# Nonlinear Guided Waves

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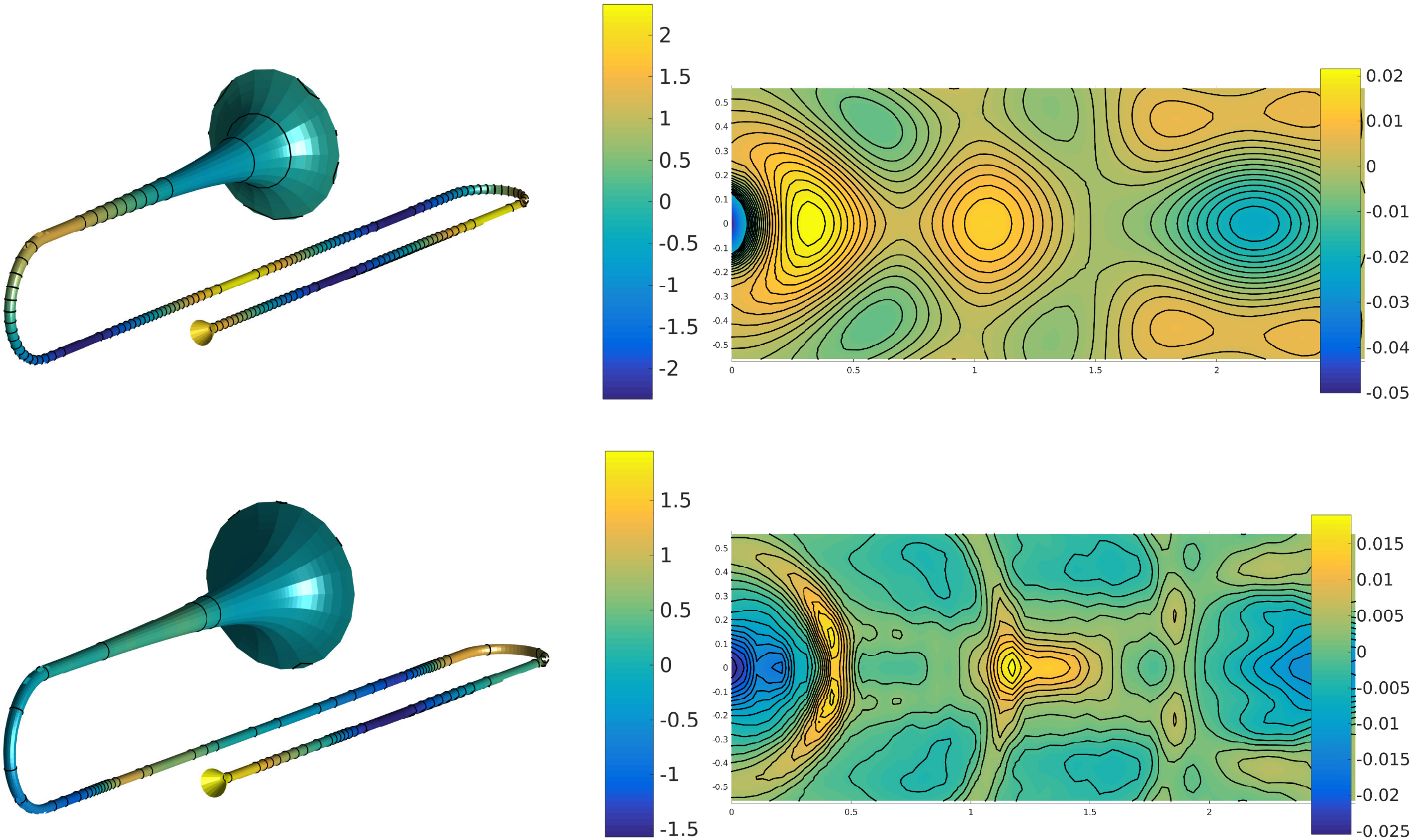
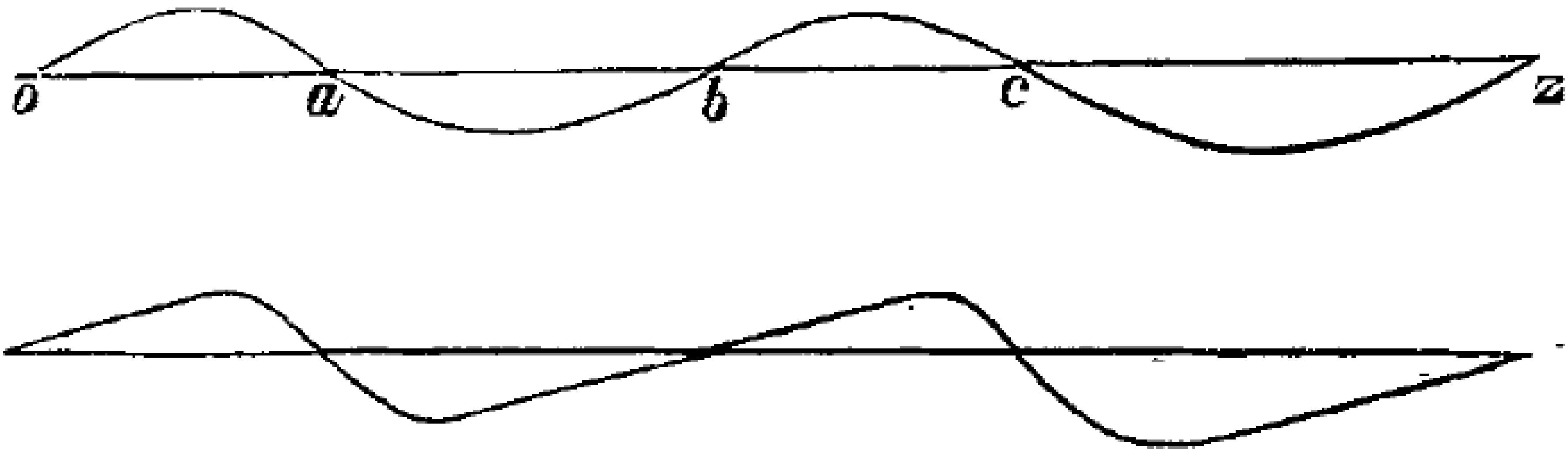
# Work by Cambridge PhD student James McTavish

