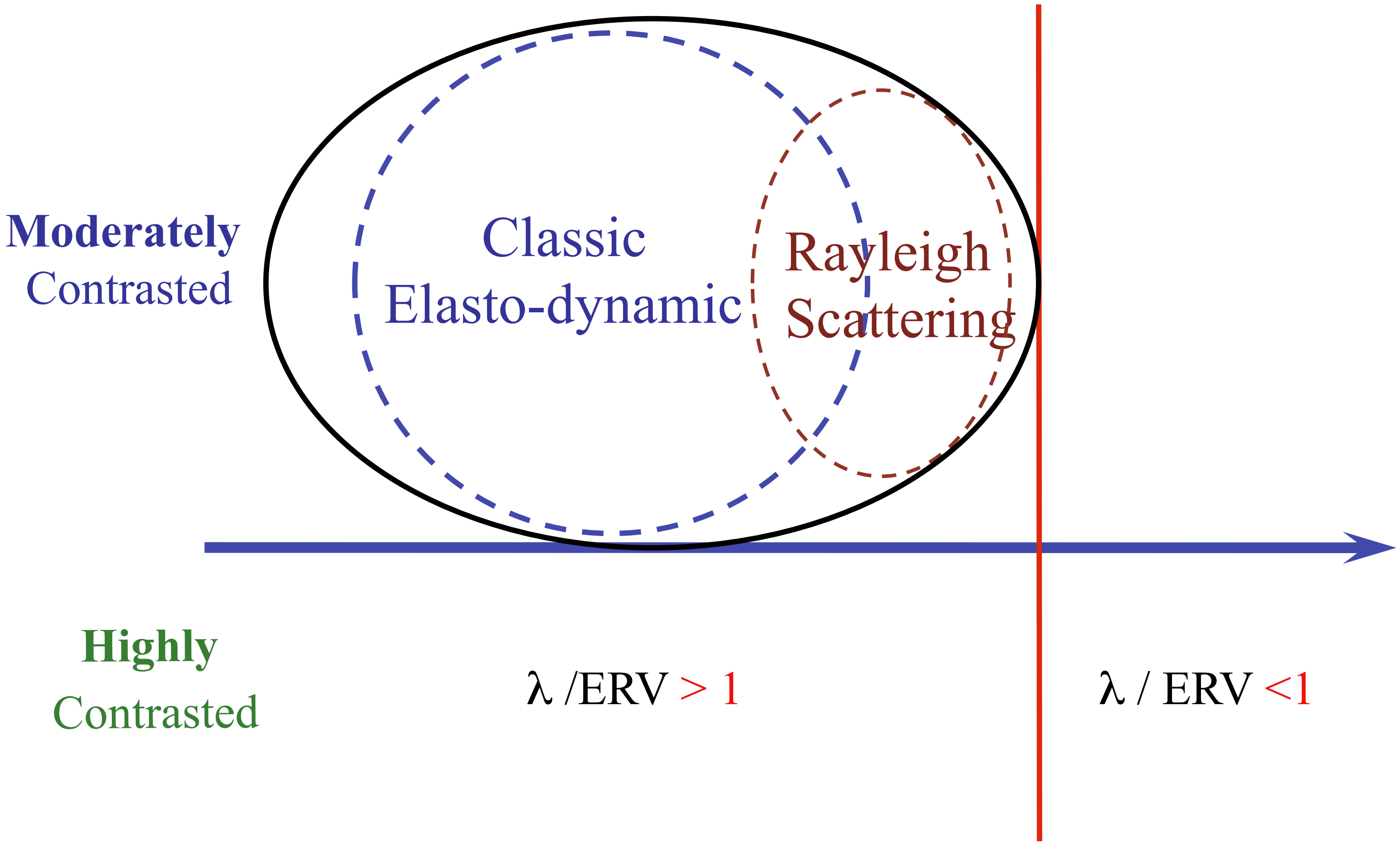


Fig. 8. From the distance δp, mode conversion appears at the second order.

Learning : Enriched Elastodynamics



Content

Inner resonance media ?

