# Modelling of Meta-surfaces: homogenization and approximate boundary conditions

*Wave propagation in complex and microstructured media, Insitut d'Études Scientifiques de Cargèse, 2019*

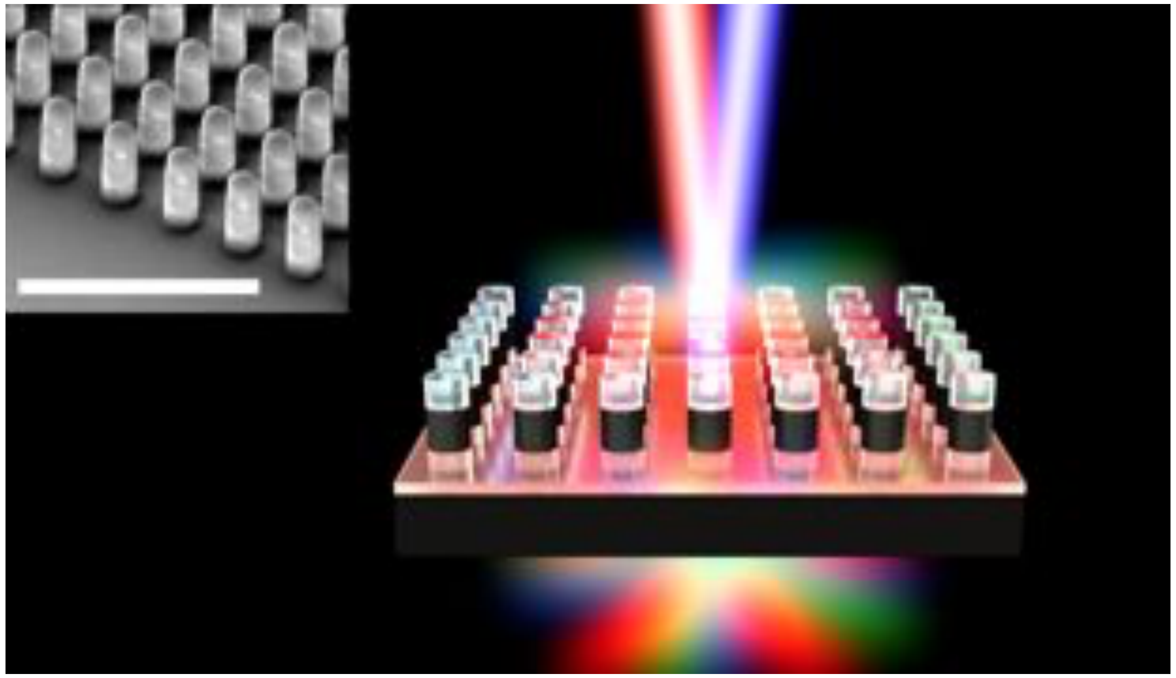
*Bérangère Delourme*

UNIVERSITÉ PARIS 13

# Context : generalities about metasurfaces

## What is a meta-surface ?

A meta-surface *(also called metafilm)* is a plane material, made of thin and densely packed planar arrays of sub wavelength elements.



Sheng Liu, Polina P. Vabishchevich, Aleksandr Vaskin, John L. Reno, Gordon A. Keeler, Michael B. Sinclair, Isabelle Staude & Igal Brener, Nature Communications, 2018

This is the 2D version of periodic meta-material

*Holloway-Kuester-Gordon-O'Hara-Booth-Smith 12, Glybovski-Tretyakov-Belov-Kivshar-Simovski 16…*

# Context : generalities about metasurfaces

## Why are the metasurfaces interesting ?

Meta-surfaces take less physical space than 3D metamaterial structures.

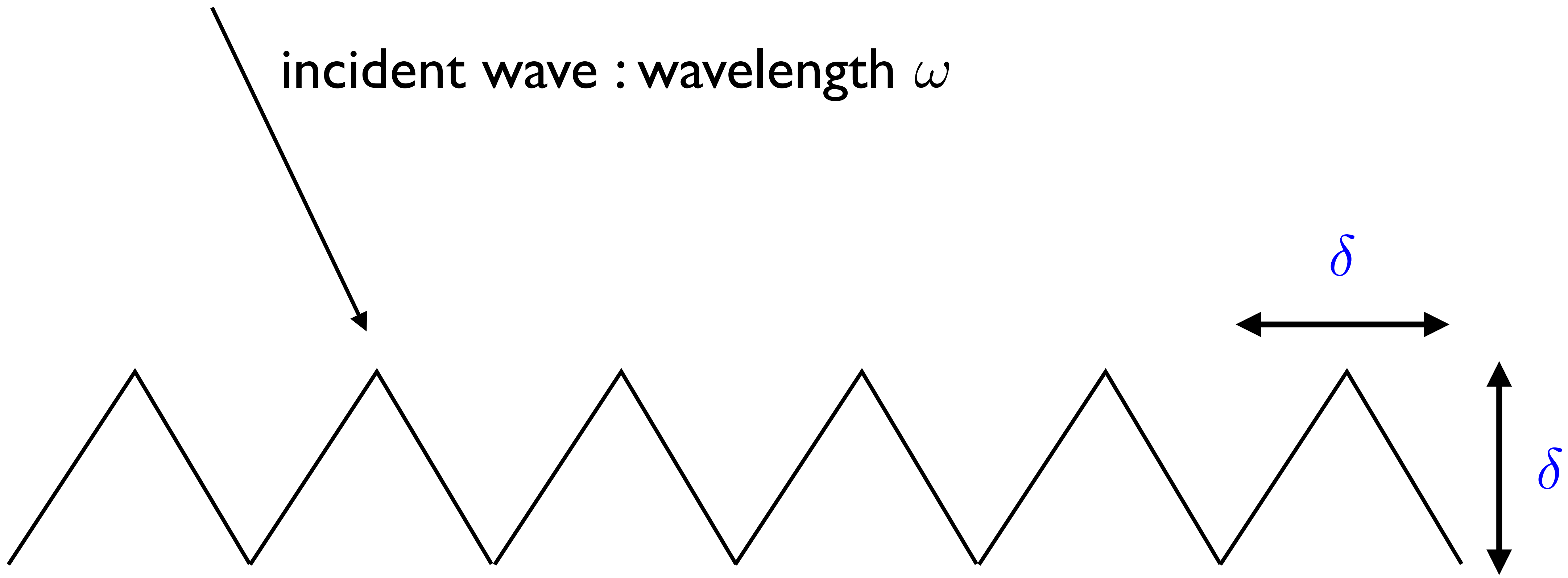
Possible applications:

- angular-independent surfaces,
- absorbers,
- controllable smart surfaces,
- wave guiding structures.

*Kuester-Holloway and co-authors 03-05-09-10-11-14….*

# Context : generalities about metasurfaces

## Why is it important to model meta-surfaces ?




$$\omega \ll \delta$$

What is the macroscopic effect of the micro-structure ?

- Numerical issues
- 3D periodic homogenization does not apply directly.