# Aim of this work

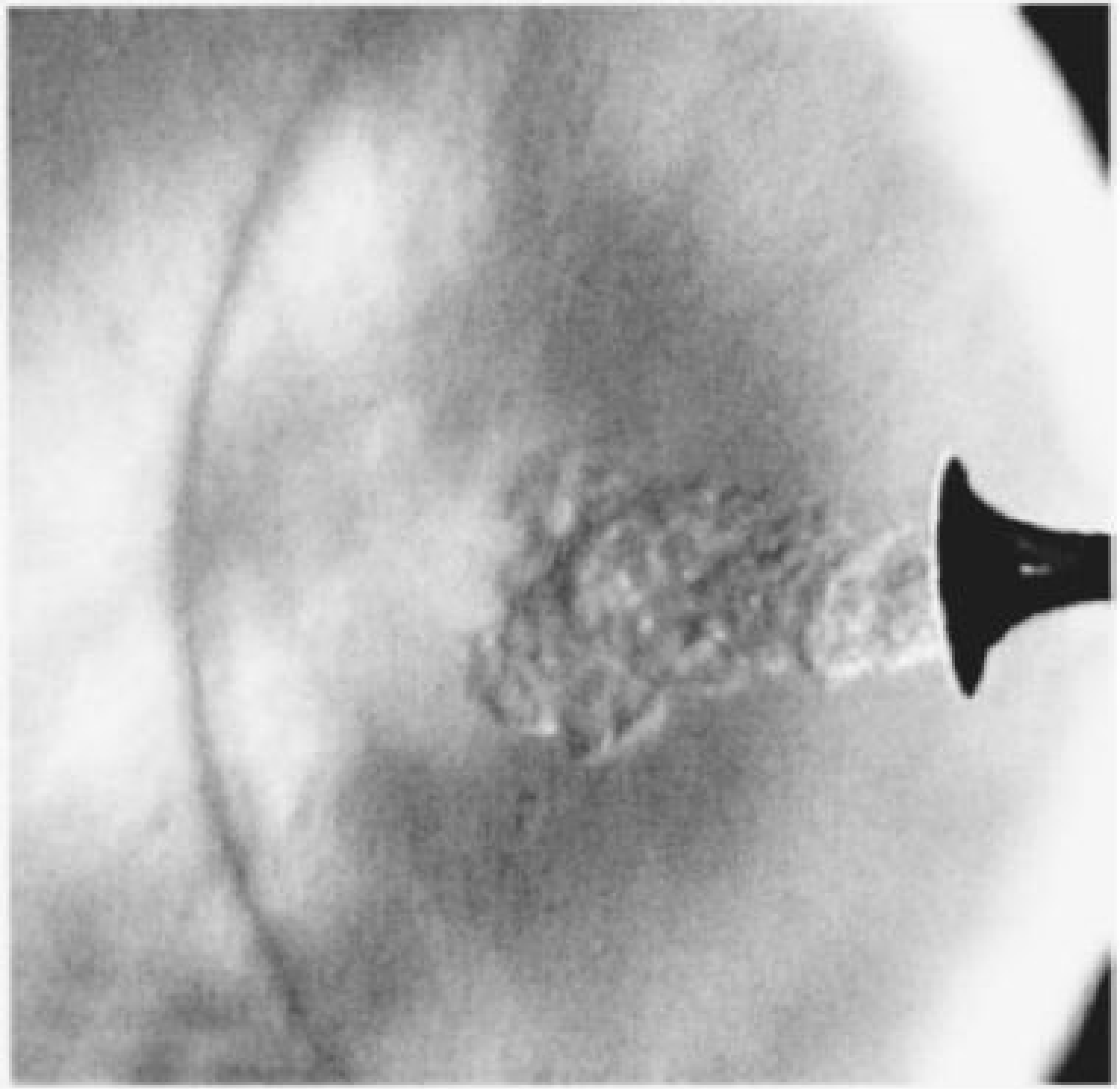


# Wave steepening




The original illustration from Stokes (1848) showing waveform steepening.

# Musical Shocks



A shock wave radiating from the bell of a trumpet. Taken from Pandya, Settles & Miller (2003, JASA).

# Governing equations I

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}) = 0, \qquad \text{(mass)}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) + \boldsymbol{\nabla}p = 0, \qquad \text{(momentum)}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}S = 0. \qquad \text{(entropy)}$$

We will take entropy to be constant everywhere, consistent with (entropy).

Perturbations about a stationary fluid:

$$p = p_0 + \rho_0 c_0^2 p', \qquad \rho = \rho_0 + \rho_0 \rho', \qquad \boldsymbol{u} = c_0 \boldsymbol{u}',$$

Magnitude of perturbation given by acoustic Mach number $M$

$$\max\|\boldsymbol{u}\| = M c_0 \qquad \Rightarrow \qquad \text{SPL} \approx (194 + 20\log_{10} M)\,\mathrm{dB},$$

# Governing equations II

Entropy is constant, so $p = p(\rho)$. Expanding in powers of $M$,

$$\rho' = p' - \frac{B}{2A} p'^2 + O(M^3), \quad \text{where} \quad A = \rho_0 \left.\frac{\partial p}{\partial \rho}\right|_s = \rho_0 c_0^2 \quad \text{and} \quad B = \rho_0^2 \left.\frac{\partial^2 p}{\partial \rho^2}\right|_s .$$

For a perfect gas, $B/A = \gamma - 1$, where $\gamma$ is the ratio of specific heats.

Governing equations become

$$\frac{1}{c_0}\frac{\partial p'}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{u'} = -\boldsymbol{\nabla} \cdot \left(p' \boldsymbol{u'}\right) + \frac{1}{c_0}\frac{B}{2A}\frac{\partial p'^2}{\partial t} + O(M^3),$$

$$\frac{1}{c_0}\frac{\partial \boldsymbol{u'}}{\partial t} + \boldsymbol{\nabla} p' = -\boldsymbol{u'} \cdot \boldsymbol{\nabla} \boldsymbol{u'} + p' \boldsymbol{\nabla} p' + O(M^3).$$

# Governing equations III

Would like time-harmonic $\exp\{-\mathrm{i}\omega t\}$, but nonlinearity gives higher harmonics. Expand as a Fourier series in time (*note: upper indices are for temporal expansion*),

$$p' = \sum_{a=-\infty}^{\infty} P^a(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}a\omega t}, \qquad \boldsymbol{u}' = \sum_{a=-\infty}^{\infty} \boldsymbol{U}^a(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}a\omega t},$$

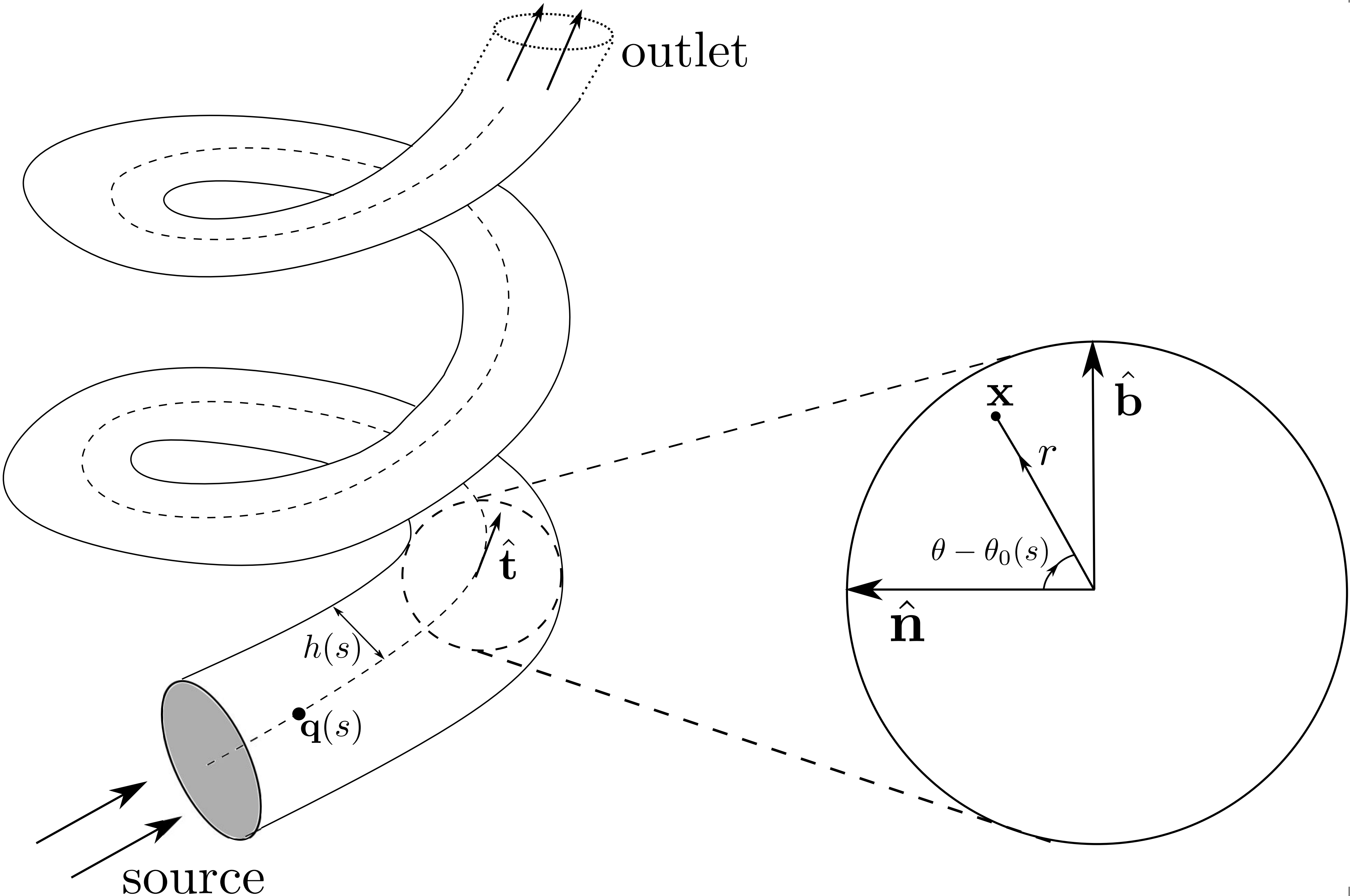
Need $P^{-a} = P^{a*}$ and $\boldsymbol{U}^{-a} = \boldsymbol{U}^{a*}$ for real solutions. Find $P^0 \equiv U^0 \equiv 0$.