Dynamic phenomena

at micro and macro scales

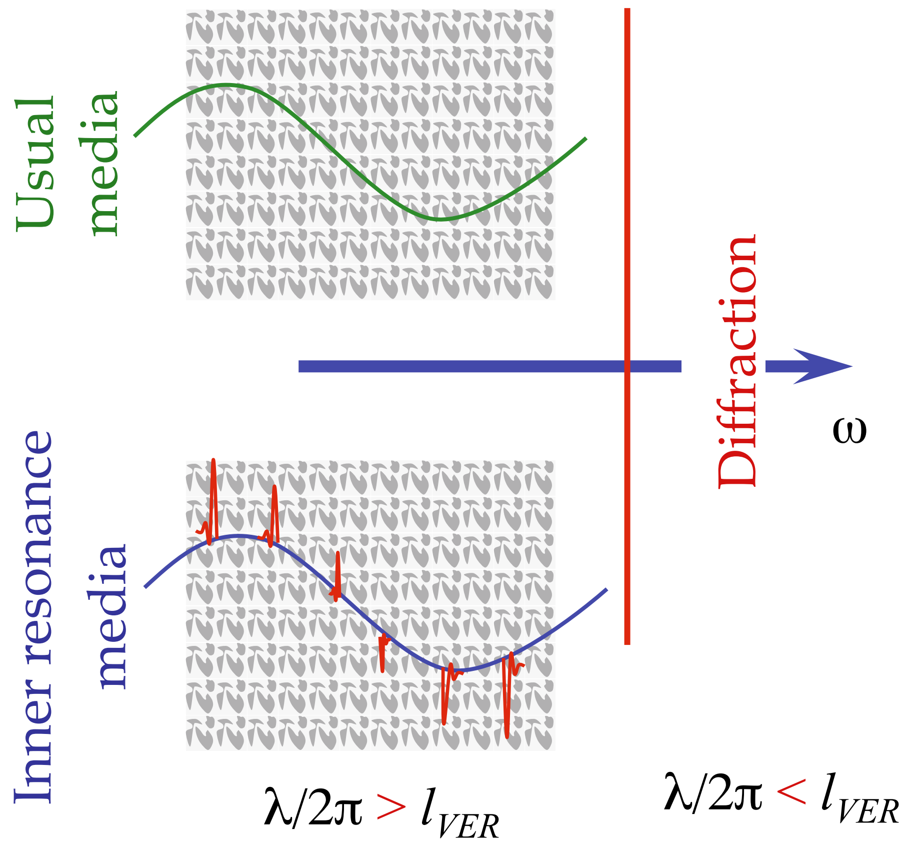
"Co-dynamics" regime

only possible in heterogeneous media

Unconventional effective parameters

Properties impossible to reach

with classical materials.



An « old » Idea

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ON SOME GENERALIZATIONS OF BOUSSINESQ AND KdV SYSTEMS

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Presented by J. Engelbrecht

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Abstract. We present basic arguments of quasi-particles (endowed with mass, momentum, and energy) to study the essential solitonic features of the Boussinesq–Korteweg–de Vries model and some of its generalizations: regularized long-wave equation, generalized Boussinesq model, and nonlinear Maxwell–Rayleigh model. Hamiltonian descriptions and associated global conservation laws are given for all these systems. The so-called wave momentum (also called pseudo-momentum, or canonical momentum in the absence of dissipation) plays a prominent role in this formulation as it provides the equation of motion of solitons or soliton-like structures.

Key words: KdV equation, solitons, conservation laws.

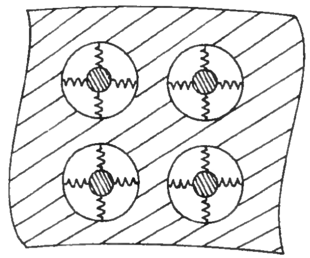
1. INTRODUCTION

In describing the typical methodology of the nineteenth-century English school of mathematical physics, Duhem [1] emphasizes William Thomson's (later Lord Kelvin) lack of understanding before he has conceived of any mechanical model to represent the physical object under study (the original quotation in Thomson [2], p. 270; the general attitude of Thomson towards mechanical models see in [3]). The same applies to James Clerk Maxwell, and this is perfectly illustrated in Whittaker's *History of the Theories of Aether and Electricity* [4], when the latter comments on various attempts at introducing dispersion in the vibration of atomic systems (see pp. 260–265). The linear version of the model developed below was initially prescribed by Maxwell for the mathematical tripos at Cambridge [5] (nowadays we would say final examinations in mathematical physics) with further elaboration by Lord Rayleigh [6], but the illustration itself (see below) could only belong to Thomson (according to Whittaker [4], p. 262). The mechanical modelling of atomic structures thus outlined was to become the basis of lattice dynamics in the spirit of

Born and Karman (see, e.g., [7, 8]) as also, in some sense, that of finite-difference methods (if we reverse the reasoning in passing from the discrete to the continuum and vice versa). As we know from Fermi et al. [9] and Zabusky and Kruskal [10], this was to lead to the introduction of solitons whenever dispersion and nonlinearity would exactly compensate one another. In particular, the most celebrated equation giving rise to this remarkable dynamic phenomenon is the Korteweg–de Vries (for short KdV) equation which was introduced by Korteweg and de Vries [11] in fluid dynamics. On the centennial celebration of this discovery we prefer here to see the KdV equation as the one-directional, or evolution, equation associated to the no less celebrated Boussinesq (for short B) equation via the reductive perturbation method [12]. Here below we shall introduce a more general model based on Maxwell's and Rayleigh's proposal to describe the phenomenon of anomalous dispersion. We shall also comment on (i) several equations of the B–KdV type that occur in various branches of applied physics and (ii) considerations of quasi-particles to interpret some of the dynamical behaviours of these systems, as true solitons may indeed be considered as quasi-particles verifying a set of equations of motion characteristic of "particles".

2. MAXWELL–RAYLEIGH MODEL OF ANOMALOUS DISPERSION

We use a modern jargon and notation and will later on give the correspondence with the original Maxwell–Rayleigh model, a picture of which, in Thomson's view, is given in the Figure. We work in the material description of continuum mechanics in order to accommodate easily nonlinear phenomena. The material point $\mathbf{X}$ has for image $\mathbf{x}$ such that $\mathbf{x} = \chi(\mathbf{X}, t)$ where t is time. This defines the deformation of the elastic matrix, of which the displacement is $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$. But there is a continuous distribution of "atoms" at each $\mathbf{X}$ with relative displacement



The Maxwell–Rayleigh model of anomalous dispersion (foreign inclusions linearly or nonlinearly elastically connected to the elastic matrix).

