Dynamic behavior of slender structures. Low-frequency asymptotic models of bridges: waveguides, by-pass systems and damage propagation

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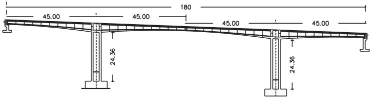
The S'Adde bridge in Macomer



The structures is a prestressed continuous hollow box-girder bridge launched by segmental construction technique (over head method). The cantilever launching gantry can move along the viaduct.

Two-dimensional beam as a multi-structure

Multi-structure — Ciarlet (1990), Kozolov et al. (1999)

Set involving subdomains of different limit dimensions connected through junction regions

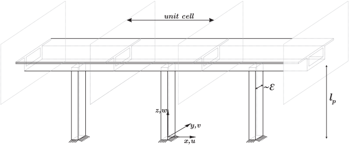
- First eigenfrequency: bridge deck as a rigid solid, supporting pillars as thin flexural elastic beams.

$$\omega_1 = \sqrt{\frac{12\,N_p\,E_p J_p}{M_T\,l_p^3}} \sim O(\varepsilon^2)$$

- Higher frequencies: bridge deck elastic flexural beam interacting with thinner supporting pillars: analysis of Bloch waves in an infinite periodic structure.

Two-dimensional beam as a multi-structure

For sufficiently low frequencies ω of vibrations, the upper deck is treated as a one-dimensional massive elastic beam resting on concentrated elastic supports disposed periodically with span length d.



Leading approximation of the elastic displacement field

$$\mathbf{u} \sim u(x,t)\mathbf{e}_x + v(x,t)\mathbf{e}_y + w(x,t)\mathbf{e}_z,$$

Decoupled vibration modes

- Vertical bending mode $[E J_y(x) w_{xx}]_{xx} + \rho A(x) w_{tt} = q(x,t)$

- Horizontal bending mode $[E J_z(x) v_{xx}]_{xx} + \rho A(x) v_{tt} = p(x,t)$

- Longitudinal mode $[E A(x) u_x]_x - \rho A(x) u_{tt} = r(x,t)$

Green's function

- Time-harmonic vibrations

$$u(x,t)\mathbf{e}_x + v(x,t)\mathbf{e}_y + w(x,t)\mathbf{e}_z = [U(x)\mathbf{e}_x + V(x)\mathbf{e}_y + W(x)\mathbf{e}_z]e^{i\omega t}$$

- Equation of motion for concentrated load at $x = x_0$

$$Dg^{Tot}_{xxxx}(x,x_0;\omega) - \rho\omega^2 g^{Tot}(x,x_0;\omega) = \delta(x - x_0)$$

where $D = E\bar{J}_j/\bar{A}$ $(j = x, y)$

- Fourier transform $g(x,x_0;\omega) \to \tilde{g}(k,x_0;\omega)$, $(x \to k)$

$$(Dk^4 - \rho\omega^2)\tilde{g}(k,x_0;\omega) = \frac{e^{ikx_0}}{\sqrt{2\pi}}$$

Green's function

- Solution in the Fourier space

$$\tilde{g}(x, x_0; \omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{ikx_0}}{2D\alpha^2} \left(\frac{1}{k^2 + \alpha^2} - \frac{1}{k^2 - \alpha^2} \right), \qquad \alpha = \left(\frac{\rho\omega^2}{D} \right)^{\frac{1}{4}}$$

- Inverse transform

$$g^{Tot}(x, x_0; \omega) = -\frac{1}{4D\alpha^3} \left(e^{-\alpha|x-x_0|} + ie^{-i\alpha|x-x_0|} \right)$$

Green's function

$$g(x, x_0; \omega) = \mathcal{R}[g^{Tot}(x, x_0; \omega)] = \frac{-1}{4D\alpha^3} \left[e^{-\alpha|x-x_0|} + \sin(\alpha|x - x_0|) \right]$$

-

Quasi-periodic Green's function

- Define

$$G^{Tot}(x,x_0;\omega,k) = g^{Tot}(x,x_0;\omega) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} g^{Tot}(x,x_0+nd;\omega)e^{iknd}$$

for $-\frac{d}{2} < x, x_0 < \frac{d}{2}$.

- It is quasi-periodic

$$G^{Tot}(x+md,x_0;\omega,k) = G^{Tot}(x,x_0;\omega,k)e^{ikmd}$$

- Recast in the form

$$G^{Tot}(x,x_0;\omega,k) = -\frac{1}{4D\alpha^3}\Big[e^{\alpha|x-x_0|}\alpha_1(\omega,k) + e^{-\alpha|x-x_0|}(\alpha_2(\omega,k)+1) + e^{i\alpha|x-x_0|}\beta_1(\omega,k) + e^{-i\alpha|x-x_0|}(\beta_2(\omega,k)+i)\Big].$$

where

$$\alpha_1(\omega,k) = \sum_{n=1}^{+\infty} e^{(-\alpha+ik)nd}, \quad \alpha_2(\omega,k) = \sum_{n=1}^{+\infty} e^{-(\alpha+ik)nd} = \bar{\alpha}_1(\omega,k),$$
$$\beta_1(\omega,k) = i\sum_{n=1}^{+\infty} e^{-i(\alpha-k)nd}, \qquad \beta_2(\omega,k) = i\sum_{n=1}^{+\infty} e^{-i(\alpha+k)nd}.$$