Governing equations become, with $k = \omega/c_0$,

$$-\mathrm{i}akP^a + \boldsymbol{\nabla}\cdot\boldsymbol{U^a} = \sum_{b=-\infty}^{\infty}\left(-P^{a-b}\boldsymbol{\nabla}\cdot\boldsymbol{U^b} - \boldsymbol{U^{a-b}}\cdot\boldsymbol{\nabla}P^b - \frac{B}{2A}\mathrm{i}akP^bP^{a-b}\right),$$

$$-\mathrm{i}ak\boldsymbol{U^a} + \boldsymbol{\nabla}P^a = \sum_{b=-\infty}^{\infty}\left(-\boldsymbol{U^{a-b}}\cdot\boldsymbol{\nabla}\boldsymbol{U^b} + P^{a-b}\boldsymbol{\nabla}P^b\right).$$

# Geometry



outlet
$\hat{\mathbf{t}}$
$h(s)$
$\mathbf{q}(s)$
source
$\mathbf{x}$
$r$
$\hat{\mathbf{b}}$
$\theta - \theta_0(s)$
$\hat{\mathbf{n}}$

# Geometry II

$s$ is arc length. Vectors given by Frenet–Serret formulae

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}s} = \boldsymbol{t}, \qquad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \kappa\boldsymbol{n}, \qquad \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \qquad \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s} = -\tau\boldsymbol{n}$$

Coordinate system $\quad \boldsymbol{x} = \boldsymbol{q}(s) + r\cos(\theta-\theta_0)\boldsymbol{n} + r\sin(\theta-\theta_0)\boldsymbol{b},$

$$\Rightarrow \qquad \mathbf{d}\boldsymbol{x} = \mathrm{d}s\Big(\boldsymbol{t} + r\cos(\theta-\theta_0)\big((\tau-\theta_0')\boldsymbol{b} - \kappa\boldsymbol{t}\big) - r\sin(\theta-\theta_0)(\tau-\theta_0')\boldsymbol{n}\Big)$$
$$+\,\mathrm{d}r\Big(\cos(\theta-\theta_0)\boldsymbol{n} + \sin(\theta-\theta_0)\boldsymbol{b}\Big)$$
$$+\,\mathrm{d}\theta\Big(-r\sin(\theta-\theta_0)\boldsymbol{n} + r\cos(\theta-\theta_0)\boldsymbol{b}\Big),$$

To get an orthogonal coordinate system, take $\theta_0' = \tau(s)$ (Germano 1982, JFM), so that

$$h_s = 1 - \kappa r\cos\phi, \qquad \boldsymbol{e_s} = \boldsymbol{t},$$
$$h_r = 1, \qquad \boldsymbol{e_r} = \cos\phi\,\boldsymbol{n} + \sin\phi\,\boldsymbol{b},$$
$$h_\theta = r, \qquad \boldsymbol{e_\theta} = -\sin\phi\,\boldsymbol{n} + \cos\phi\,\boldsymbol{b}.$$

# Geometry III

In this coordinate system,

$$
- \mathrm{i}akP^a + \frac{1}{1-\kappa\cos\phi}\frac{\partial U^a}{\partial s} + \frac{\partial V^a}{\partial r} + \frac{1}{r}V^a - \frac{\kappa\cos\phi}{1-\kappa r\cos\phi}V^a + \frac{1}{r}\frac{\partial W^a}{\partial\theta} + \frac{\kappa\sin\phi}{1-\kappa r\cos\phi}W^a
$$

$$
= \sum_{b=-\infty}^{\infty}\left(-\mathrm{i}bkP^{a-b}P^b - \mathrm{i}bkU^{a-b}U^b - \mathrm{i}bkV^{a-b}V^b - \mathrm{i}bkW^{a-b}W^b - \frac{B}{2A}\mathrm{i}akP^{a-b}P^b\right),
$$

$$
- \mathrm{i}akU^a + \frac{1}{1-\kappa r\cos\phi}\frac{\partial P^a}{\partial s}
$$

$$
= \sum_{b=-\infty}^{\infty}\left(\frac{-U^{a-b}}{1-\kappa r\cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}\right.
$$

$$
\left. + \frac{\kappa\cos\phi}{1-\kappa r\cos\phi}U^{a-b}V^b - \frac{\kappa\sin\phi}{1-\kappa r\cos\phi}U^{a-b}W^b + \mathrm{i}bkP^{a-b}U^b\right),
$$

$$
- \mathrm{i}akV^a + \frac{\partial P^a}{\partial r}
$$

$$
= \sum_{b=-\infty}^{\infty}\left(\frac{-U^{a-b}}{1-\kappa r\cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial\theta}\right.
$$

$$
\left. - \frac{\kappa\cos\phi}{1-\kappa r\cos\phi}U^{a-b}U^b + \frac{W^{a-b}W^b}{r} + \mathrm{i}bkP^{a-b}V^b\right),
$$

# Geometry IV

$$
\begin{aligned}
&-\mathrm{i}akW^a + \frac{1}{r}\frac{\partial P^a}{\partial \theta} \\
&\qquad = \sum_{b=-\infty}^{\infty}\Bigg(\frac{-U^{a-b}}{1-\kappa r\cos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \\
&\qquad\qquad + \frac{\kappa\sin\phi}{1-\kappa r\cos\phi}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} + \mathrm{i}bkP^{a-b}W^b\Bigg) .
\end{aligned}
$$

# Expansion in terms of straight duct modes

Follow Félix & Pagneux (2001 JASA; 2002 WM) and McTavish & Brambley (2019 JFM) and expand spatially in straight duct modes. E.g.

$$P^a = \sum_{\alpha=0}^{\infty} P^a_\alpha(s)\psi_\alpha(s,r,\theta).$$

Modes $\psi_\alpha$ satisfy the straight duct equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_\alpha}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi_\alpha}{\partial\theta^2} + \lambda_\alpha^2\psi_\alpha = 0, \qquad \text{with} \qquad \left.\frac{\partial\psi_\alpha}{\partial r}\right|_{r=h} = 0,$$

normalized so that $$\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta\, r\mathrm{d}r\mathrm{d}\theta = \delta_{\alpha\beta}.$$

The solution is $$\psi_\alpha = C_\alpha J_p\left(\frac{j_{pq}r}{h}\right)\cos\left(p\phi - \frac{\xi\pi}{2}\right), \qquad \text{with} \qquad \xi\in\{0,1\},$$

with eigenvalues $\lambda_\alpha = j_{pq}/h$ (where $J_p'(j_{pq}) = 0$) and normalization factor $C_\alpha$

$$C_\alpha = \begin{cases} \left(\pi h^2 J_0^2(j_{0q})\right)^{-1/2} & p = 0,\\ \left(\frac{\pi h^2}{2}\left(1 - \frac{p^2}{j_{pq}^2}\right)J_p^2(j_{pq})\right)^{-1/2} & p > 0.\end{cases}$$

# Algebra I

Substituting in this expansion into the governing equations gets MESSY!