# Outline of the talk

1- Investigation of a 2D-model problem

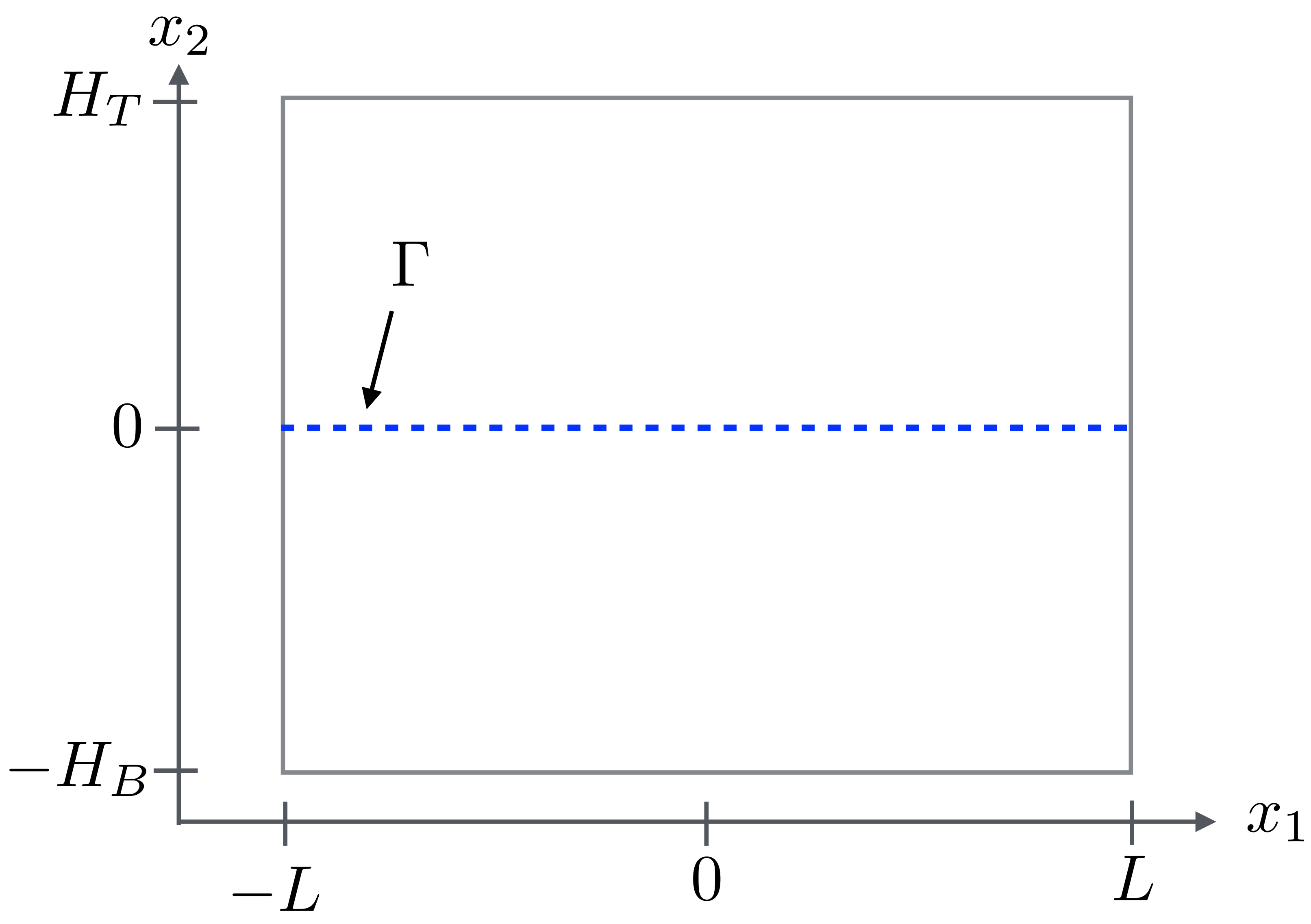
2- Extensions and numerical illustrations