Quasi-periodic Green's function

Configuration where $x = x_0 = 0$

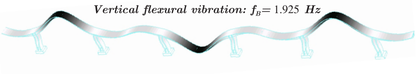
$$G_0(\omega, k) = G(0,0;\omega,k) = \mathcal{R}[G^{Tot}(0,0;\omega,k)] =$$
$$= -\frac{1}{4D\alpha^3}\left[1 + 2\sum_{n=1}^{+\infty} e^{-\alpha n d}\cos(knd) + 2\sum_{n=1}^{+\infty}\sin(\alpha n d)\cos(knd)\right]$$
$$= -\frac{1}{4D\alpha^3}\left[1 + \frac{\cos(kd) - e^{-\alpha d}}{\cosh(\alpha d) - \cos(kd)} - \frac{\sin(\alpha d)}{\cos(\alpha d) - \cos(kd)}\right]$$

Dispersion diagrams

- Vertical bending mode, vertical displacement $W(x)$ in the deck and longitudinal displacement in the pillar

$$G_0(\omega, k) = -\frac{1}{\gamma_z}$$

Equivalent vertical stiffness $\gamma_z = 20.79\,10^3$ MPa m



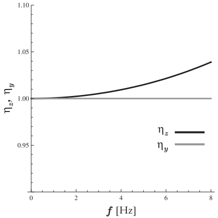
- Transverse bending mode, horizontal displacement $V(x)$ of the deck and transverse displacement of the pillar

$$G_0(\omega, k) = -\frac{1}{\gamma_y}$$

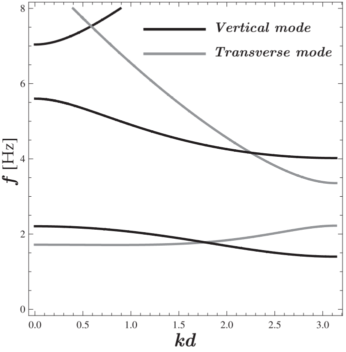
Equivalent transverse stiffness $\gamma_y = 1.50\,10^3$ MPa m.

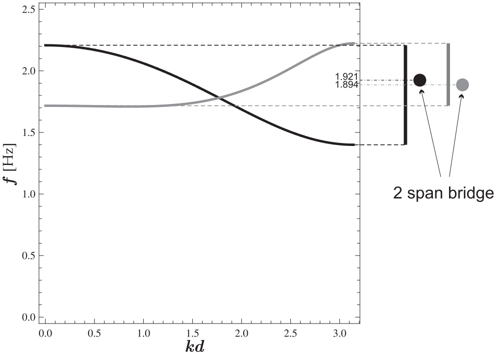
Pillar: inertial or quasi-static?

- Vertical mode $W_p(z) = F_1 \cos\left(\frac{\omega}{c_0} z\right) + F_2 \sin\left(\frac{\omega}{c_0} z\right)$
  Dispersion relation: $G_0(\omega, k) = -\frac{\eta_z(\omega)}{\gamma_z}$
  with *inertial factor* $\eta_z(\omega) = \frac{\tan\left(\frac{\omega}{c_0} l\right)}{\frac{\omega}{c_0} l_p} \to 1$ as $\omega \to 0$.
- Transverse mode
  $V_p(z) = G_1 e^{\alpha_p z} + G_2 e^{-\alpha_p z} + G_3 e^{i\alpha_p z} + G_4 e^{-i\alpha_p z}$
  Dispersion relation: $G_0(\omega, k) = -\frac{\eta_y(\omega)}{\gamma_y}$,
  with *inertial factor*
  $\eta_y(\omega) = \frac{12}{(\alpha_p l_p)^3} \frac{1 - \cos(\alpha_p l_p)\cosh(\alpha_p l_p)}{\cosh(\alpha_p l_p)\sin(\alpha_p l_p) + \sinh(\alpha_p l_p)\cos(\alpha_p l_p)} \to 1$ as $\omega \to 0$.

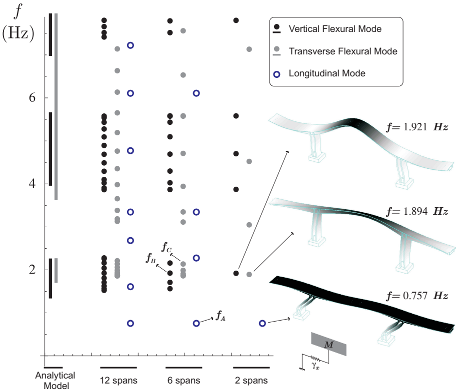


Pass bands for Bloch waves versus eigenfrequencies of finite systems

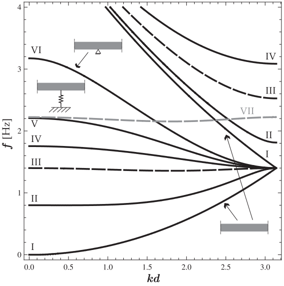


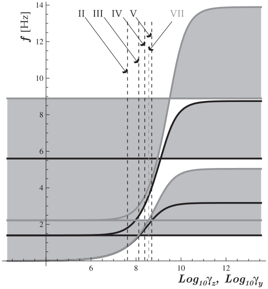


Pass bands for Bloch waves versus eigenfrequencies of finite systems



Structural optimisation



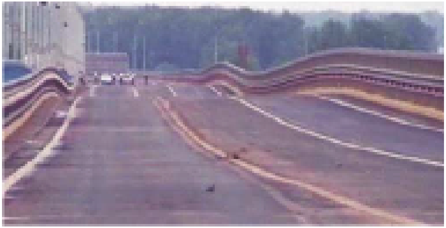


Lower bound	*Upper bound*	*Optimal stiffness*
$Dk^4 - \rho\omega^2 = 0$	$G_0(\omega, k) = 0$	$\gamma_{Opt} = \dfrac{1}{G_0\left(\sqrt{\frac{DD}{\rho}}\frac{\pi^2}{d^2}, 0\right)}$

Volgograd bridge

- Concrete girder bridge 7.110 m long
- October 2009 inauguration, May 2010 strong oscillations
- Large vibrations induced by relatively small external forces

Volgograd bridge

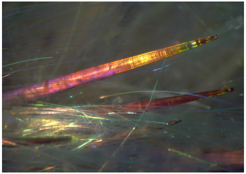


How to fix?