Introduce some shorthand notation (McTavish & Brambley, 2019 JFM):

$$\Psi_{\alpha(\beta)} = \int_0^{2\pi}\int_0^h \psi_\alpha \frac{\partial \psi_\beta}{\partial \theta}\,\mathrm{d}r\mathrm{d}\theta,$$

$$\Psi_{[\alpha]\beta}[r] = \int_0^{2\pi}\int_0^h \frac{\partial \psi_\alpha}{\partial r}\psi_\beta\, r\mathrm{d}r\mathrm{d}\theta,$$

$$\Psi_{\{\alpha\}\beta\gamma}[r(1-\kappa r\cos\phi)] = \int_0^{2\pi}\int_0^h \frac{\partial \psi_\alpha}{\partial s}\psi_\beta\psi_\gamma\, r(1-\kappa r\cos\phi)\,\mathrm{d}r\mathrm{d}\theta$$

Using this notation, expanding the mass governing equation gives, eventually, e.g.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}U_\alpha^a - \mathrm{i}ak\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]P_\beta^a \\
&\quad - \Psi_{\{\alpha\}\beta}[r]U_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)]V_\beta^a - \Psi_{(\alpha)\beta}[1-\kappa r\cos\phi]W_\beta^a \\
&\quad = \Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]\Big( \\
&\qquad - ibkP_\beta^{a-b}P_\gamma^b - ibkU_\beta^{a-b}U_\gamma^b - ibkV_\beta^{a-b}V_\gamma^b - ibkW_\beta^{a-b}W_\gamma^b - iak\frac{B}{2A}P_\beta^{a-b}P_\gamma^b\Big)
\end{aligned}$$

# Algebra II

Expanding all governing equations in terms of straight duct modes.

Eliminate $V_\alpha^a$ and $W_\alpha^a$, to get the FINAL governing equations (with $' = \mathrm{d}/\mathrm{d}s$)

$$\boldsymbol{u}' + \mathsf{M}\boldsymbol{p} + \mathsf{G}\boldsymbol{u} = \mathcal{A}[\boldsymbol{u}, \boldsymbol{u}] + \mathcal{B}[\boldsymbol{p}, \boldsymbol{p}] + \mathcal{E}[\boldsymbol{u}, \boldsymbol{p}]$$

$$\boldsymbol{p}' - \mathsf{N}\boldsymbol{u} - \mathsf{H}\boldsymbol{p} = \mathcal{C}[\boldsymbol{u}, \boldsymbol{p}] + \mathcal{D}[\boldsymbol{u}, \boldsymbol{u}]$$

In this new notation, $\boldsymbol{p}$ and $\boldsymbol{u}$ are vectors representing $P_\alpha^a$ and $U_\alpha^a$.

$\mathsf{M}$ acts like a matrix,

$$(\mathsf{M}\boldsymbol{x})_\alpha^a = \sum_{\beta=0}^{\infty} \mathsf{M}_{\alpha\beta}^a \boldsymbol{x}_\beta^a$$

$\mathcal{A}$ combines a quadratic form and a convolution,

$$\big(\mathcal{A}[\boldsymbol{x}, \boldsymbol{y}]\big)_\alpha^a = \sum_{b=-\infty}^{\infty} \sum_{\beta,\gamma=0}^{\infty} A_{\alpha\beta\gamma}^{ab} \boldsymbol{x}_\beta^{a-b} \boldsymbol{y}_\gamma^b$$

We will also use the notation $\mathcal{B} = \mathcal{A}[\mathsf{X}, \mathsf{Y}]$ to mean $\mathcal{B}[\boldsymbol{x}, \boldsymbol{y}] = \mathcal{A}[\mathsf{X}\boldsymbol{x}, \mathsf{Y}\boldsymbol{y}]$. In components,

$$(\mathcal{A}[\mathsf{X}, \mathsf{Y}])_{\alpha\beta\gamma}^{ab} = \sum_{\delta,\epsilon=0}^{\infty} A_{\alpha\delta\epsilon}^{ab} \mathsf{X}_{\delta\beta}^{a-b} \mathsf{Y}_{\epsilon\gamma}^b$$

# Can calculate explicitly coefficients, e.g.:

$$
\begin{aligned}
\mathsf{V}^a_{\alpha\beta} &= \frac{1}{\mathrm{i}ak}\Psi_{\alpha[\beta]}[r] \\
\mathsf{W}^a_{\alpha\beta} &= -\frac{1}{\mathrm{i}ak}\Psi_{(\alpha)\beta} = \frac{1}{\mathrm{i}ak}\Psi_{\alpha(\beta)} \\
\mathsf{M}^a_{\alpha\beta} &= -\mathrm{i}ak\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)] - \Psi_{[\alpha]\delta}[r(1-\kappa r\cos\phi)]\mathsf{V}^a_{\delta\beta} - \Psi_{(\alpha)\delta}[1-\kappa r\cos\phi]\mathsf{W}^a_{\delta\beta} \\
\mathsf{N}^a_{\alpha\beta} &= \mathrm{i}ak\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)] \\
\mathsf{G}^a_{\alpha\beta} &= -\Psi_{\{\alpha\}\beta}[r] \\
\mathsf{H}^a_{\alpha\beta} &= -\Psi_{\alpha\{\beta\}}[r] \\
\mathcal{B}^a_{\alpha\beta\gamma} &= -\frac{B}{2A}\mathrm{i}ak\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)] - \mathrm{i}bk\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)] \\
&\quad - \mathrm{i}bk\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{V},\mathsf{V}] - \mathrm{i}bk\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{W},\mathsf{W}] \\
&\quad - \Psi_{[\alpha]\delta}[r(1-\kappa r\cos\phi)](\mathsf{N}^{-1})^a_{\delta\epsilon}\Big(\Psi_{\epsilon\beta\gamma}[r][\mathsf{M},\mathsf{V}] + \mathrm{i}ak\Psi_{\epsilon\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{I},\mathsf{V}] \\
&\qquad\qquad + \Psi_{[\epsilon]\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{V},\mathsf{V}] + \Psi_{(\epsilon)\beta\gamma}[1-\kappa r\cos\phi][\mathsf{W},\mathsf{V}] \\
&\qquad\qquad + \Psi_{\epsilon\beta\gamma}[1-\kappa r\cos\phi][\mathsf{W},\mathsf{W}]\Big) \\
&\quad - \Psi_{(\alpha)\delta}[1-\kappa r\cos\phi](\mathsf{N}^{-1})^a_{\delta\epsilon}\Big(\Psi_{\epsilon\beta\gamma}[r][\mathsf{M},\mathsf{W}] + \mathrm{i}ak\Psi_{\epsilon\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{I},\mathsf{W}] \\
&\qquad\qquad + \Psi_{[\epsilon]\beta\gamma}[r(1-\kappa r\cos\phi)][\mathsf{V},\mathsf{W}] + \Psi_{(\epsilon)\beta\gamma}[1-\kappa r\cos\phi][\mathsf{W},\mathsf{W}] \\
&\qquad\qquad - \Psi_{\epsilon\beta\gamma}[1-\kappa r\cos\phi][\mathsf{W},\mathsf{V}]\Big)
\end{aligned}
$$