3- 3D time-harmonic Maxwell's equations

4- Homogenization in presence of corners

# I- Investigation of a 2D-model problem

## The domain of interest :

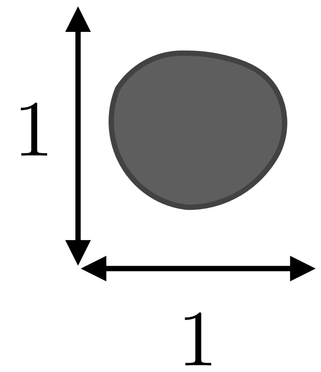



$$\Omega_p = (-L, L) \times (-H_B, H_T)$$

# I- Investigation of a 2D-model problem

## The domain of interest :

✓ **canonical obstacle** $\hat{\Omega}_{\mathrm{hole}} \subset (-1/2, 1/2)^2$

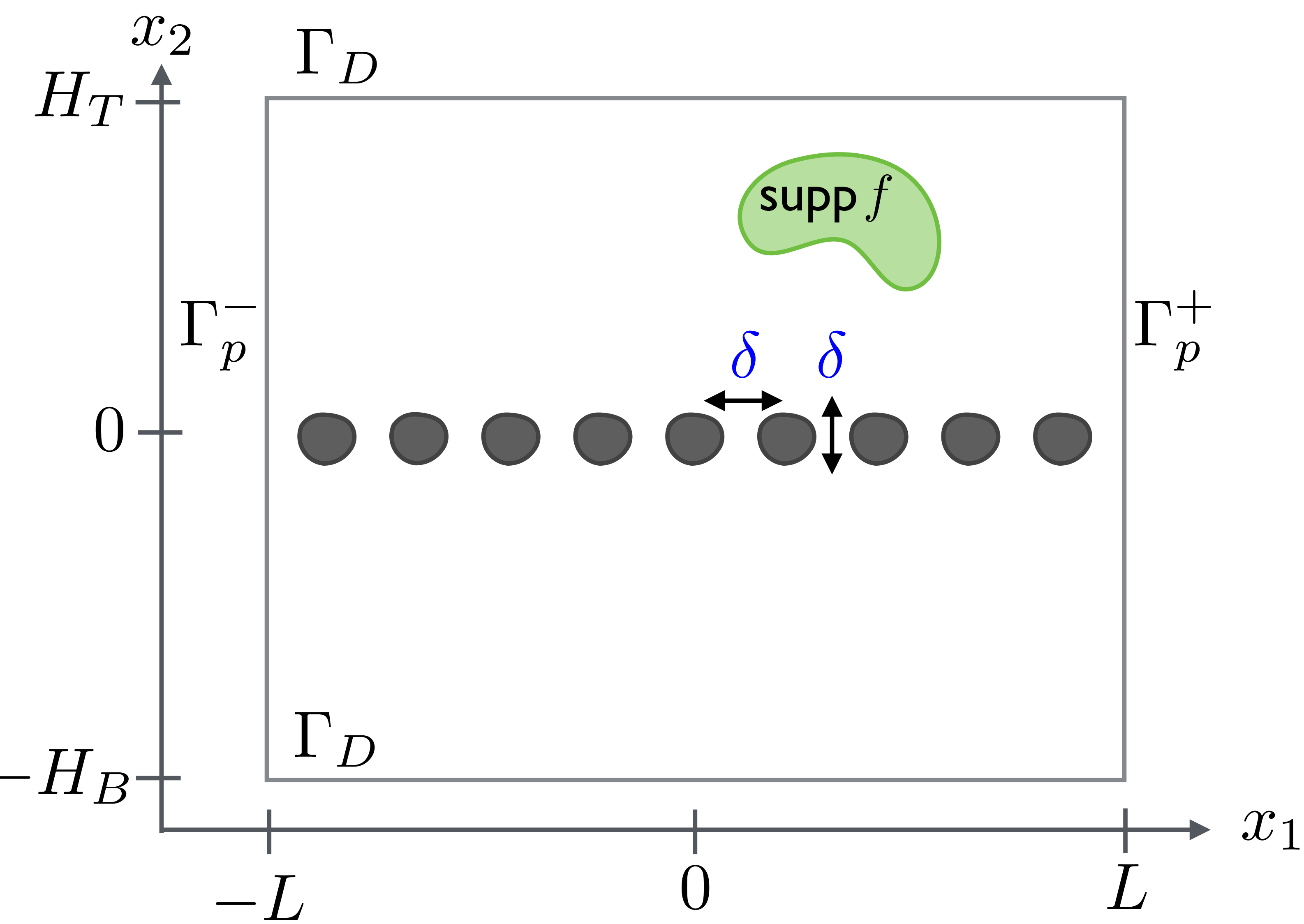



✓ **Translation + scaling:** $\delta > 0 \ (2L/\delta \in \mathbb{N})$

$$\Omega^{\delta}_{\mathrm{hole}} = \bigcup_{\ell=0}^{2L/\delta-1} \left\{ -Le_1 + \delta \left\{ \hat{\Omega}_{\mathrm{hole}} + (\ell + 1/2) e_1 \right\} \right\}$$

# I- Investigation of a 2D-model problem

## The domain of interest :




$$
(\mathcal{P}) \begin{cases} -\Delta u^\delta = f \text{ in } \Omega_p^\delta \\ u^\delta = 0 \text{ on } \Gamma_D \\ \partial_n u^\delta = 0 \text{ on } \partial\Omega_{\text{hole}}^\delta \\ u^\delta \ 2L\text{-periodic} \end{cases}
$$

$$\Omega_p = (-L, L) \times (-H_B, H_T)$$

$$\Omega_p^\delta = \Omega_p \setminus \overline{\Omega_{\text{hole}}^\delta}$$

$$\partial\Omega_p = \Gamma_p^- \cup \Gamma_p^+ \cup \Gamma_D$$

## Well-posedness and stability property

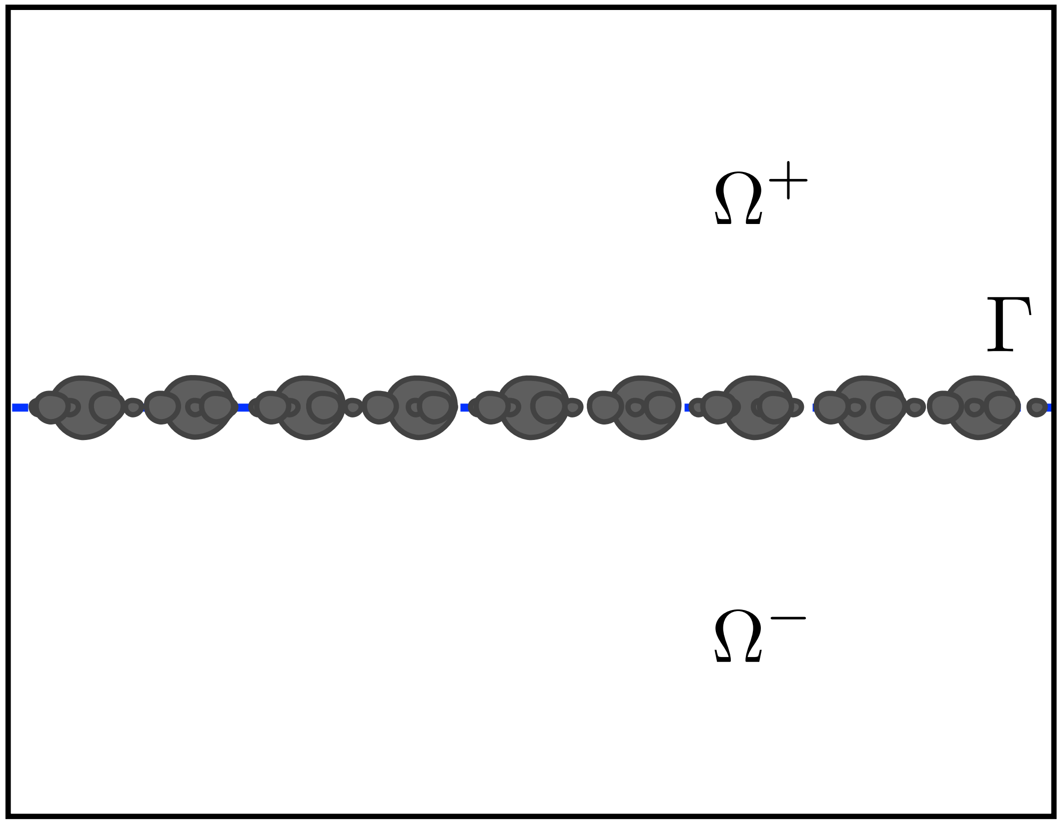
**Proposition:** let $f \in L^2(\Omega^\delta)$. Problem $(\mathcal{P})$ has a unique solution $u^\delta \in H^1(\Omega^\delta)$ that satisfies the following stability estimate: $\exists C > 0,$

$$\|u^\delta\|_{H^1(\Omega^\delta)} \;\leq\; C\;\|f\|_{L^2(\Omega^\delta)}$$

# I- Investigation of a 2D-model problem

**Objective:**

- behavior of $u^\delta$ with respect to $\delta$ as $\delta$ tends to $0$
- replacement of the periodic layer with an approximate transmission condition posed on the limit interface $\Gamma$