Maxwell J.C. 1869 ; L. Rayleigh, 1899

$\zeta(\mathbf{X},t)$ with respect to the matrix. That is, the instantaneous physical position of these atoms is given by $\mathbf{x}_I(\mathbf{X},t) = \mathbf{X} + \mathbf{u}(\mathbf{X},t) + \zeta(\mathbf{X},t)$. We may view this as a microstructure giving rise to a continuum of inclusions. Let ρ and r be the mass densities of the matrix and the "inclusions", respectively. Then the density of kinetic energy is given by

$$K = \frac{1}{2}\rho\left(\frac{\partial \mathbf{u}}{\partial t}\right)^2 + \frac{1}{2}r\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \varsigma}{\partial t}\right)^2 . \tag{2.1}$$

We consider a one-dimensional model, so that we indeed have a composite lattice in the form of a one-dimensional chain, but we do not allude further to any discrete structure. Each inclusion being supposed to be maintained in its placement in the matrix by an attractive force $r\omega_0^2\zeta$, where ω_0 is a characteristic frequency, with a linear elastic matrix of elasticity coefficient E, we have a density of potential energy given by

$$V = \frac{1}{2}E\left(\frac{\partial u}{\partial X}\right)^2 + \frac{1}{2}r\omega_0^2\varsigma^2 . \tag{2.2}$$

The associated Euler–Lagrange equations of motion are:

$$\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} + r\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 \varsigma}{\partial t^2}\right) - E\frac{\partial^2 u}{\partial X^2} &= 0, \\ r\left(\frac{\partial^2 \varsigma}{\partial t^2} + \frac{\partial^2 u}{\partial t^2}\right) + r\omega_0^2\varsigma &= 0. \end{aligned} \tag{2.3}$$

By applying the operator $1 + \omega_0^{-2}(\partial^2/\partial t^2)$ to the first of these and substituting for the second, we eliminate the internal degree of freedom ζ and deduce the following wave equation for the matrix displacement u:

$$(1 + \nu)\, u_{tt} - c_0^2 u_{xx} + \omega_0^{-2} u_{tttt} - k_0^{-2} u_{ttxx} = 0, \tag{2.4}$$

wherein $\nu = r/\rho$ is the ratio of densities, $c_0 = (E/\rho)^{1/2}$ is the characteristic elastic speed, $k_0 = \omega_0/c_0$ is a characteristic wave number, and we have used the applied-mathematics notation for partial derivatives with respect to t and X. Equation (2.4) is the Maxwell–Rayleigh equation for anomalous dispersion, but written in modern elasticity notation. It contains two dispersion terms, but either of these would be sufficient to produce the required dispersion. For further comparison, after appropriate scaling we can rewrite it in fully nondimensional form as

$$u_{tt} - u_{xx} + \varepsilon\,(u_{tttt} - u_{ttxx}) = 0, \tag{2.4'}$$

where the ordering parameter ε emphasizes the eventual smallness of dispersion effects.

In the original works of Maxwell and Rayleigh the model considered in fact is an elastic aether (nowadays spelled ether), the substratum of light waves, and the above "inclusions" are "atoms": each such "atom" is composed of a single massive particle, which is supported symmetrically by springs (remember Thomson's mental mechanical images) from the interior of a massless spherical shell, i.e., in other words, the "atoms" occupy small spherical cavities in the aether, the outer shell of each atom being in contact with the aether at all points and participating in its motion. In Whittaker's own words, "the medium as a whole is fine-grained". Forgetting about this nineteenth-century picture, the model obtained appears as a special model of diatomic lattice (in the long wave approximation), in which one degree of freedom has been eliminated in favour of the other [12]. The linear version of the Boussinesq equation for elastic crystals (that one which gives rise to the KdV equation) reads [13]

$$u_{tt} - u_{xx} - \varepsilon\, u_{xxxx} = 0, \tag{2.5}$$

while the Love–Rayleigh equation for rods accounting for lateral inertia reads (cf. Love [14], p. 428)

$$u_{tt} - u_{xx} - \varepsilon u_{ttxx} = 0. \tag{2.6}$$

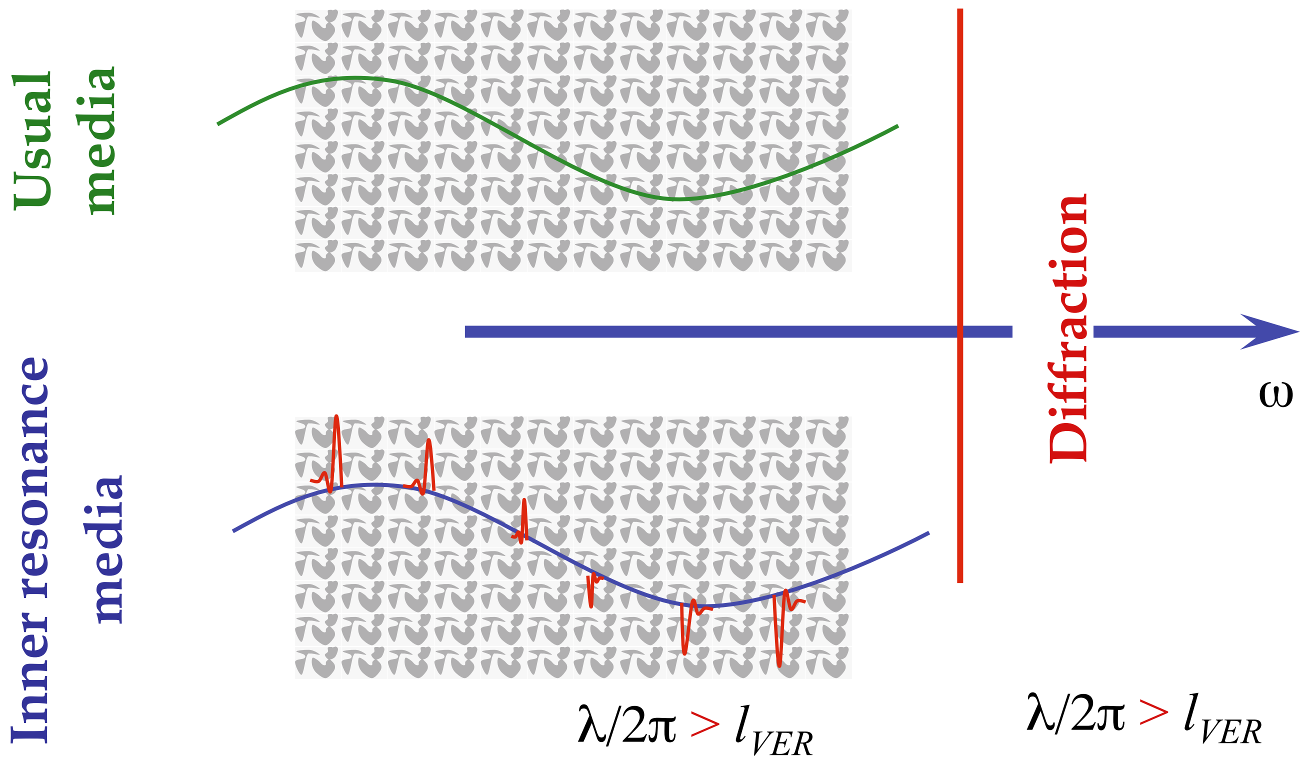
It is clear that Eqs. (2.5) and (2.6) do not have the same dispersion characteristics. However, Eq. (2.4′), where the two dispersion terms concur, has dispersion characteristics akin to those of (2.6). Finally, Eqs. (2.3) and (2.4′) are also to be compared to simple equations governing porous media and granular media, but we shall not do it here. It is of greater interest to introduce nonlinearity.