# Solution using impedance/admittance

Governing equations are

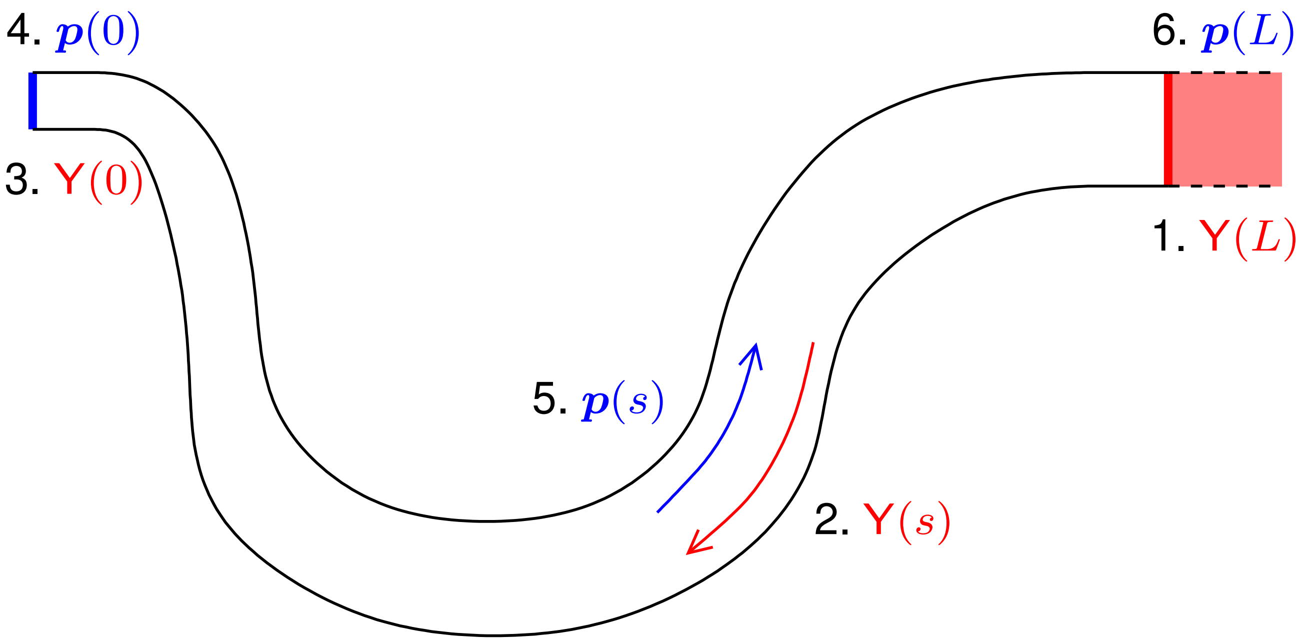
$$\boldsymbol{u}' + \mathsf{M}\boldsymbol{p} + \mathsf{G}\boldsymbol{u} = \mathcal{A}[\boldsymbol{u}, \boldsymbol{u}] + \mathcal{B}[\boldsymbol{p}, \boldsymbol{p}] + \mathcal{E}[\boldsymbol{u}, \boldsymbol{p}]$$

$$\boldsymbol{p}' - \mathsf{N}\boldsymbol{u} - \mathsf{H}\boldsymbol{p} = \mathcal{C}[\boldsymbol{u}, \boldsymbol{p}] + \mathcal{D}[\boldsymbol{u}, \boldsymbol{u}]$$

Use idea from Félix & Pagneux (2001 JASA; 2002 WM) for linear terms:

$$\boldsymbol{u} = \mathsf{Y}\boldsymbol{p} \qquad \Rightarrow \qquad \mathsf{Y}' + \mathsf{YNY} + \mathsf{M} + \mathsf{YH} + \mathsf{GY} = 0$$

$\mathsf{Y}(s)$ is the linear admittance. Once known, reconstruct $\boldsymbol{p}(s)$ using $\boldsymbol{p}' = \mathsf{NY}\boldsymbol{p} + \mathsf{H}\boldsymbol{p}$.



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Use idea from Félix & Pagneux (2001 JASA; 2002 WM) for linear terms:

$$\boldsymbol{u} = \mathsf{Y}\boldsymbol{p} + \mathcal{Y}[\boldsymbol{p},\boldsymbol{p}] \qquad \Rightarrow \qquad \mathsf{Y}' + \mathsf{YNY} + \mathsf{M} + \mathsf{YH} + \mathsf{GY} = 0$$

$\mathcal{Y}(s)$ is the nonlinear admittance (McTavish & Brambley, 2019 JFM),

$$\mathcal{Y}' + \mathcal{Y}[\mathsf{NY},\mathsf{I}] + \mathcal{Y}[\mathsf{I},\mathsf{NY}] + \mathsf{YN}\mathcal{Y} + \mathsf{Y}\mathcal{C}[\mathsf{Y},\mathsf{I}] - \mathcal{A}[\mathsf{Y},\mathsf{Y}] - \mathcal{B}$$
$$+ \mathcal{Y}[\mathsf{H},\mathsf{I}] + \mathcal{Y}[\mathsf{I},\mathsf{H}] + \mathsf{G}\mathcal{Y} + \mathsf{Y}\mathcal{D}[\mathsf{Y},\mathsf{Y}] - \mathcal{E}[\mathsf{Y},\mathsf{I}] = 0$$

Once $\mathsf{Y}(s)$ and $\mathcal{Y}(s)$ known, reconstruct $\boldsymbol{p}(s)$ using

$$\boldsymbol{p}' = \mathsf{NY}\boldsymbol{p} + \mathsf{H}\boldsymbol{p} + \mathsf{N}\mathcal{Y}[\boldsymbol{p},\boldsymbol{p}] + \mathcal{C}[\mathsf{Y}\boldsymbol{p},\boldsymbol{p}] + \mathcal{D}[\mathsf{Y}\boldsymbol{p},\mathsf{Y}\boldsymbol{p}]$$

*Note that both $\mathsf{Y}(s)$ and $\mathcal{Y}(s)$ are properties of the duct, and don't depend on the sound field $p(s)$ or its amplitude $M$.*

# Exit admittance into a straight uniform duct

Eliminate $\boldsymbol{u}$ to get $\boldsymbol{p}'' + \mathsf{NM}\boldsymbol{p} = 0$, with solution

$$\boldsymbol{p} = \boldsymbol{p}^{+} + \boldsymbol{p}^{-} = \sum_{j=1}^{\infty} \left( c_j^{+} \boldsymbol{v}_j e^{\mathrm{i}\lambda_j s} + c_j^{-} \boldsymbol{v}_j e^{-\mathrm{i}\lambda_j s} \right),$$

where $\boldsymbol{v}_j$ and $\lambda_j^2$ are eigenvectors and eigenvalues of $\mathsf{NM}$.

Want only outward propagating waves. Let $\mathsf{E} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$ and

$$\Lambda = \mathrm{diag}\big(\mathrm{i}\lambda_1, \mathrm{i}\lambda_2, \ldots\big) \qquad \text{with} \qquad \mathrm{Re}\lambda_j > 0 \qquad \text{or} \qquad \mathrm{Im}\lambda_j > 0.$$

Then $\boldsymbol{p}^{\pm\prime} = \mathrm{i}\sqrt{\mathsf{NM}}\boldsymbol{p}^{\pm} = \pm\mathsf{E}\Lambda\mathsf{E}^{-1}\boldsymbol{p}^{\pm} = \mathsf{N}\boldsymbol{u}^{\pm}$, with $\pm$ denoting the propagation direction.

Hence $\boldsymbol{u}^{\pm} = \mathsf{Y}^{\pm}\boldsymbol{p}^{\pm} \Rightarrow \mathsf{Y}^{\pm} = \pm\mathsf{N}^{-1}\mathsf{E}\Lambda\mathsf{E}^{-1} = \pm\mathrm{i}\mathsf{N}^{-1}\sqrt{\mathsf{NM}}$.

Check: for a constant duct, $\mathsf{Y}$ should be constant, so

$$\mathsf{Y}' = -\mathsf{YNY} - \mathsf{M} = \mathsf{N}^{-1}\sqrt{\mathsf{NM}}\mathsf{N}\mathsf{N}^{-1}\sqrt{\mathsf{NM}} - \mathsf{M} = 0$$

By a similar argument, the nonlinear characteristic admittance $\mathcal{Y}^{\pm}$ is given by

$$\mathcal{Y}^{\pm}[\boldsymbol{x}, \boldsymbol{y}] = \mathsf{N}^{-1}\mathsf{E}\tilde{\mathcal{Y}}^{\pm}[\mathsf{E}^{-1}\boldsymbol{x}, \mathsf{E}^{-1}\boldsymbol{y}].$$

where

$$\left(\tilde{\mathcal{Y}}^{\pm}\right)^{ab}_{pqr} = \frac{\left(\mathsf{E}^{-1}\mathsf{N}\mathcal{A}[\mathsf{Y}^{\pm}\mathsf{E}, \mathsf{Y}^{\pm}\mathsf{E}] + \mathsf{E}^{-1}\mathsf{N}\mathcal{B}[\mathsf{E}, \mathsf{E}] - \mathsf{E}^{-1}\mathsf{N}\mathsf{Y}^{\pm}\mathcal{C}[\mathsf{Y}^{\pm}\mathsf{E}, \mathsf{E}]\right)^{ab}_{pqr}}{\pm i\lambda_p^a \pm i\lambda_q^{a-b} \pm i\lambda_r^b}.$$