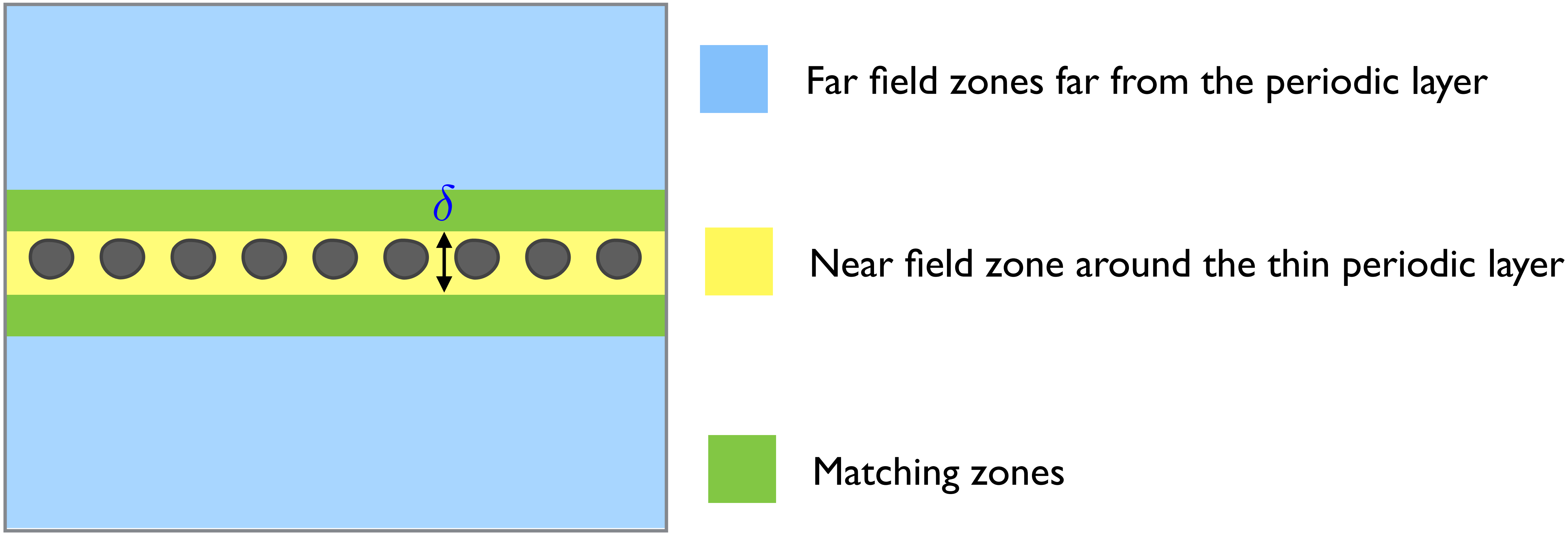

**Method:**

construction of an asymptotic expansion of $u^\delta$ w.r.t $\delta$

derivation of an approximate problem

# I- Investigation of a 2D-model problem

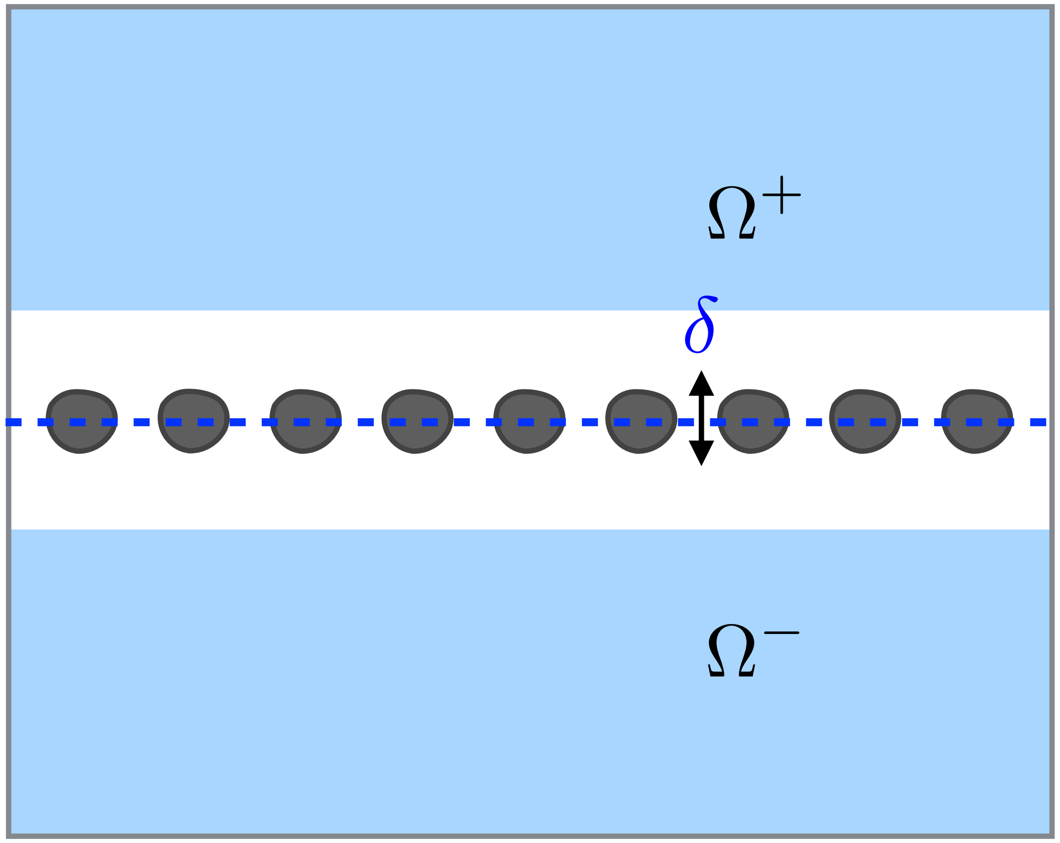
## Method of matched asymptotic expansions




*Van Dyke 64, Il'in 92,Maz'ya-Nazarov-Plamenevskij 00, Tordeux-Joly 06, Claeys 08, Hewett-Hewitt 16, Marigo-Maurel 16-18, Maurel-Marigo-Pham 18-19, Mercier-Maurel-Marigo 17…*

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: far field equations




Far field zones far from the periodic layer

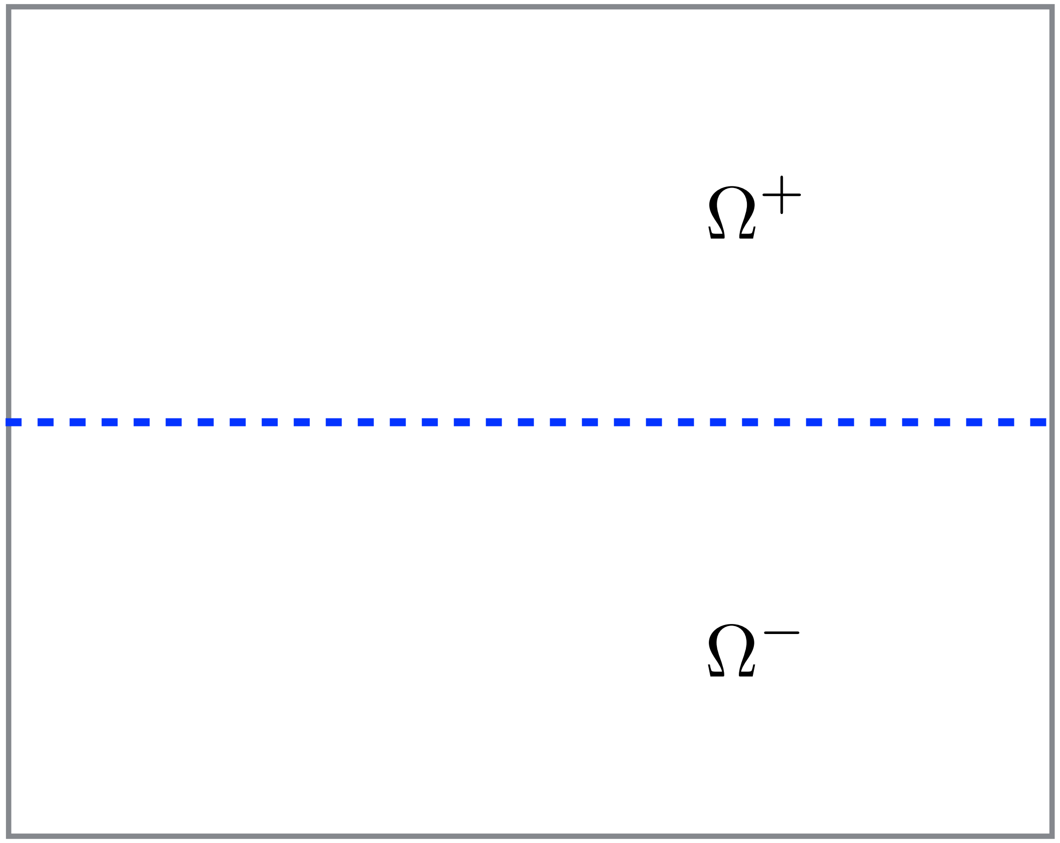
$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, u_q^\pm(\mathbf{x})$$