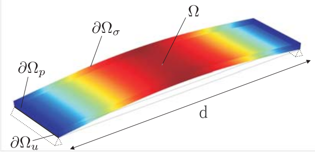
1. Strong alteration: increase total stiffness and/or inertia and made structure capable to support external actions
2. Lightweight alteration: by-pass system for elastic waves, channel waves around some parts of the structure

Lightweight resonant structures

Re-route the waves

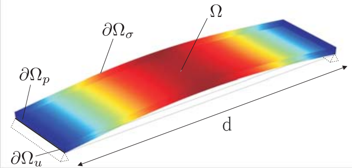


Change the eigenmode!!



Modelling

Finite structure composed by N identical spans $\rightarrow$ infinite periodic structure subjected to Bloch-Floquet waves

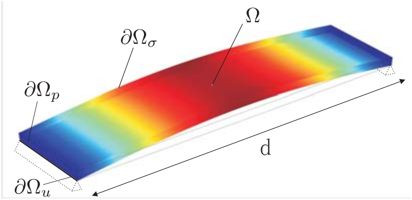


Advantages:

1. Analysis of only a single unit
2. Dispersion properties characterize all possible propagating and non propagation waves
3. Methodology for the design of resonant structures and band gap opening

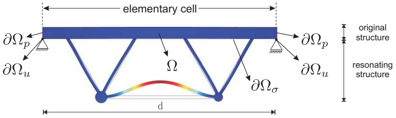
Setting of the problem

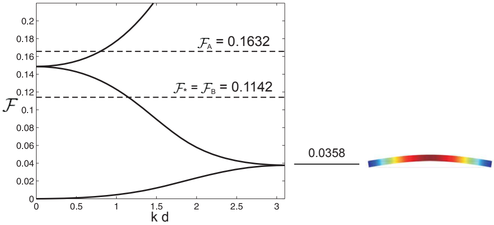
Simplified FEM model of a unit cell



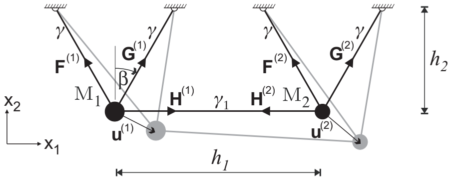
- Lamé equations: $\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\omega^2\mathbf{u} = 0 \quad \text{in } \Omega$
- Boundary conditions: $\mathbf{t}^{(n)}(\mathbf{u}) = 0$ on $\partial\Omega_\sigma$ and $\mathbf{u} = 0$ on $\partial\Omega_u$
- Quasy-periodicity conditions $\mathbf{u}(\mathbf{x} + d\mathbf{e}^{(1)}) = \mathbf{u}(\mathbf{x})e^{ikd}$

2D simplified model: analysis of dispersion properties





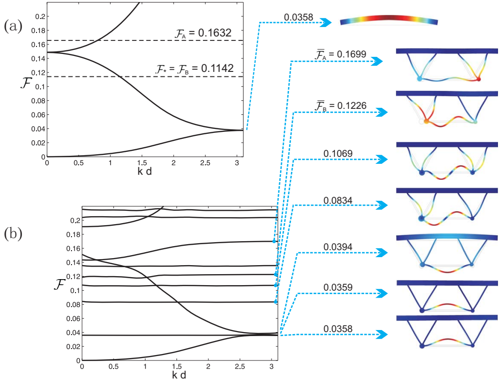
2D simplified model: pre-design of the resonant lightweight structure



$$f_{A,B}^2 = \frac{1}{8\pi^2}\left(\frac{1}{M_1}+\frac{1}{M_2}\right)\left[\gamma_1 + 2\gamma\sin^2\beta \pm \sqrt{\gamma_1^2 + 4\left(\frac{M_1-M_2}{M_1+M_2}\right)^2 \gamma\sin^2\beta(\gamma_1+\gamma\sin^2\beta)}\right]$$

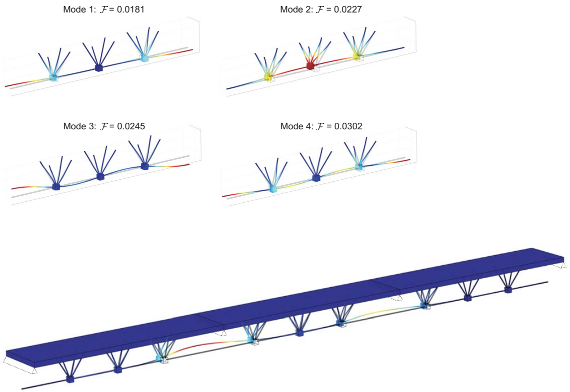
$$f_C^2 = \frac{\gamma}{2\,M_1\pi^2}\cos^2\beta, \quad f_D^2 = \frac{\gamma}{2\,M_2\pi^2}\cos^2\beta,$$

2D simplified model: variation of dispersion properties



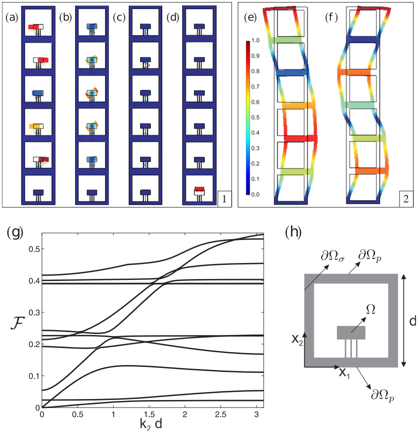
3D simplified model

Change of the eigenmode by the introduction of the lightweight resonators



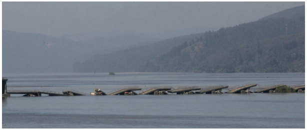
Suppression of lateral vibrations of a skyscraper