# Other nonlinear generalizations

- Impedance:

$$\boldsymbol{p} = \mathsf{Z}\boldsymbol{u} + \mathcal{Z}[\boldsymbol{u}, \boldsymbol{u}] \qquad \Rightarrow \qquad \mathsf{Z} = \mathsf{Y}^{-1}, \qquad \mathcal{Z} = -\mathsf{Z}\mathcal{Y}[\mathsf{Z}, \mathsf{Z}].$$

- Reflection coefficient between downstream $\boldsymbol{p}^{+}$ and upstream $\boldsymbol{p}^{-}$ sound:

$$\boldsymbol{p}^{-} = \mathsf{R}\boldsymbol{p}^{+} + \mathcal{R}[\boldsymbol{p}^{+}, \boldsymbol{p}^{+}]$$

$$\Rightarrow \qquad \mathsf{R} = (\mathsf{Y} - \mathsf{Y}^{-})^{-1}(\mathsf{Y}^{+} - \mathsf{Y})$$

and $$\mathcal{R} = (\mathsf{Y} - \mathsf{Y}^{-})^{-1}(\mathcal{Y}^{+} + \mathcal{Y}^{-}[\mathsf{R}, \mathsf{R}] - \mathcal{Y}[\mathsf{I} + \mathsf{R}, \mathsf{I} + \mathsf{R}]).$$

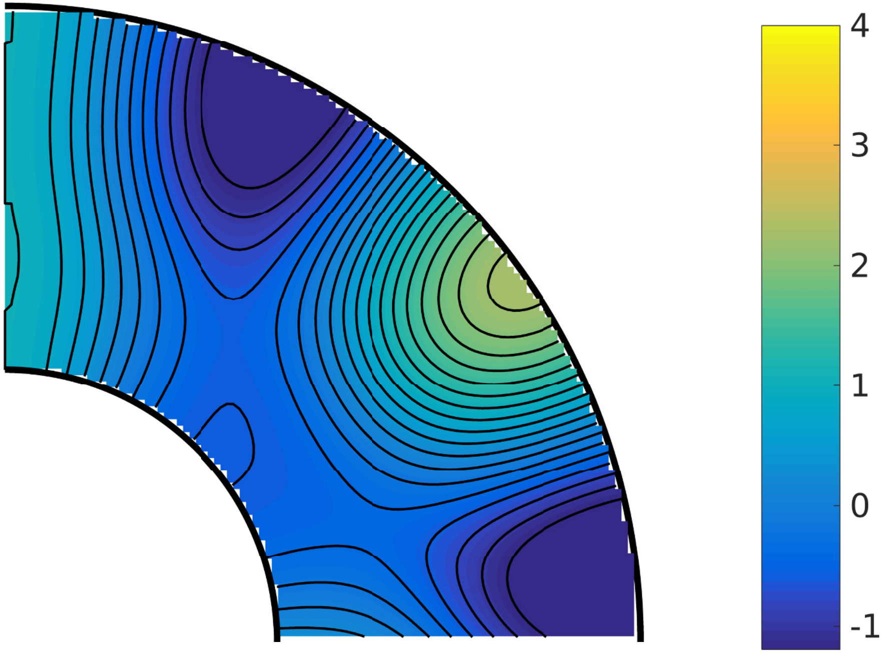
- Transmission coefficient across the duct:

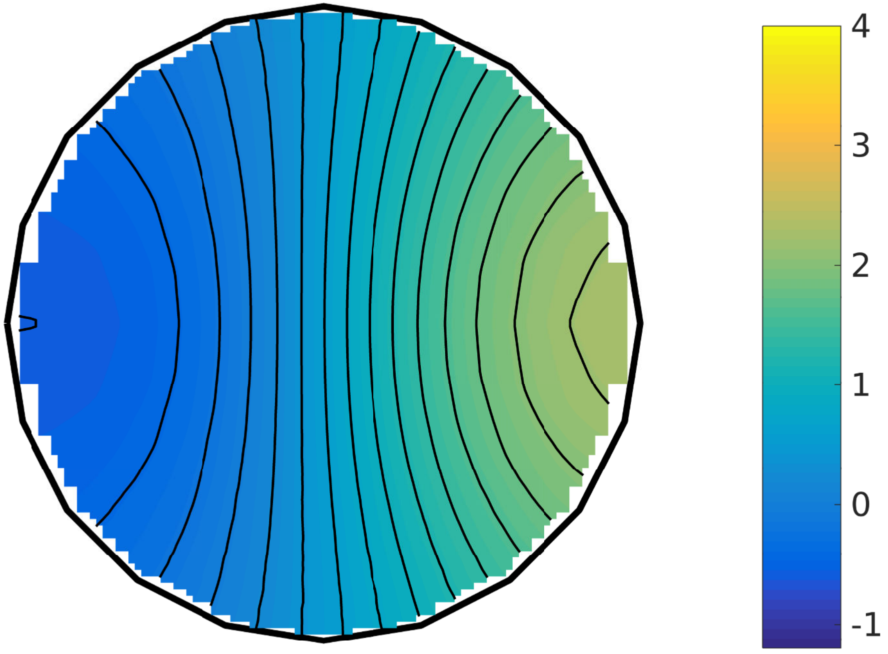
$$\boldsymbol{p}(L) = \mathsf{T}(L)\boldsymbol{p}(0) + \mathcal{T}(L)[\boldsymbol{p}(0), \boldsymbol{p}(0)]$$

$$\Rightarrow \qquad \mathsf{T}' = \mathsf{NYT} + \mathsf{HT}$$

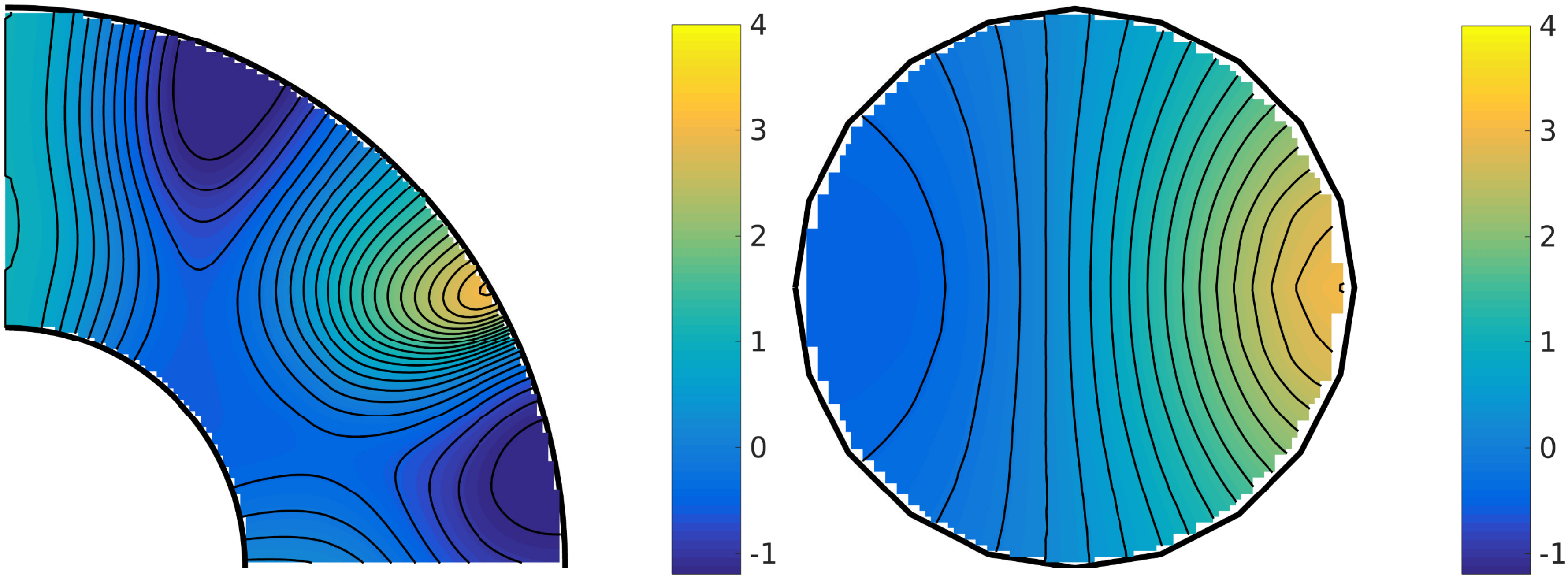
and $$\mathcal{T}' = (\mathsf{NY} + \mathsf{H})\mathcal{T} + \mathsf{N}\mathcal{Y}[\mathsf{T}, \mathsf{T}] + \mathcal{C}[\mathsf{YT}, \mathsf{T}] + \mathcal{D}[\mathsf{YT}, \mathsf{YT}].$$