3. GENERALIZATIONS OF BOUSSINESQ AND KdV EQUATIONS

The modelling (2.1)–(2.5) may be complicated in three ways. First, as remarked by Whittaker ([4], p. 264), one may be tempted to introduce a dissipative force varying like the relative velocity $\partial\zeta/\partial t$ and opposing the motion of "atoms" relative to their shell. This is sufficient to prevent an annoying phenomenon of resonance that would occur if the applied (light) frequency matched the natural frequency ω_0. Second, the restoring force applied to atoms may be modelled by nonlinear springs, so that the second contribution in (2.2) takes on the form

$$V_I = \frac{1}{2}r\omega_0^2\left(\varsigma^2 + \frac{2}{3}k\varsigma^3\right), \tag{3.1}$$

Inner resonance media



Condition for a « Co-dynamic » regime

Two Phase media (C, R)

Long **and** short wave lengths at ω

C-constituent

$\Lambda_C >> l$

Scale separation $\varepsilon = 2\pi\, l/\Lambda << 1$

➔ Quasi static local regime

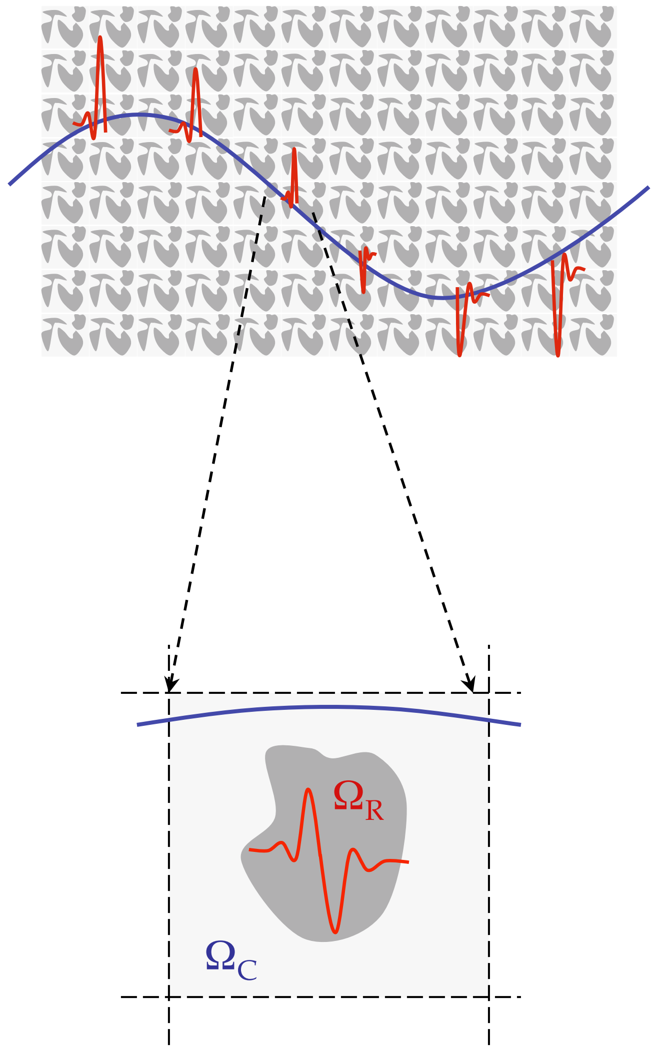
C-carrying constituent (connected)

R-constituent

$\lambda_R \approx l$

➔ Dynamic local regime

R-resonating constituent



Scale separation and large contrast

$\varepsilon = 2\pi\, l/\Lambda \approx \lambda_R / \Lambda_C$ Depends on the physics

High contrasts ?

Change the basic/implicit assumptions

Contrast of properties : O(1)

Inner Geometry : O(1)

Possible candidates

Contrast $\neq$ O(1)
Geometry : O(1)
→ **Highly contrasted continuous media**

Contrast : O(1)
Geometry $\neq$ O(1)
→ **Discrete Reticulated media**

Arch. Mech., 37, 4–5, pp. 269–284, Warszawa 1985

Dynamique des composites elastiques periodiques

J. L. AURIAULT et G. BONNET (GRENOBLE)

On étudie le comportement macroscopique équivalent de composites élastiques sous sollicitation dynamique, en présence ou non de fortes discontinuités des propriétés élastiques des constituants. On suppose que la structure fine (microscopique) est périodique. Pour obtenir les paramétres effectifs on utilise la méthode de l'homogénéisation.