Macroscopic (far field) terms

- ✓ The macroscopic terms $u_q$ are defined in $\Omega^+ \cup \Omega^-$
- ✓ They are not necessarily continuous across $\Gamma$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: far field equations




$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q u_q^\pm(\mathbf{x})$$

Macroscopic (far field) terms

✓ **Far field equations**

$$-\Delta u_q^\pm = \begin{cases} f & q = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{in } \Omega^+ \cup \Omega^- \text{ + B.C}$$

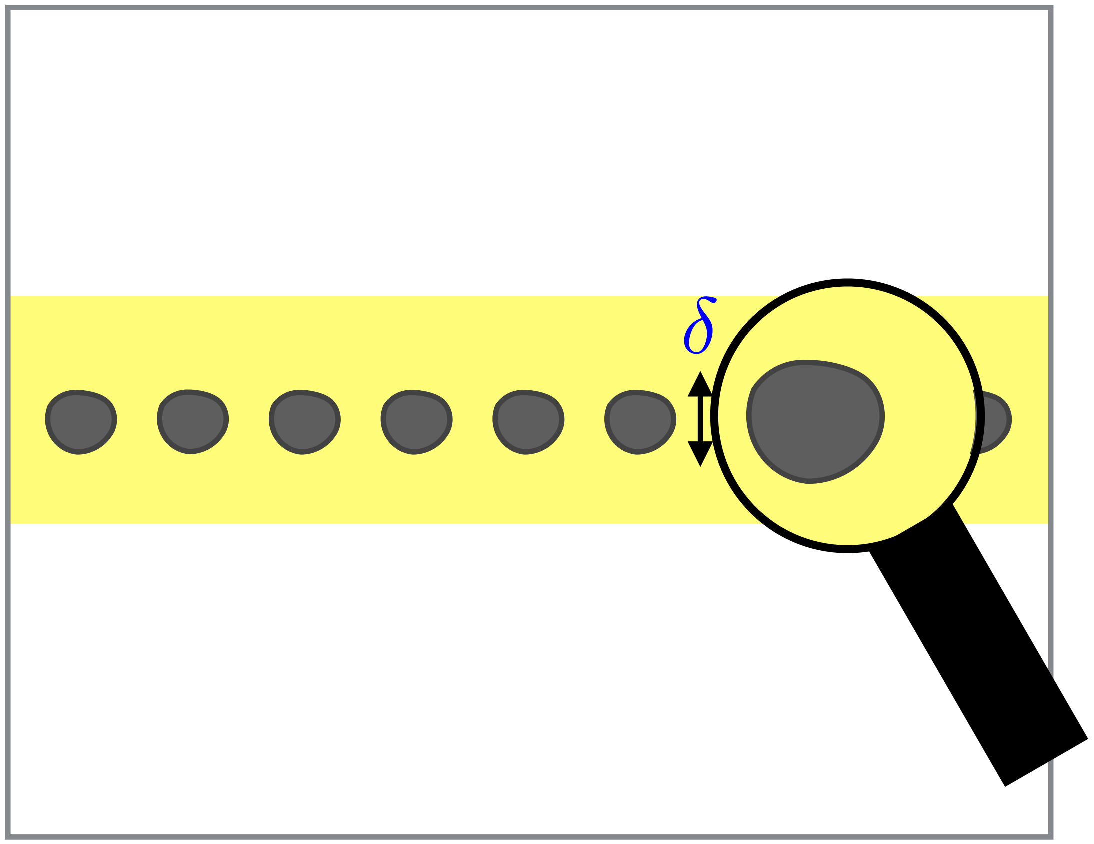
Missing information: transmission conditions across $\Gamma$ (two functions of $x_1$ )

$$[u_q](x_1) = u_q^+(x_1, 0) - u_q^-(x_1, 0)$$

$$[\partial_{x_2} u_q](x_1) = \partial_{x_2} u_q^+(x_1, 0) - \partial_{x_2} u_q^-(x_1, 0)$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: near field equations




Near field zone around the thin periodic layer

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, U_q\left(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta}\right)$$