# Curved cylinder (linear)

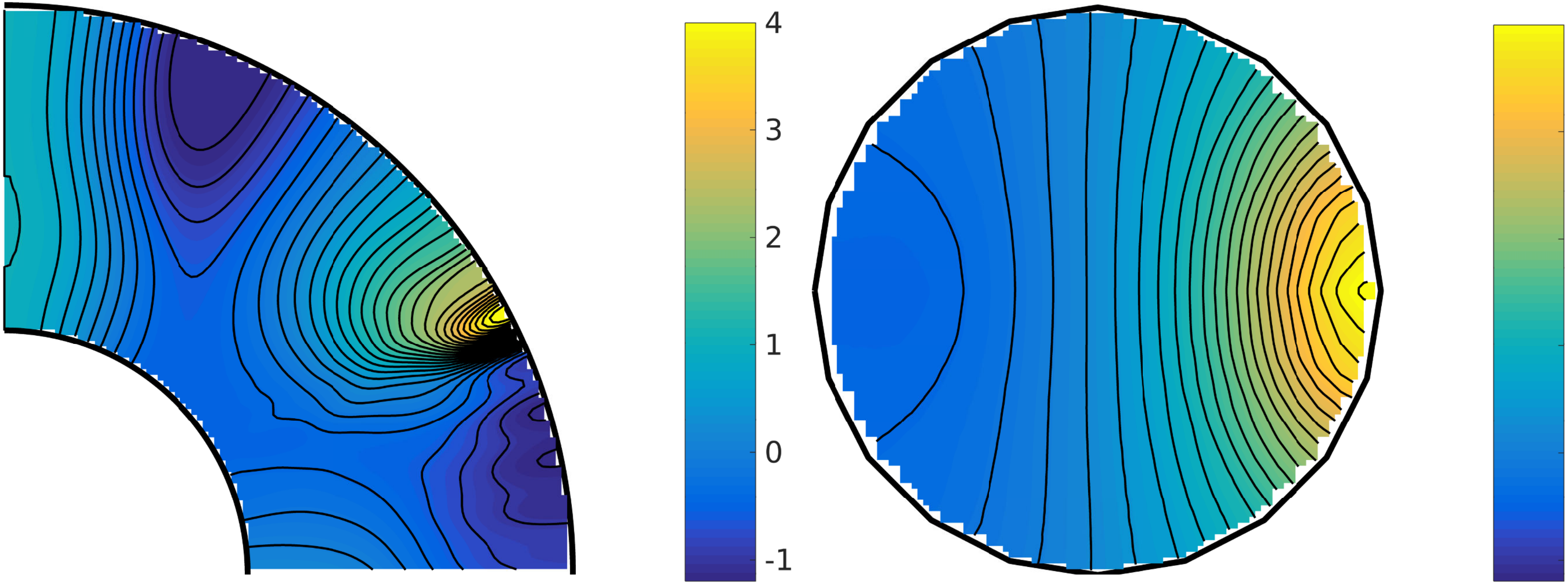


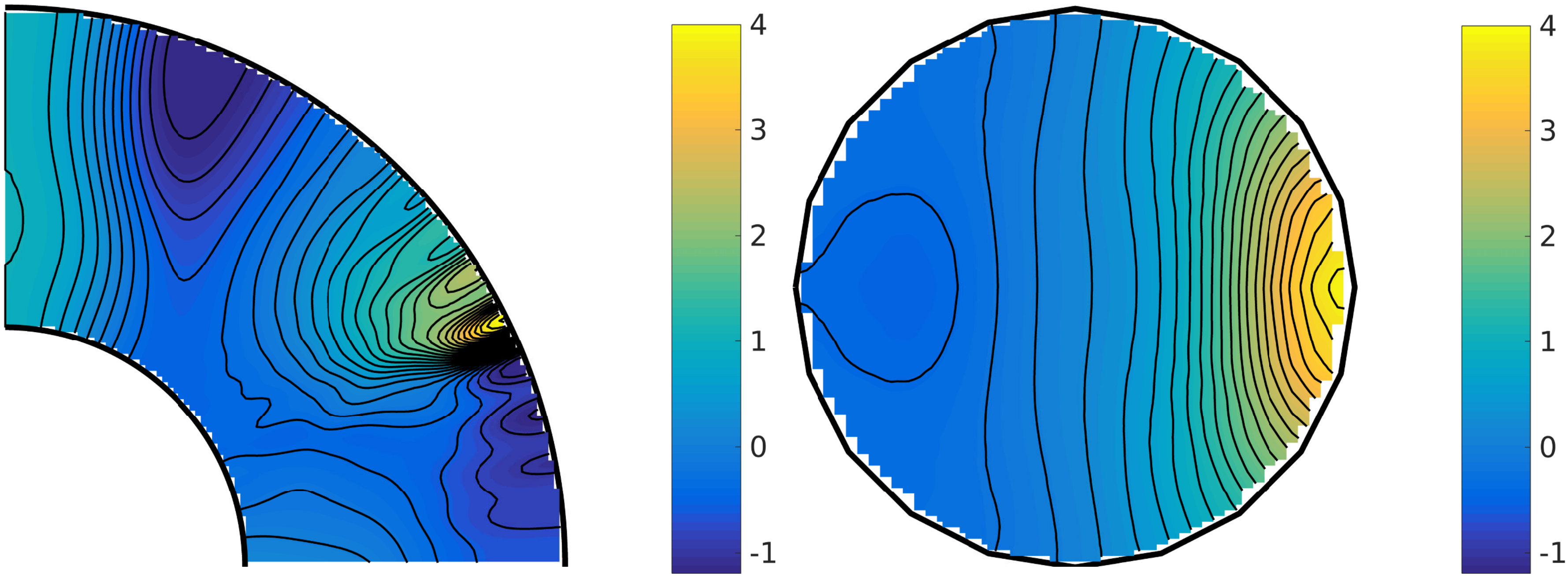
# Curved cylinder ($M = 0.05$)