Zbadano makroskopowe zachowanie się kompozytów sprężystych poddanych działaniu obciążeń dynamicznych. Uwzględniono przypadek silnych nieciągłości własności sprężystych składników kompozytu. Przyjęto, że struktura mikroskopowa jest periodyczna. Dla otrzymania wielkości efektywnych zastosowano metodę homogenizacji.

Исследовано макроскопическое поведение упругого композита под динамической нагрузкой. Учтен случай сильных разрывов упругих свойств компонентов композита. Принято, что микроскопическая структура является периодической. Для получения эффективных параметров применен метод гомогенизации.

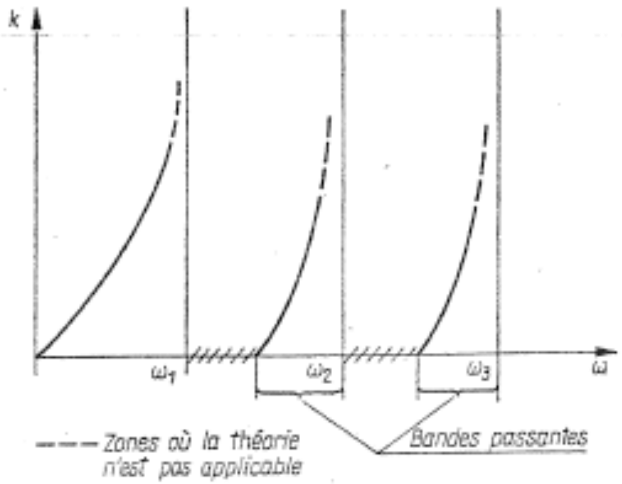


Fig. 3. Milieu composite bilaminé: nombre d'ondes fonction de la pulsation.

La formulation variationnelle correspondante s'écrit:

$$\forall \mathbf{w} \in E_1, \quad \int_{\Omega_1} \mathbf{a}_1 \left(\mathbf{e}_y(\mathbf{u}_1^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)})\right) \mathbf{e}_y(\mathbf{w})\, d\Omega = 0,$$

formulation qui admet une solution unique. Ainsi:

$$\mathbf{u}_1^{(1)} = \xi_1\, \mathbf{e}_x(\mathbf{u}^{(0)}) + \tilde{\mathbf{u}}_1^{(1)}(x),$$

où les ξ_{1ijk} sont des solutions particulières du problème correspondant à

$$e_{x_{lm}}(\mathbf{u}^{(0)}) = \delta_{lj}\,\delta_{mk}.$$

Les équations (3.2) et (3.4) à ε^0 donnent $\mathbf{u}_2^{(0)}$:

(3.5)
$$\nabla_y(\mathbf{a}_2'\, \mathbf{e}_y(\mathbf{u}_2^{(0)})) = -\varrho_2 \omega^2 \mathbf{u}_2^{(0)},$$
$$\mathbf{u}_2^{(0)} = \mathbf{u}^{(0)}(x) \text{ sur } \Gamma, \quad \mathbf{u}_2^{(0)},\ \Omega\text{ — périodique.}$$

Pratiquons la translation:

$$\mathbf{u}_2^{(0)} = \mathbf{u}^{(0)}(x) + \mathbf{v}.$$

Le déplacement relatif $\mathbf{v}$ est solution de:

(3.6)
$$\nabla_y(\mathbf{a}_2'\, \mathbf{e}_y(\mathbf{v})) = -\varrho_2 \omega^2 (\mathbf{u}^{(0)}(x) + \mathbf{v}),$$
$$\mathbf{v} = 0 \quad \text{sur } \Gamma.$$

Soit E_2 l'espace des vecteurs $\mathbf{v}$ Ω — périodiques, nuls sur Γ avec le produit scalaire:

$$(\mathbf{v}, \mathbf{w}) = \int_{\Omega_2} e_{ij}(\mathbf{v})\, e_{ij}(\mathbf{w})\, d\Omega.$$

Le spectre de l'opérateur du premier membre de (3.5) dans E_2, donné par:

$$\nabla_y(\mathbf{a}_2'\, \mathbf{e}_y(\mathbf{v})) = -\lambda \mathbf{v}, \quad \mathbf{v} \neq 0,$$

est discret réel et positif [2]. Les valeurs propres sont λ_i telles que:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots.$$

Dans ces conditions il est clair que la solution de (3.6) existe pour les valeurs de ω différentes de $\omega_i = (\lambda_i/\varrho_2)^{1/2}$, $i = 1, 2 \ldots$. Si $\omega = \omega_i$ la solution existe si et seulement si la fonction propre correspondante $\mathbf{v}_i$ est orthogonale à $\varrho_2 \omega^2 \mathbf{u}^{(0)}(x)$ c–a–d si $\langle \mathbf{v}_i \rangle_{\Omega_2} = 0$. Dans le cas contraire la solution $\mathbf{v}$ n'est pas bornée au voisinage de ω_i et change de signe au passage de cette valeur si λ_i est une valeur propre simple.