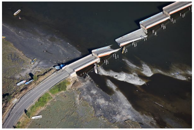
(d) 0.4446 ⟶
(c) 0.3936 ⟶
(b) 0.2254 ⟶
(a) 0.0225 ⟶



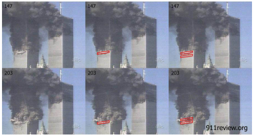
Failure wave in elastic waveguides

- Localized damage in uniform or periodic waveguide may cause a failure wave
- Propagating wave generated by earthquake (Chile 2010)





- Failure wave in World Trade Center



Bažant and Zhou (2002), Bažant and Verdure (2007), Bažant et al (2008)

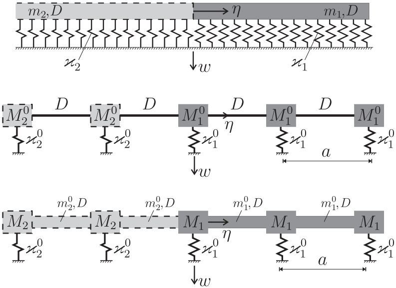
Failure wave in elastic waveguides

- Failure wave accompanied by energy release that overcomes an energy barrier

- Analogy with phase transition
- Plane crushing waves: Galin & Cherepanov (1966), Grigoryan (1967), Slepyan (1968,1977), Slepyan & Troyankina (1969), Slepyan (2002)
- Higher-order derivative formulation: Truskinovsky (1994,1997), Ngan & Truskinovsky (1999)
- Discrete chain model: Slepyan & Troyankina (1984,1988), Puglisi & Truskinovsky (2000), Slepyan (2000,2001), Balk et al (2001a,b), Cherkaev et al (2005), Slepyan et al (2005), Slepyan & Ayzenberg-Stepanenko (2004)

Transition wave in a supported heavy beam

Three models

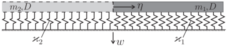


$$\eta = x - v\,t$$

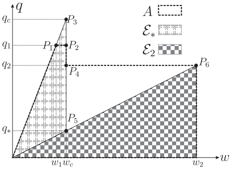
$$\varkappa^0 = \varkappa_1^0, \quad M = M_1 \quad \text{(before damage, } \eta > 0)$$
$$\varkappa^0 = \varkappa_2^0 < \varkappa_1^0, \quad M = M_2 \quad \text{(after damage, } \eta < 0)$$

$$w|_{\eta=0} \gtrless w_c?$$

Uniform continuous model: a beam on an elastic foundation



Energy considerations

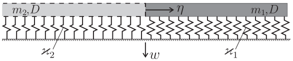


- Energy excess per unit length:

$$\mathcal{E}_0 = A - \mathcal{E}_2 - \mathcal{E}_* > 0 \quad \text{if} \quad w_c < w_c^* = w_1 \sqrt{\varkappa_1 / \varkappa_2}$$

- Energy Balance:
$\mathcal{E}_0 - U_1(c_1/v - 1) - U_2(1 - c_2/v) = 0$

Uniform continuous model: a beam on an elastic foundation



- Equation of motion

$$D\frac{\partial^4 w(x,t)}{\partial x^4} + m_{1,2}\,\frac{\partial^2 w(x,t)}{\partial t^2} + \varkappa_{1,2}\, w(x,t) = m_{1,2}\; g$$

- Steady state regime $\eta = x - vt$ (v is failure wave speed)

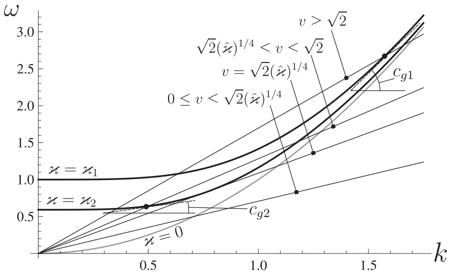
$$D w(\eta)^{IV} + m_{1,2}\, v^2 w''(\eta) + \varkappa_{1,2}\, w(\eta) = m_{1,2}\; g$$

- Normalization: introduce $\xi = (D/\varkappa_1)^{1/4}$, $\tau = \sqrt{m_1/\varkappa_1}$ (and $\hat{\varkappa} = \varkappa_2/\varkappa_1$, $\hat{m} = m_2/m_1$)

$$\begin{aligned} \tilde{w}^{IV}(\eta) + \tilde{v}^2\,\tilde{w}''(\eta) + \tilde{w}(\eta) &= \tilde{g} \quad (\eta > 0)\,, \\ \tilde{w}^{IV}(\eta) + \hat{m}\,\tilde{v}^2\,\tilde{w}''(\eta) + \hat{\varkappa}\,\tilde{w}(\eta) &= \hat{m}\tilde{g} \quad (\eta < 0)\,, \end{aligned}$$

- Separate the initial static displacement $\tilde{w} = \bar{w} + \tilde{g}$

$$\begin{aligned} \bar{w}^{IV}(\eta) + \tilde{v}^2\,\bar{w}''(\eta) + \bar{w}(\eta) &= 0 \quad (\eta > 0) \\ \bar{w}^{IV}(\eta) + \hat{m}\,\tilde{v}^2\,\bar{w}''(\eta) + \hat{\varkappa}\bar{w}(\eta) &= \tilde{g}\,(\hat{m} - \hat{\varkappa}) \quad (\eta < 0) \end{aligned}$$