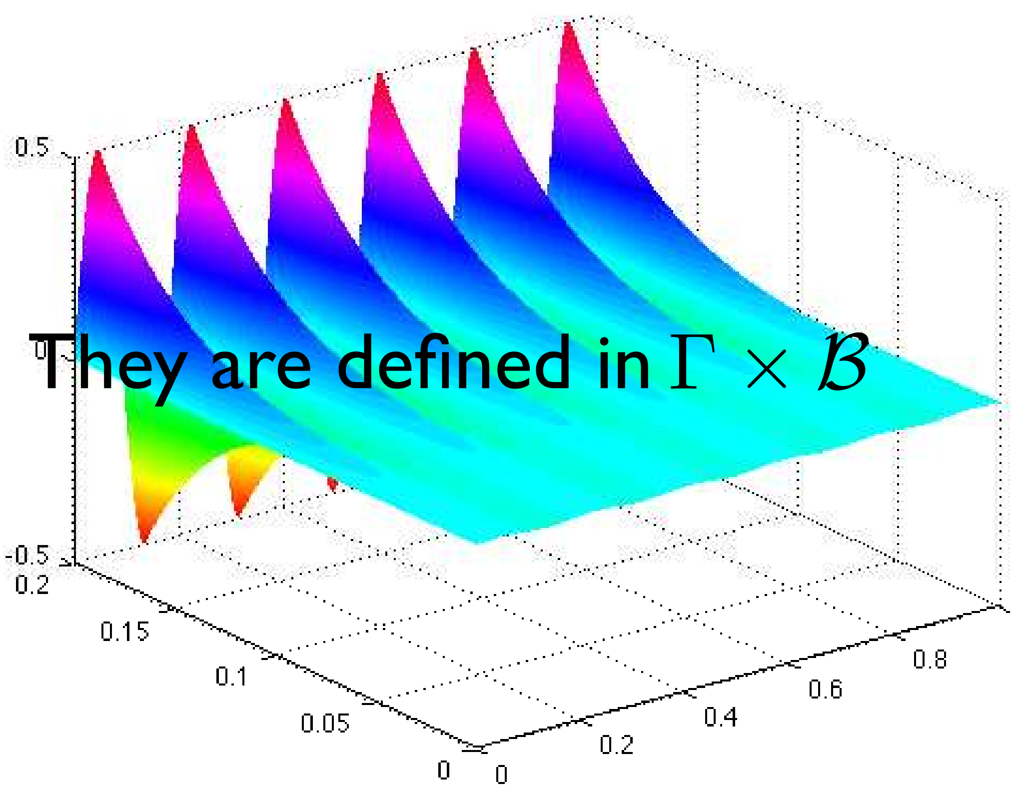
Near field terms

slow variation

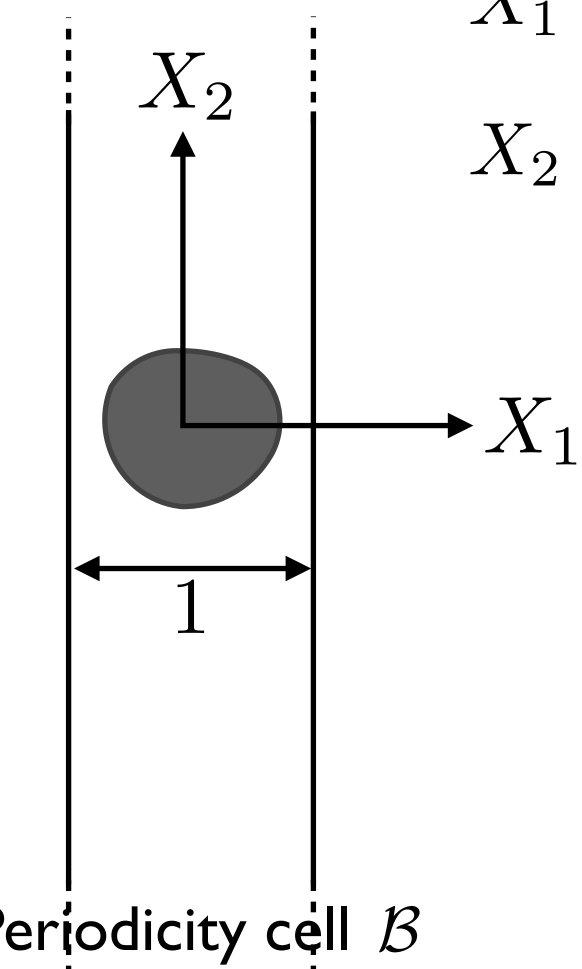
1-periodic w.r.t $X_1$

$$X_1 = \frac{x_1}{\delta}$$

$$X_2 = \frac{x_2}{\delta}$$




✓ They are defined in $\Gamma \times \mathcal{B}$




Periodicity cell $\mathcal{B}$

$$\mathcal{B} = \{(-1/2, 1/2) \times \mathbb{R}\} \setminus \hat{\Omega}_{\text{hole}}$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: near field equations

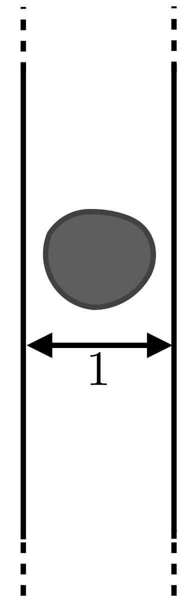
$$\nabla\left(U_q(x_1,\frac{x_1}{\delta},\frac{x_2}{\delta})\right) = \left(\partial_{x_1}U_q\mathbf{e_1} + \frac{1}{\delta}\nabla_X U_q\right)(x_1,\frac{x_1}{\delta},\frac{x_2}{\delta})$$

✓ **Near field equations**

parameter

$$\begin{cases} -\Delta_{\mathbf{X}} U_q(x_1,\mathbf{X}) = G_q \quad \text{in } \mathcal{B} \\ \partial_n U_q = 0 \quad \text{on } \partial\widehat{\Omega}_{\text{hole}} \\ U_q \quad 1\text{-periodic} \end{cases}$$

$$G_q = \partial^2_{x_1}U_{q-2} + 2\partial_{x_1}\partial_{X_1}U_{q-1}$$




We assume that the near field terms are not exponentially growing at infinity

These equations determined $U_q$ up to the determination of the kernel $\mathcal{K}$ of the Laplacian operator in the periodicity cell $\mathcal{B}$ (with homogeneous Neumann boundary conditions)

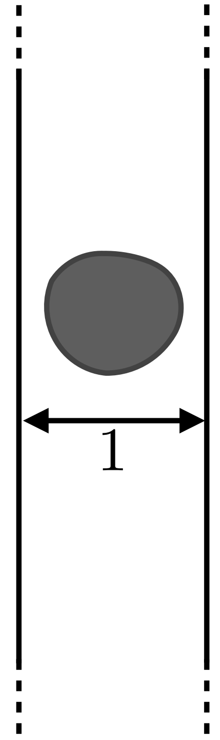
# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: near field equations

**Proposition:**

$$\mathcal{K} = \operatorname{span}\{1, \mathcal{N}\}$$

$$\begin{cases} -\Delta_X \mathcal{N} = 0 \\ \partial_n \mathcal{N} = 0 \\ \mathcal{N} \sim \begin{cases} X_2 + \mathcal{N}_\infty & X_2 \to +\infty \\ X_2 - \mathcal{N}_\infty & X_2 \to -\infty \end{cases} \end{cases}$$