La linéarité nous permet d'écrire la solution sous la forme:

$$\mathbf{v} = \mathbf{u}_2^{(0)} - \mathbf{u}^{(0)}(x) = \mathbf{g}(\omega)\mathbf{u}^{(0)}(x) - \mathbf{u}^{(0)}(x).$$

L'équation (3.1) à l'ordre ε^0 s'écrit:

$$\nabla_y\left(\mathbf{a}_1\left(\mathbf{e}_y(\mathbf{u}_1^{(2)}) + \mathbf{e}_x(\mathbf{u}_1^{(1)})\right)\right) + \nabla_x\left(\mathbf{a}_1\left(\mathbf{e}_y(\mathbf{u}_1^{(1)}) + \mathbf{e}_x(\mathbf{u}^{(0)})\right)\right) = -\varrho_1 \omega^2 \mathbf{u}^{(0)}.$$

Intégrée sur Ω_1 et jointe à l'équation (3.5) intégrée sur Ω_2, elle conduit, avec (3.3) à l'ordre ε, à l'équation de compatibilité donnant la description macroscopique:

$$\nabla_x\left(\mathbf{c}\mathbf{e}_x(\mathbf{u}^{(0)})\right) = -\left(\langle \varrho_1 \rangle + \langle \mathbf{g}(\omega) \rangle \varrho_2\right) \omega^2 \mathbf{u}^{(0)},$$

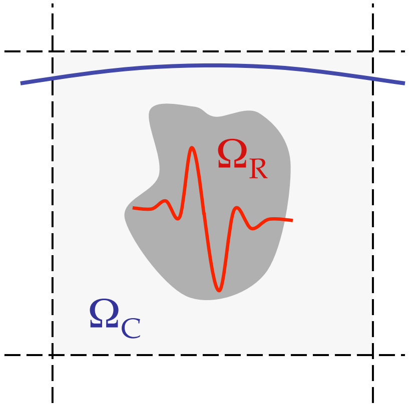
Inner Resonance by Stiffness Contrast

Contrasted Elastic Composites

$\varepsilon = 2\pi\, l/\Lambda \approx \lambda_R / \Lambda_C$

$$\frac{\Lambda_C}{2\pi} = \frac{1}{\omega}\sqrt{\frac{|\mathbf{a}_c|}{\rho_c}} \qquad \frac{\lambda_R}{2\pi} = \frac{1}{\omega}\sqrt{\frac{|\mathbf{a}_r|}{\rho_R}}$$

$$\rho_c = O(\rho_r) \quad \Rightarrow \quad \frac{|\mathbf{a}_R|}{|\mathbf{a}_C|} = \varepsilon^2 \ll 1$$



Homogenization

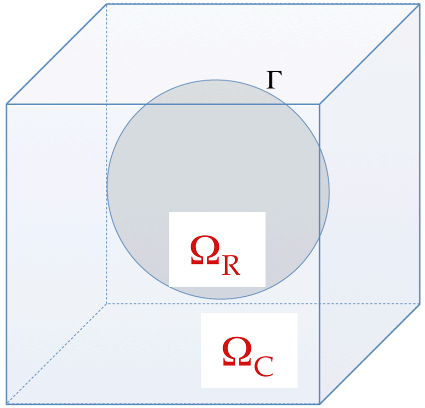
Auriault et Bonnet, Arch. Mech, 1985

Auriault et Boutin, IJSS, 2012

Homogenization - 1

Scaled governing equations

$$\left\{\begin{array}{rcll} \mathrm{div}(\mathbf{a}_\mathrm{c} : \mathbf{e}(\mathbf{u}_\mathrm{c})) &=& -\omega^2 \rho_\mathrm{c} \mathbf{u}_\mathrm{c} & \text{in } \Omega_\mathrm{c} \\ \mathrm{div}(\varepsilon^2 \mathbf{a}_\mathrm{r} : \mathbf{e}(\mathbf{u}_\mathrm{r})) &=& -\omega^2 \rho_\mathrm{r} \mathbf{u}_\mathrm{r} & \text{in } \Omega_\mathrm{r} \\ (\mathbf{a}_\mathrm{c} : \mathbf{e}(\mathbf{u}_\mathrm{c}) - \varepsilon^2 \mathbf{a}_\mathrm{r} : \mathbf{e}(\mathbf{u}_\mathrm{r})).\mathbf{n} &=& 0 & \text{over } \Gamma \\ \mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{c} &=& 0 & \text{over } \Gamma \end{array}\right.$$



C constituent

$$\mathbf{u}_\mathrm{c}^{(0)} = \mathbf{U}^{(0)}(\mathbf{x})$$

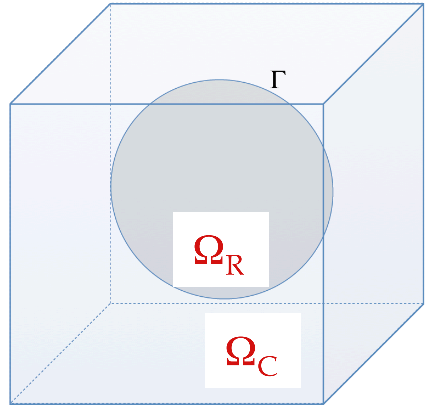
$$\left\{\begin{array}{ll} \mathrm{div}_y(\mathbf{a}_\mathrm{c} : (\mathbf{e}_y(\mathbf{u}_\mathrm{c}^{(1)}) + \mathbf{e}_x(\mathbf{U}^{(0)})) = 0 & \text{in } \Omega_\mathrm{c} \\ (\mathbf{a}_\mathrm{c} : \mathbf{e}_y(\mathbf{u}_\mathrm{c}^{(1)}) + \mathbf{e}_x(\mathbf{U}^{(0)})) \cdot \mathbf{n} = 0 & \text{over } \Gamma \end{array}\right.$$