Three regimes ($0<\hat{\varkappa} = \varkappa_2/\varkappa_1<1$)

- *subsonic* range: $0 \leq v < v_2 = \sqrt{2}(\hat{\varkappa})^{1/4}$
- *intersonic* range: $v_2 < v < v_1 = \sqrt{2}$
- *supersonic* range: $v > v_1$

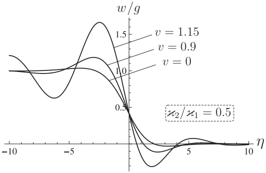
Subsonic regime

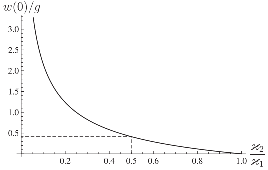
$$\text{Subsonic regime: } 0 \leq v < v_2 = \sqrt{2}(\hat{\varkappa})^{1/4}$$

$$w(\eta) = e^{-\alpha_1 \eta}(A_1 \cos \beta_1 \eta + B_1 \sin \beta_1 \eta) \quad (\eta > 0)$$

$$w(\eta) = e^{\alpha_2 \eta}(A_2 \cos \beta_2 \eta + B_2 \sin \beta_2 \eta) + \mathcal{Q}/\hat{\varkappa} \quad (\eta < 0)$$

$$w(0) = \left(\sqrt{\frac{\varkappa_1}{\varkappa_2}} - 1\right) g > 0$$





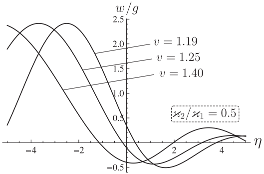
Intersonic regime

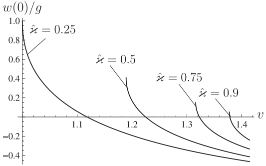
Intersonic regime: $v_2 = \sqrt{2}(\hat{\varkappa})^{1/4} < v < v_1 = \sqrt{2}$

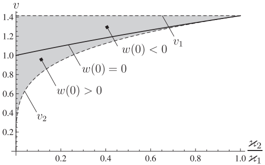
$$w(\eta) = \mathrm{e}^{-\alpha_1 \eta}(A_1 \cos \beta_1 \eta + B_1 \sin \beta_1 \eta) \quad (\eta > 0)$$

$$w(\eta) = A_2 \cos \beta_2 \eta + B_2 \sin \beta_2 \eta + \mathcal{Q}/\hat{\varkappa} \quad (\eta < 0)$$

$$w(0) = \frac{v^2 - 2\hat{\varkappa} - \sqrt{v^4 - 4\hat{\varkappa}}}{2\hat{\varkappa}} g$$







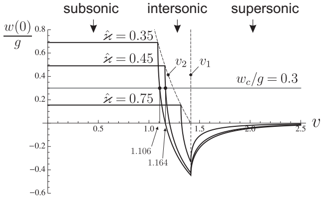
Suersonic regime

Supersonic regime: $v > v_1 = \sqrt{2}$

$$w(\eta) = A_1 \cos \beta_1 \eta + B_1 \sin \beta_1 \eta \quad (\eta > 0)$$

$$w(\eta) = A_2 \cos \beta_2 \eta + B_2 \sin \beta_2 \eta + \mathcal{Q}/\hat{\varkappa} \quad (\eta < 0)$$

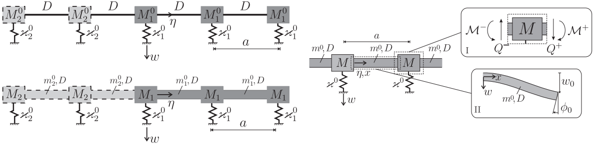
$$w(0) = -\frac{v^2 - \sqrt{v^4 - 4\hat{\varkappa}}}{\sqrt{v^4 - 4} + \sqrt{v^4 - 4\hat{\varkappa}}} \, \frac{1 - \hat{\varkappa}}{\hat{\varkappa}} g < 0$$



Three regimes ($\hat{\varkappa} = \varkappa_2/\varkappa_1$)

- *subsonic* range: $w(0) = \left(\sqrt{\frac{\varkappa_1}{\varkappa_2}} - 1\right) g > 0$
- *intersonic* range: $w(0) = \frac{v^2 - 2\hat{\varkappa} - \sqrt{v^4 - 4\hat{\varkappa}}}{2\hat{\varkappa}} g$
- *supersonic* range: $w(0) = -\frac{v^2 - \sqrt{v^4 - 4\hat{\varkappa}}}{\sqrt{v^4 - 4 + \sqrt{v^4 - 4\hat{\varkappa}}}} \frac{1-\hat{\varkappa}}{\hat{\varkappa}} g < 0$

Discrete continuous model



- $x \neq an$:

$$D\frac{\partial^4 w(x,\eta)}{\partial x^4} + m^0_{1,2}\frac{\partial^2 w(x,\eta)}{\partial t^2} = m^0_{1,2}\, g$$

- $x = an,\ \eta > 0$:

$$M_1\frac{\partial^2 w(\eta)}{\partial t^2} + \varkappa^0_1 w(\eta) - Q^+(\eta) + Q^-(\eta) = M_1\, g$$

$$\mathcal{M}^+(\eta) - \mathcal{M}^-(\eta) = 0$$

- $x = an,\ \eta < 0$:

$$M_2\frac{\partial^2 w(\eta)}{\partial t^2} + \varkappa^0_2 w(\eta) - Q^+(\eta) + Q^-(\eta) = M_2\, g$$

$$\mathcal{M}^+(\eta) - \mathcal{M}^-(\eta) = 0$$

Non dimensional form

Introduce $\xi = (D/\varkappa_1)^{1/4}, \quad \tau = \sqrt{m_1/\varkappa_1} \quad (\varkappa_1 = \varkappa_1^0/a)$