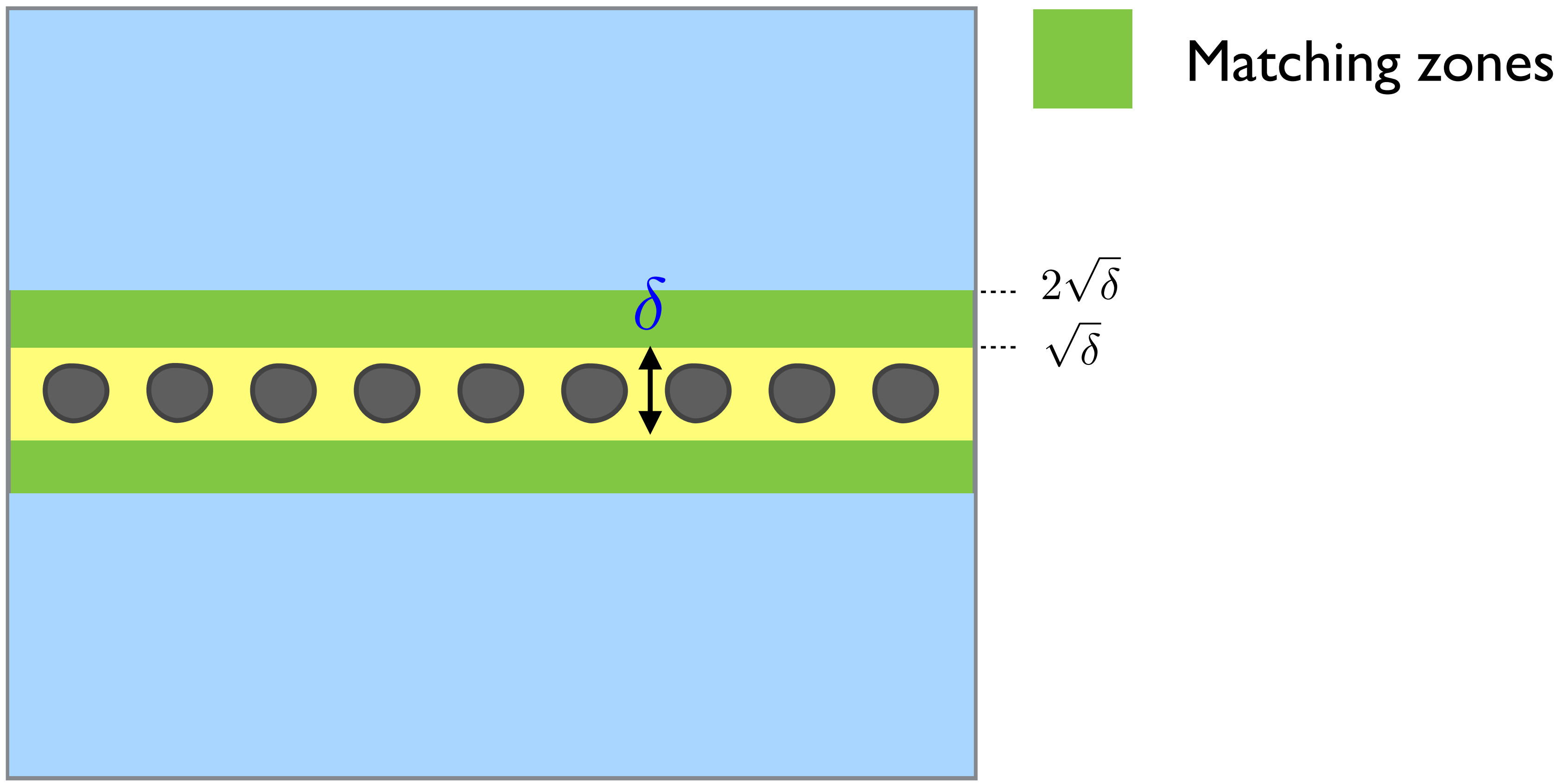

The near field terms $U_q$ are defined up to the specification of two functions of $\mathcal{K}$, linked to their behaviour at infinity.

$$\alpha(x_1) + \beta(x_1)\mathcal{N}$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: matching conditions

The missing information (4 functions of $x_1$) will be provided by the matching conditions




Far and near field series coincide in the matching zones

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, u_q(\mathbf{x}) \qquad u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, U_q(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta})$$

Neighborhood of $\Gamma$ for the far field ($x_2$ small)

Behavior at infinity of the near field ($X_2$ large)

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: matching conditions

**Proposition:**

$$U_q(x_1, X_1, X_2) = a_q^+(x_1) + b_q^+(x_1) X_2 + (X_2)^2 \underbrace{p_q^+(X_2, x_1)}_{\text{polynomial w.r.t } X_2} + \overbrace{O(e^{-X_2})}^{\text{periodic exponentially decaying}}$$

$$u_q(x_1, \delta X_2) = u_q^+(x_1, 0) + X_2 \, \delta \, \partial_{x_2} u_q^+(x_1, 0) + \sum_{k=2}^{+\infty} \frac{(X_2)^k \delta^k}{k!} \partial_{x_2}^k u_q^+(x_1, 0)$$

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q u_q(x_1, x_2) = \sum_{q \in \mathbb{N}} \delta^q \, U_q(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta})$$

$\longrightarrow$

$$a_q^+(x_1) = u_q^+(x_1, 0) \qquad b_q^+(x_1) = \partial_{x_2} u_{q-1}^+(x_1, 0)$$

$$a_q^-(x_1) = u_q^-(x_1, 0) \qquad b_q^-(x_1) = \partial_{x_2} u_{q-1}^-(x_1, 0)$$

## Method of matched asymptotic expansions: construction of the first terms

**Notation:** mean value and jump value across the limit interface $\Gamma$

mean value $$\langle u\rangle(x_1) := \frac{u^+(x_1,0)+u^-(x_1,0)}{2}$$

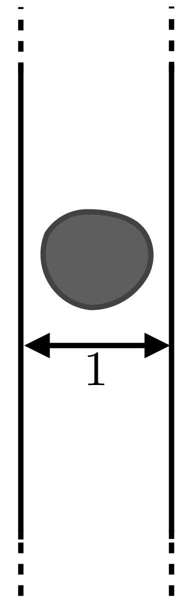
jump value $$[u]\,(x_1) := u^+(x_1,0) - u^-(x_1,0)$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: construction of the first terms

**Near field term of order 0:**

$$\begin{cases} -\Delta_{\mathbf{X}} U_0(x_1, \mathbf{X}) = 0 \quad \text{in } \mathcal{B} \\ \partial_n U_0 = 0 \text{ on } \partial\widehat{\Omega}_{\text{hole}} \\ U_0 \ 1\text{-periodic} \\ U_0 \sim u_0^{\pm}(x_1, 0) \quad X_2 \to \pm\infty \end{cases}$$




→ $U_0 \in \mathcal{K}$

$$U_0 = \alpha(x_1) + \beta(x_1)\mathcal{N}$$

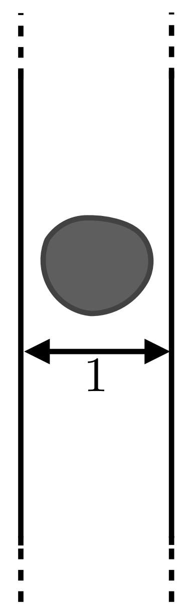
$$U_0 \sim (\alpha(x_1) \pm \beta(x_1)\mathcal{N}_\infty) + \beta(x_1)X_2$$

→ $[u_0](x_1) = 0 \qquad U_0(x_1, X_1, X_2) = \langle u_0 \rangle(x_1)$

## Method of matched asymptotic expansions: construction of the first terms

**Near field term of order 1:**

$$\begin{cases} -\Delta_{\mathbf{X}} U_1(x_1, \mathbf{X}) = 0 \quad \text{in } \mathcal{B} \\ \partial_n U_1 = 0 \text{ on } \partial\widehat{\Omega}_{\text{hole}} \\ U_1 \ 1\text{-periodic} \\ U_1 \sim u_1^{\pm}(x_1, 0) + X_2\, \partial_{x_2} u_0^{\pm}(x_1, 0) \quad X_2 \to \pm\infty \end{cases}$$




→ $U_1 \in \mathcal{K}$

$$U_1 = \alpha(x_1) + \beta(x_1)\mathcal{N}$$

$$U_1 \sim (\alpha(x_1) \pm \beta(x_1)\mathcal{N}_\infty) + \beta(x_1) X_2$$

→

$$[\partial_{x_2} u_0](x_1) = 0 \qquad [u_1](x_1) = 2\,\mathcal{N}_\infty \,\langle \partial_{x_2} u_0 \rangle(x_1)$$

$$U_0(x_1, X_1, X_2) = \langle u_1 \rangle(x_1) + \langle \partial_{x_2} u_0 \rangle(x_1)\, \mathcal{N}(X_1, X_2)$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: construction of the first terms

**Near field term of order 2:**

$$\begin{cases} -\Delta_{\mathbf{X}} U_2(x_1, \mathbf{X}) = \partial^2_{x_1} \langle u_0 \rangle (x_1) + 2\partial_{x_1} \langle \partial_{x_2} u_0 \rangle (x_1) \partial_{X_1} \mathcal{N} \\ \partial_n U_2 = 0 \ \text{ on } \partial\widehat{\Omega}_{\text{hole}} \\ U_2 \ 1\text{-periodic} \\ U_2 \sim u_2^{\pm}(x_1, 0) \ + \ X_2 \, \partial_{x_2} u_1^{\pm}(x_1, 0) - \dfrac{(X_2)^2}{2} \partial^2_{x_1} \langle u_0 \rangle (x_1) \quad X_2 \to \pm\infty \end{cases}$$

$$\begin{cases} -\Delta_{\mathbf{X}} \mathcal{N}_{20} = 1 \\ \partial_n \mathcal{N}_{20} = 0 \\ \mathcal{N}_{20} \ \sim \ \pm \mathcal{N}_{20}^{\infty} \ \pm \ C_{20}^{\infty} X_2 \ - \ \dfrac{(X_2)^2}{2} \end{cases}
\qquad
\begin{cases} -\Delta_{\mathbf{X}} \mathcal{N}_{21} = 2\partial_{X_1} \mathcal{N} \\ \partial_n \mathcal{N}_{21} = 0 \\ \mathcal{N}_{21} \ \sim \ \pm \mathcal{N}_{21}^{\infty} \ \pm \ C_{21}^{\infty} X_2 \end{cases}$$