➔ Local quasi-statics - As without R

$$\left\{\begin{array}{rcl} \mathbf{u}_\mathrm{c}^{(1)} &=& \chi(\mathbf{y}) : \mathbf{e}_x(\mathbf{U}_\mathrm{s}^{(0)}) + \mathbf{U}_\mathrm{s}^{(1)}(\mathbf{x}) \\ \boldsymbol{\sigma}_\mathrm{c}^{(o)} &=& [\mathbf{a} : \mathbf{e}_y(\chi) + \mathbf{a}] : \mathbf{e}_x(\mathbf{U}^{(0)}) \end{array}\right.$$

Homogenization - 2

Next order - C constituent



$$\begin{cases} \mathrm{div}_y(\boldsymbol{\sigma}_c^{(1)}) + \mathrm{div}_x(\boldsymbol{\sigma}_c^{(o)}) = -\omega^2 \rho_c \mathbf{U}^{(0)} & \text{in } \Omega_c \\ \boldsymbol{\sigma}_c^{(1)} \cdot \mathbf{n} = (\mathbf{a}_r : \mathbf{e}_y(\mathbf{u}_r^{(0)}) \cdot \mathbf{n} & \text{over } \Gamma \end{cases}$$

➔ Partial balance

$$\mathrm{div}_x(\mathbf{C} : \mathbf{e}_x(\mathbf{U}^{(0)})) = -\omega^2 \rho_c \frac{|\Omega_c|}{|\Omega|} \mathbf{U}^{(0)} - \frac{1}{|\Omega|} \int_\Gamma (\mathbf{a}_r : \mathbf{e}_y(\mathbf{u}_r^{(0)})) \cdot \mathbf{n} \, d\Gamma$$

Leading order - R constituent

$$\begin{cases} \mathrm{div}_y(\mathbf{a}_r : \mathbf{e}_y(\mathbf{u}_r^{(0)})) = -\omega^2 \rho_r \mathbf{u}_r^{(0)} & \text{in } \Omega_r \\ \mathbf{u}_r^{(0)} = \mathbf{U}^{(0)}(\mathbf{x}) & \text{over } \Gamma \end{cases}$$

➔ Local dynamics - Eigen modes of R

$$\mathbf{u}_r^{(0)} = \mathbf{U}^{(0)} + \boldsymbol{\alpha}(\mathbf{y}, \omega)$$

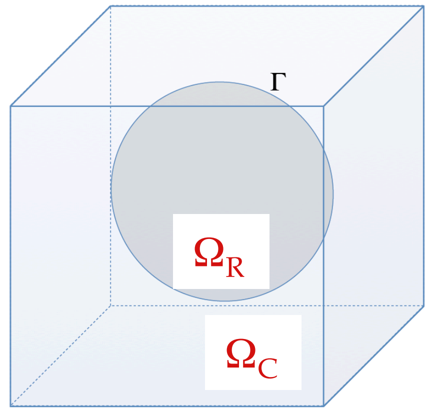
$$\boldsymbol{\alpha}(\mathbf{y}, \omega) = \sum_{i=1}^{\infty} \frac{\mathbf{U}^{(0)} . \int_{\Omega_r} \phi^i d\Omega}{\int_{\Omega_r} \|\phi^i\|^2 d\Omega} \frac{\phi^i}{\frac{\omega_i^2}{\omega^2} - 1}$$

Homogenization - 3

Inner resonant field α

$$\mathbf{u}_r^{(0)} = \mathbf{U}^{(0)} + \boldsymbol{\alpha}(\mathbf{y}, \omega)$$

$$\begin{cases} \mathrm{div}_y(a_r : e_y(\alpha)) = -\omega^2 \rho(\alpha + U^{(0)}) & \text{in } \Omega_r \\ \alpha = 0 & \text{over } \Gamma \end{cases}$$



Eigen modes of Ω_r

$$\begin{cases} \mathrm{div}_y(a_r : e_y(\phi)) = -\omega^2 \rho\, \phi & \text{in } \Omega_r \\ \phi = 0 & \text{over } \Gamma \end{cases}$$

$\Rightarrow \{\omega_i ; \phi_i\}$; $\quad \int_{\Omega r} \phi_{i.}\, \phi_j \, dv = \delta_{ij}$

Modal decomposition of α

$$\int_{\Omega r} \phi_i \Big(\mathrm{div}_y(a_r : e_y(\alpha)) + \omega^2 \rho(\alpha + U^{(0)})\Big) dv - \int_{\Omega} \alpha \Big(\mathrm{div}_y(a_r : e_y(\phi_i)) + \omega_i^2 \rho\, \phi_i\Big) dv = 0$$

$$\alpha = \sum_j \beta_j \phi_j$$

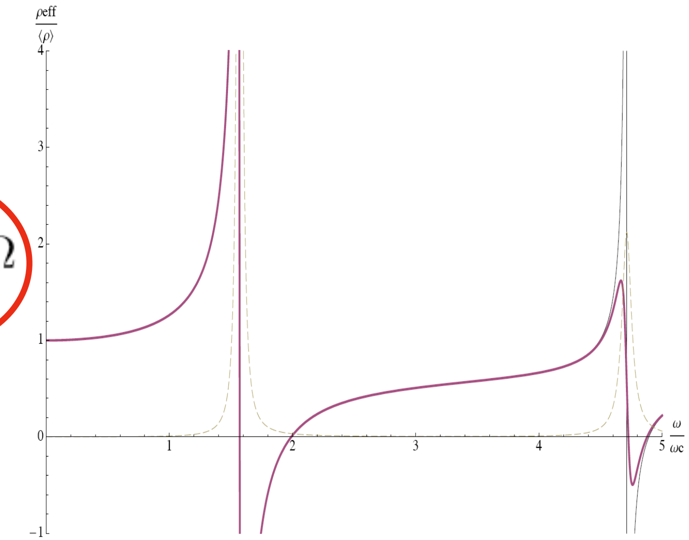
$$\omega^2 \rho \left(\beta_i + \int_{\Omega r} \phi_i . U^{(0)}\, dv\right) - \omega_i^2 \rho\, \beta_i = 0 \qquad \backslash\ \beta_i = \left(\int_{\Omega r} \phi_i . U^{(0)}\, dv\right) (\omega^2 - \omega_i^2)^{-1}$$