- $x \neq an$:

$$\frac{\partial^4 \tilde{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{x}^4} + \frac{m_{1,2}^0}{m_1}\frac{\partial^2 \tilde{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{t}^2} = \frac{m_{1,2}^0}{m_1}\tilde{g}$$

- $x = an$, $\eta > 0$:

$$\tilde{M}_1 \frac{\partial^2 \tilde{w}(\tilde{\eta})}{\partial \tilde{t}^2} + \tilde{w}(\eta) - \frac{1}{\tilde{a}}\left[\tilde{Q}^+(\tilde{\eta}) - \tilde{Q}^-(\tilde{\eta})\right] = \tilde{M}_1 \tilde{g}$$

$$\tilde{\mathcal{M}}^+(\tilde{\eta}) - \tilde{\mathcal{M}}^-(\tilde{\eta}) = 0$$

- $x = an$, $\eta < 0$:

$$\tilde{M}_2 \frac{\partial^2 \tilde{w}(\tilde{\eta})}{\partial \tilde{t}^2} + \frac{\varkappa_2}{\varkappa_1}\tilde{w}(\eta) - \frac{1}{\tilde{a}}\left[\tilde{Q}^+(\tilde{\eta}) - \tilde{Q}^-(\tilde{\eta})\right] = \tilde{M}_2 \tilde{g}$$

$$\tilde{\mathcal{M}}^+(\tilde{\eta}) - \tilde{\mathcal{M}}^-(\tilde{\eta}) = 0$$

Separation of the static contribution

$$\tilde{w}(\tilde{x},\tilde{\eta}) = \tilde{g} + \frac{m_1^0(\tilde{x}-\tilde{a}n)^2(\tilde{x}-\tilde{a}(n+1))^2}{m_1}\tilde{g} + \bar{w}(\tilde{x},\tilde{\eta})$$

- $\eta > 0$:

$$x \neq an: \quad \frac{\partial^4 \bar{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{x}^4} + \frac{m_1^0}{m_1}\frac{\partial^2 \bar{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{t}^2} = 0$$

$$x = an: \quad \tilde{M}_1 \frac{\partial^2 \bar{w}(\tilde{\eta})}{\partial \tilde{t}^2} + \bar{w}(\tilde{\eta}) - \frac{1}{\tilde{a}}\left[\bar{Q}^+(\tilde{\eta}) - \bar{Q}^-(\tilde{\eta})\right] = 0$$

$$\bar{\mathcal{M}}^+(\tilde{\eta}) - \bar{\mathcal{M}}^-(\tilde{\eta}) = 0$$

- $\eta < 0$:

$$x \neq an: \quad \frac{\partial^4 \bar{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{x}^4} + \frac{m_2^0}{m_1}\frac{\partial^2 \bar{w}(\tilde{x},\tilde{\eta})}{\partial \tilde{t}^2} = \frac{m_2^0 - m_1^0}{m_1}\tilde{g}$$

$$x = an: \quad \tilde{M}_2 \frac{\partial^2 \bar{w}(\eta)}{\partial \tilde{t}^2} + \frac{\varkappa_2}{\varkappa_1}\bar{w}(\tilde{\eta}) - \frac{1}{\tilde{a}}\left[\bar{Q}^+(\tilde{\eta}) - \bar{Q}^-(\tilde{\eta})\right] = \left(1 - \frac{\varkappa_2}{\varkappa_1} + \tilde{M}_2 - \tilde{M}_1\right)\tilde{g}$$

$$\bar{\mathcal{M}}^+(\tilde{\eta}) - \bar{\mathcal{M}}^-(\tilde{\eta}) = 0$$

Interaction of neighboring cross section

Steady-state regime:

$$w^F(x,k) = W(x)w_0 + \Phi(x)\phi_0$$

$$W(x) = \frac{(\cosh\lambda a - \cos\lambda a)(\cosh\lambda x - \cos\lambda x) - (\sinh\lambda a + \sin\lambda a)(\sinh\lambda x - \sin\lambda x)}{2(1 - \cosh\lambda a \cos\lambda a)}$$

$$\Phi(x) = \frac{(\cosh\lambda a - \cos\lambda a)(\sinh\lambda x - \sin\lambda x) - (\sinh\lambda a - \sin\lambda a)(\cosh\lambda x - \cos\lambda x)}{2\lambda(1 - \cosh\lambda a \cos\lambda a)}$$

$$\lambda = (m_1^0/m_1)^{1/4}\sqrt{kv - \mathrm{i}0}$$

Interaction of neighboring cross section

$$Q^F(0,\eta) = Q_{w0}w_0 + Q_{\phi 0}\phi_0\,, \quad Q^F(a,\eta) = Q_{wa}w_0 + Q_{\phi a}\phi_0$$
$$\mathcal{M}^F(0,\eta) = \mathcal{M}_{w0}w_0 + \mathcal{M}_{\phi 0}\phi_0\,, \quad \mathcal{M}^F(a,\eta) = \mathcal{M}_{wa}w_0 + \mathcal{M}_{\phi a}\phi_0$$