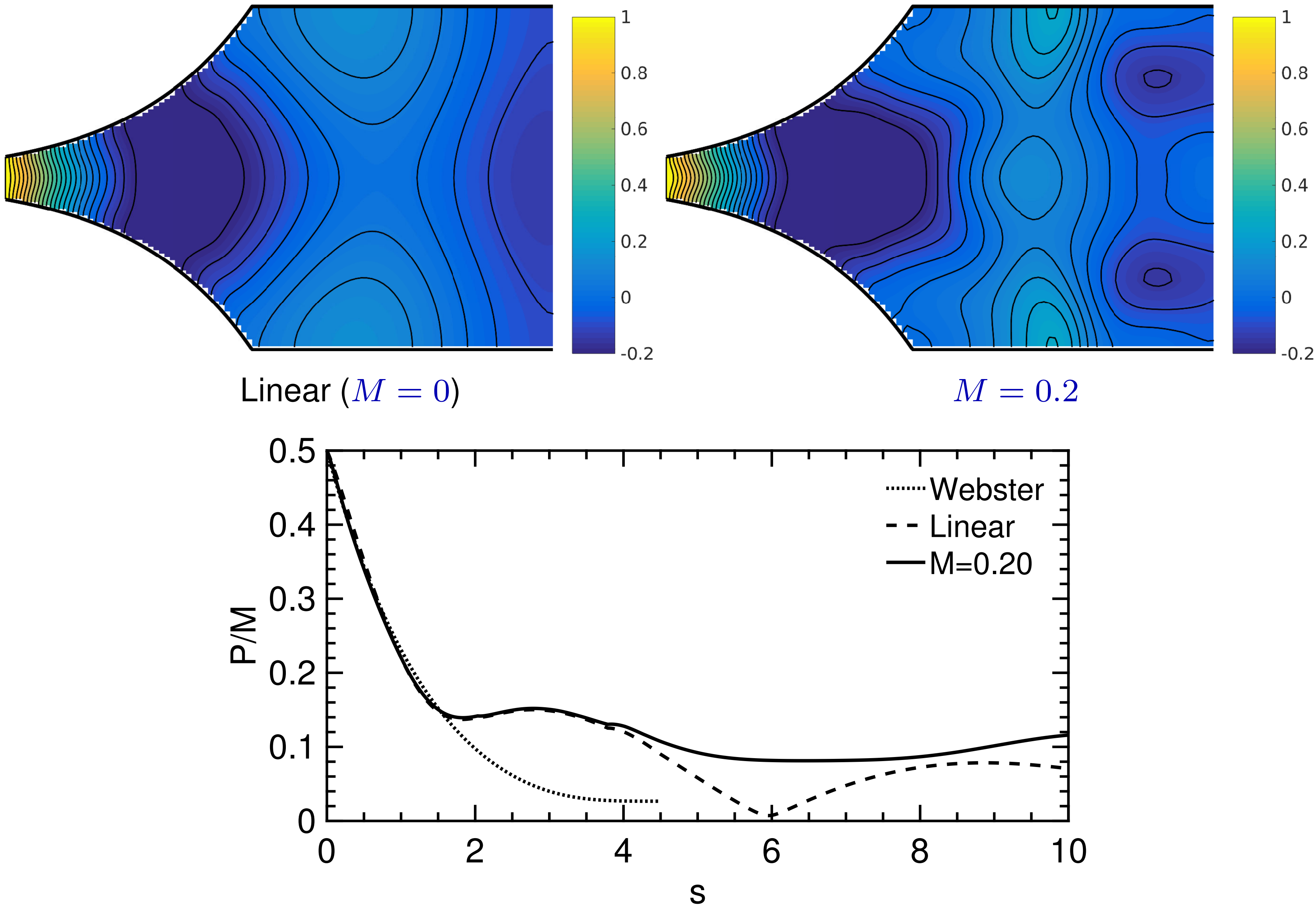
# Curved cylinder ($M = 0.10$)



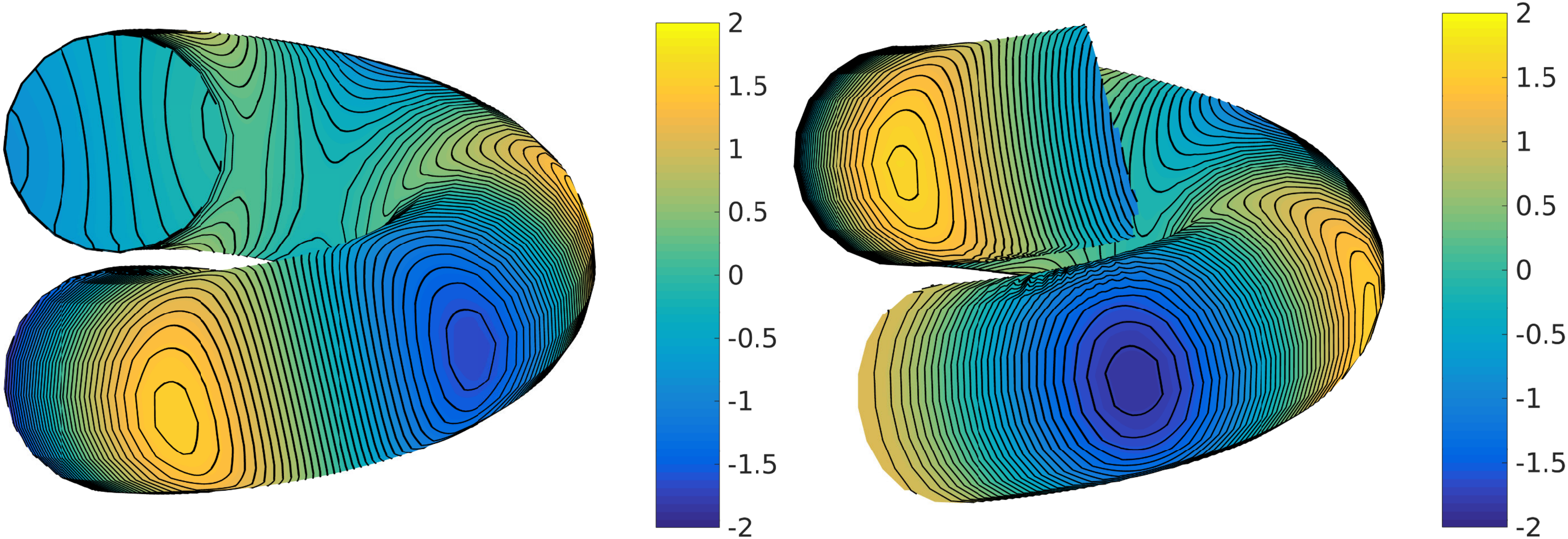
# Curved cylinder ($M = 0.15$)



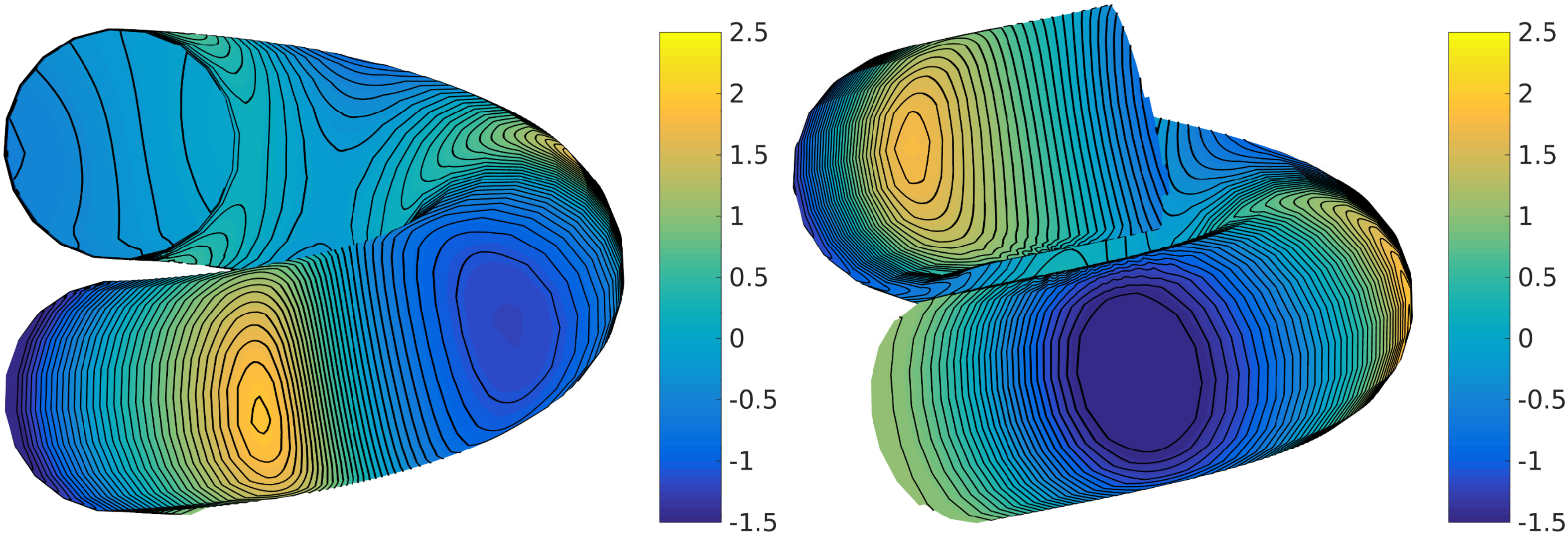
# Exponential Horn



# Helix Linear $\tau h = 0.14$



**Helix** $M = 0.05\ \tau h = 0.14$



**Helix** $M = 0.10 \ \tau h = 0.14$