Inner Resonance Description

Global R&C balance

$$\mathrm{div}_x(\mathbf{C} : \mathbf{e}_x(\mathbf{U}^{(0)})) = -\omega^2 \boldsymbol{\rho} \cdot \mathbf{U}^{(0)}$$

$$\boldsymbol{\rho}(\omega) = \frac{\rho_c |\Omega_c| + \rho_r |\Omega_r|}{|\Omega|} \mathbf{I} + \frac{\rho_r}{|\Omega|} \int_{\Omega_r} \boldsymbol{\alpha} \mathrm{d}\Omega$$

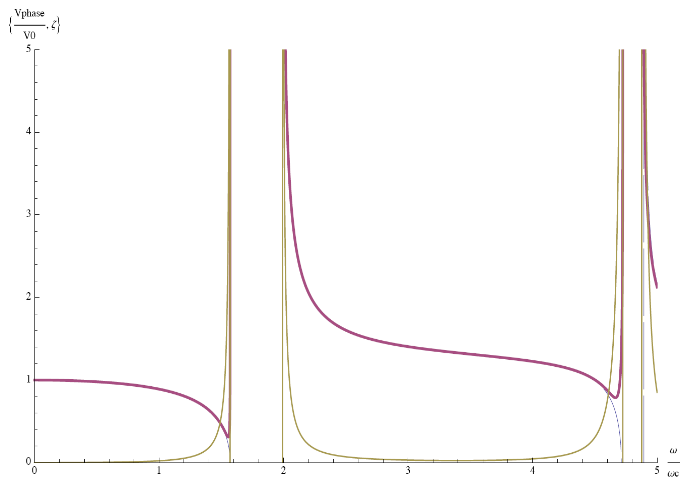


Unconventional dynamics

Usual elasticity

Apparent mass Tensor
Frequency dependent

Long wavelenght Dispersion
Band gaps



Many other possible geometries

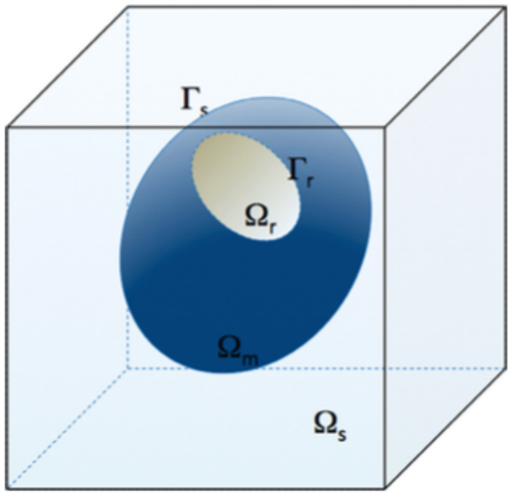
Cylindrical, spherical inclusions

Analytical solutions *[Bonnet, Monchiet JASA 2015]*

Composite inclusion in matrix

No symetry : rotation inertia

$$\mathrm{div}_x(\mathbf{A}_s^{eff} : \mathbf{e}_x(\mathbf{U}_s^{(0)})) = -\omega^2\left(\mathbf{d}(\omega)\cdot\mathbf{U}_s^{(0)} + \mathbf{e}(\omega)\cdot\widetilde{\mathbf{U}}_r^{(0)} + \mathbf{f}(\omega)\cdot\Omega_r^{(0)}\right)$$
$$\omega^2 c_r \rho_r(\mathbf{U}_s^{(0)} + \widetilde{\mathbf{U}}_r^{(0)}) = \mathbf{g}(\omega)\cdot\mathbf{U}_s^{(0)} + \mathbf{h}(\omega)\cdot\widetilde{\mathbf{U}}_r^{(0)} + \mathbf{k}(\omega)\cdot\Omega_r^{(0)},$$
$$\omega^2\mathbf{J}_r\cdot\mathbf{\Omega}_r^{(0)} = \mathbf{l}(\omega)\cdot\mathbf{U}_s^{(0)} + \mathbf{m}(\omega)\cdot\widetilde{\mathbf{U}}_r^{(0)} + \mathbf{n}(\omega)\cdot\Omega_r^{(0)},$$



Viscoelasticity

Damping , Regularisation of singularities

Learnings

General principles $\varepsilon = 2\pi\, l/\Lambda \approx \lambda_R / \Lambda_C$

Carrying constituent (connected) & Resonant constituent

Forcing motion Forced regime

High contrast

Resonant constituent

Source term on the **macroscopic balance**

Momentum balance **Atypical mass**

Local out of equilibrium regime

Elasto-inertial system : poles High dispersion and band gaps

Inner resonance media governed by hyperbolic and parabolic dynamic equations. Principles and examples

C. Boutin, J.L. Auriault, G. Bonnet

In **Generalized Models and Non-classical Approaches in Complex Materials 1-** Chap 6 pp. 87-138

Advanced structured materials, Vol. 89, Part 4, 131-141

Springer Eds. H. Altenbach, J. Pouget, M. Rousseau, B. Collet, T. Michelitsch (2018)

Merci de votre attention

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