$$Q_{w0} = \frac{\lambda^3(\sinh\lambda a + \sin\lambda a)}{1-\cosh\lambda a\cos\lambda a}\,, \quad Q_{\phi 0} = -\frac{\lambda^2(\cosh\lambda a - \cos\lambda a)}{1-\cosh\lambda a\cos\lambda a}$$
$$Q_{wa} = \frac{\lambda^3(\cosh\lambda a\sin\lambda a + \sinh\lambda a\cos\lambda a)}{1-\cosh\lambda a\cos\lambda a}\,, \quad Q_{\phi a} = -\frac{\lambda^2\sinh\lambda a\sin\lambda a}{1-\cosh\lambda a\cos\lambda a}$$
$$\mathcal{M}_{w0} = \frac{\lambda^2(\cosh\lambda a - \cos\lambda a)}{1-\cosh\lambda a\cos\lambda a}\,, \quad \mathcal{M}_{\phi 0} = -\frac{\lambda(\sinh\lambda a - \sin\lambda a)}{1-\cosh\lambda a\cos\lambda a}$$
$$\mathcal{M}_{wa} = -\frac{\lambda^2(\sinh\lambda a\sin\lambda a)}{1-\cosh\lambda a\cos\lambda a}\,, \quad \mathcal{M}_{\phi a} = \frac{\lambda(\cosh\lambda a\sin\lambda a - \sinh\lambda a\cos\lambda a)}{1-\cosh\lambda a\cos\lambda a}$$

Static limit, $v \to 0$:

$$Q_{w0} \to Q_{wa} \to \frac{12}{a^3}\,, \quad Q_{\phi 0} \to Q_{\phi a} \to -\frac{6}{a^2}$$
$$\mathcal{M}_{w0} \to -\mathcal{M}_{wa} \to \frac{6}{a^2}\,, \quad \mathcal{M}_{\phi a} \to -2\mathcal{M}_{\phi 0} \to \frac{4}{a}$$

The Wiener-Hopf equation

- One-sided Fourier transform $\eta = an \to k$

$$\{w_+(k), \phi_+(k)\} = \int_0^{\infty} \{w(\eta), \phi(\eta)\} \mathrm{e}^{ik\eta} \, \mathrm{d}\eta$$

$$\{w_-(k), \phi_-(k)\} = \int_{-\infty}^{0} \{w(\eta), \phi(\eta)\} \mathrm{e}^{ik\eta} \, \mathrm{d}\eta$$

- Governing Equation for M_1 at $\eta = 0$

$$(1 - M_1 v^2 k^2) w_+(k) + (\varkappa_2/\varkappa_1 - M_2 v^2 k^2) w_-(k) +$$

$$+ \frac{2}{a}(Q_{wa} - Q_{w0} \cos ka) w^F(k) + \frac{2\mathrm{i}}{a} Q_{\phi 0} \sin ka \, \phi^F(k) = \frac{C}{0 + \mathrm{i}k}$$

$$\mathrm{i}\mathcal{M}_{w0} \sin ka \, w^F(k) + (\mathcal{M}_{\phi a} - \mathcal{M}_{\phi 0} \cos ka) \phi^F(k) = 0$$

Eliminate $\phi^F(k)$

$$L_1(k)w_+(k) + L_2(k)w_-(k) = \frac{C}{0+\mathrm{i}k}$$

$$L_1(k) = 1 + M_1(0+\mathrm{i}kv)^2 + \frac{2}{a}\left[(Q_{wa} - Q_{w0}\cos ka) + \frac{Q_{\phi 0}\mathcal{M}_{w0}\sin^2 ka}{\mathcal{M}_{\phi a} - \mathcal{M}_{\phi 0}\cos ka}\right]$$

$$L_2(k) = \frac{\varkappa_2}{\varkappa_1} + M_2(0+\mathrm{i}kv)^2 + \frac{2}{a}\left[(Q_{wa} - Q_{w0}\cos ka) + \frac{Q_{\phi 0}\mathcal{M}_{w0}\sin^2 ka}{\mathcal{M}_{\phi a} - \mathcal{M}_{\phi 0}\cos ka}\right]$$

Wiener-Hopf equation

$$L_0(k)w_+(k) + w_-(k) = \frac{C}{(0+\mathrm{i}k)[1-\varkappa_2/\varkappa_1+(M_1-M_2)(0+\mathrm{i}kv)^2]}[L_0(k)-1]$$

$$L_0(k) = L_1(k)/L_2(k)$$

Factorization

- $\mathrm{Ind}\, L(k) = \frac{1}{2\pi}[\mathrm{Arg}L(\infty) - \mathrm{Arg}L(-\infty)] = 0$
- Cauchy type integral: $L_0(k) = \lim_{\Im k \to 0} L_+(k)L_-(k)$

$$L_\pm(k) = \exp\left[\pm\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln L(\xi)}{\xi - k}\,\mathrm{d}\xi\right] \quad (\pm\Im k > 0)$$

- Wiener-Hopf equation

$$L_+(k)w_+(k) + \frac{w_-(k)}{L_-(k)} = \left\{\frac{g}{\mathrm{i}k}[L_+(k) - L_+(0)]\right\} + \left\{\frac{g}{0+\mathrm{i}k}\left[L_+(0) - \frac{1}{L_-(k)}\right]\right\}$$

regular in the upper/lower half plane of k

- One-sided transform

$$w_+(k) = \frac{g}{\mathrm{i}k}\frac{L_+(k) - L_+(0)}{L_+(k)} \qquad w_-(k) = \frac{g}{0+\mathrm{i}k}[L_+(0)L_-(k) - 1]$$

Solution

- Critical displacement: from limiting relations

$$w(\pm 0) = \lim_{k \to \pm \mathrm{i}\infty} (\pm \mathrm{i}k) w_\pm(k)\,, \quad L_\pm(\pm \mathrm{i}\infty) = 1$$

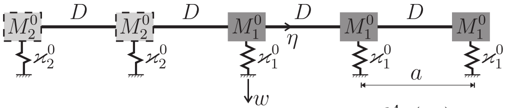
we find

$$w(+0) = w(-0) = w(0) = g[L_+(0) - 1]$$

with

$$L_\pm(0) = \sqrt{\frac{\varkappa_1}{\varkappa_2}} \exp\left[\pm \frac{1}{\pi} \int_0^\infty \frac{\mathrm{Arg} L(k)}{k}\, \mathrm{d}k\right] \quad \left(L(0) = \frac{\varkappa_1}{\varkappa_2}\right)$$

The massless beam model



Equation of motion for the massless beam: $\frac{\partial^4 w(x,\eta)}{\partial x^4} = 0$

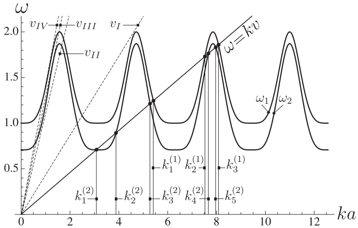
$$Q_{w0} \to Q_{wa} \to \frac{12}{a^3}, \quad Q_{\phi 0} \to Q_{\phi a} \to -\frac{6}{a^2}$$

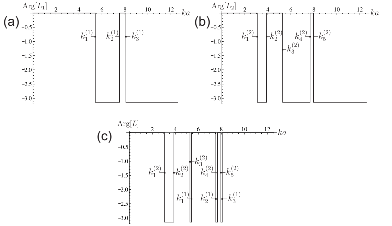
$$\mathcal{M}_{w0} \to -\mathcal{M}_{wa} \to \frac{6}{a^2}, \quad \mathcal{M}_{\phi a} \to -2\mathcal{M}_{\phi 0} \to \frac{4}{a}$$

$$L_1(k) = 1 + (0 + ikv)^2 + \frac{12}{a^4}\frac{(1-\cos ka)^2}{2+\cos ka}$$

$$L_2(k) = \varkappa_2/\varkappa_1 + (0 + ikv)^2 + \frac{12}{a^4}\frac{(1-\cos ka)^2}{2+\cos ka}$$

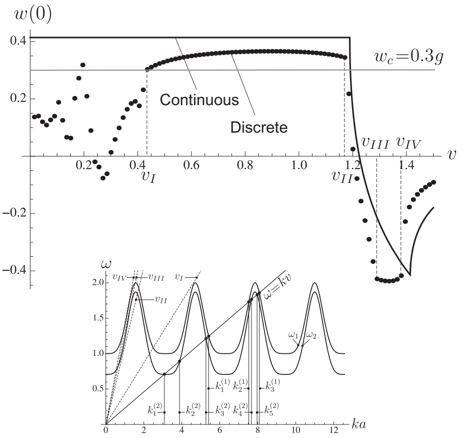
Massless beam model



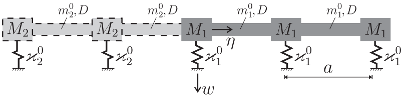


$$w(0) = g \left[\sqrt{\frac{\varkappa_1}{\varkappa_2}} \left(\prod_{i=1}^{n_1} \frac{k_{2i}^{(1)}}{k_{2i-1}^{(1)}} \prod_{j=1}^{n_2} \frac{k_{2j-1}^{(2)}}{k_{2j}^{(2)}} \right) \frac{k_{2n_2+1}^{(2)}}{k_{2n_1+1}^{(1)}} - 1 \right]$$

The critical displacement



Solution for the inertial beam model: regularization



- Introduce a small dissipation α in proportion to the strain rate in the bending moment

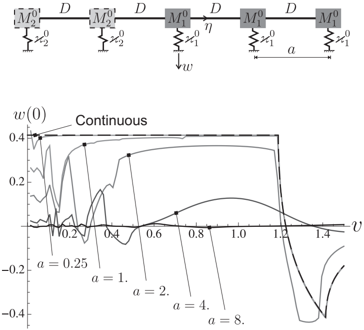
$$\mathcal{M} = \frac{\partial^2 w(x,\eta)}{\partial x^2} + \alpha \frac{\partial^3 w(x,\eta)}{\partial t \partial x^2}$$

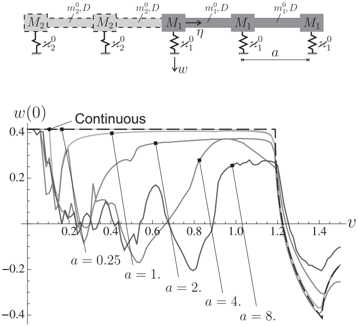
- $$w(+0) = w(-0) = w(0) = g[L_+(0) - 1]$$

  with

$$L_\pm(0) = \sqrt{\frac{\varkappa_1}{\varkappa_2}} \exp\left[\pm\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Arg}L(k)}{k}\,\mathrm{d}k\right] \quad \left(L(0) = \frac{\varkappa_1}{\varkappa_2}\right)$$

Inertial and Massless beam model



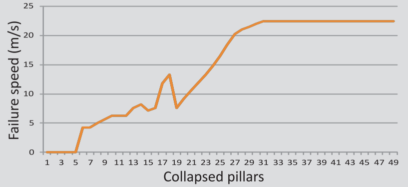


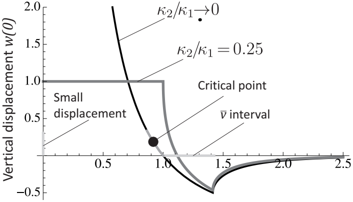
Normalised span length.

S'Adde bridge: $a = 2.02$ for vertical flexural waves and $a = 0.83$ for horizontal flexural waves

Millau viaduct: $a = 7.61$ for vertical flexural waves and $a = 0.99$ for the horizontal flexural waves

San Saba bridge in Las Plassas, Texas





$$v_{sup} = \left(\frac{D\kappa}{\rho^2}\right)^{1/4} = 24.3\text{m/s}$$

$$v = 22.4\text{m/s}$$

Conclusion

Thank you for your attention!