By linearity

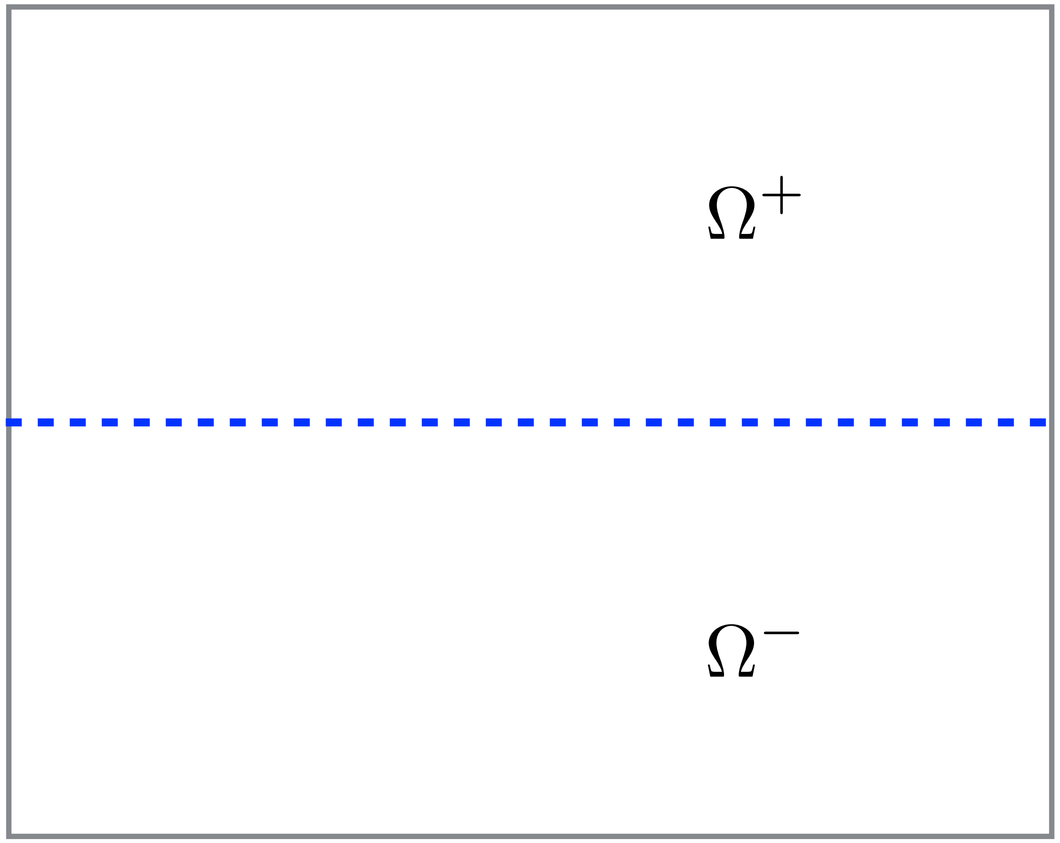
$$U_2 = \langle u_2 \rangle + \langle \partial_{x_2} u_1 \rangle \mathcal{N} + \partial^2_{x_1} \langle u_0 \rangle \mathcal{N}_{20} + \partial_{x_1} \langle \partial_{x_2} u_0 \rangle \mathcal{N}_{21}$$

$$[\partial_{x_2} u_1] = 2\, C_{20}^{\infty} \, \partial^2_{x_1} \langle u_0 \rangle + 2\, C_{21}^{\infty} \, \partial_{x_1} \langle \partial_{x_2} u_0 \rangle$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: construction of the first terms

**Far field term of order 0:**

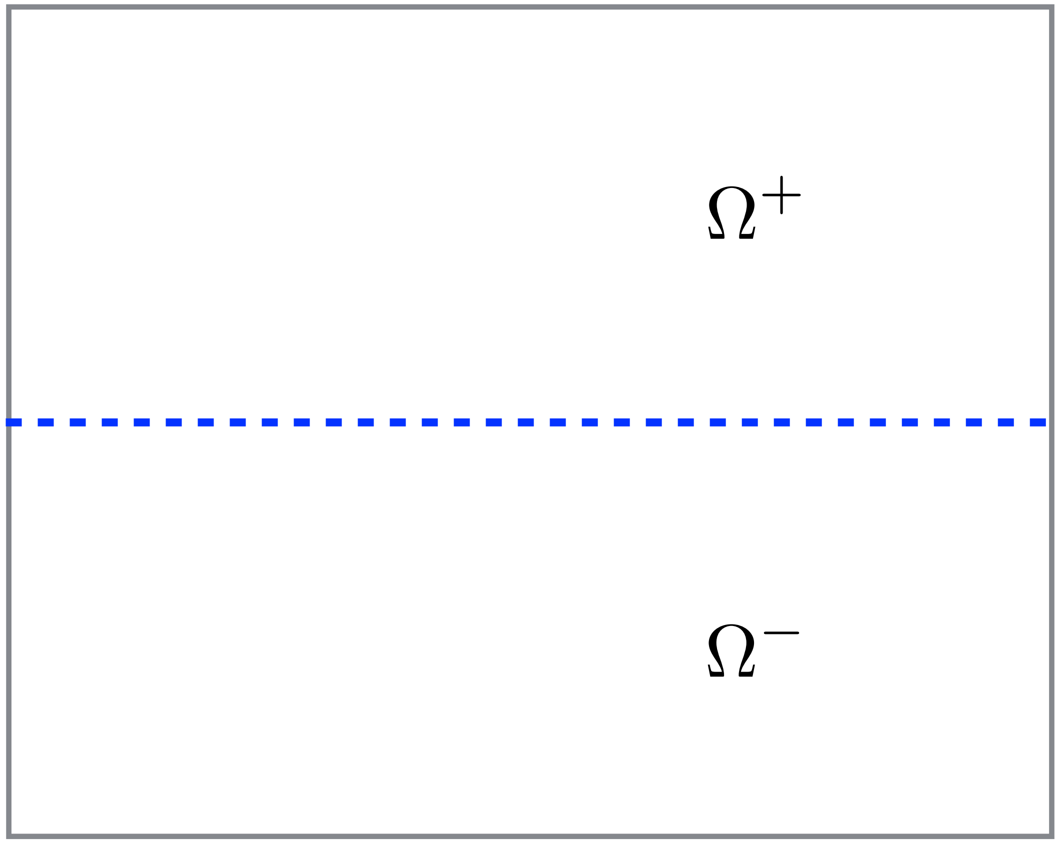



$$\begin{cases} -\Delta u_0^\pm = f \quad \text{in } \Omega^\pm \\ [u_0] = 0 \\ [\partial_{x_2} u_0] = 0 \\ + \text{ B.C} \end{cases}$$

At the limit, the thin periodic interface disappears.

## Method of matched asymptotic expansions: construction of the first terms

### Far field term of order 1:

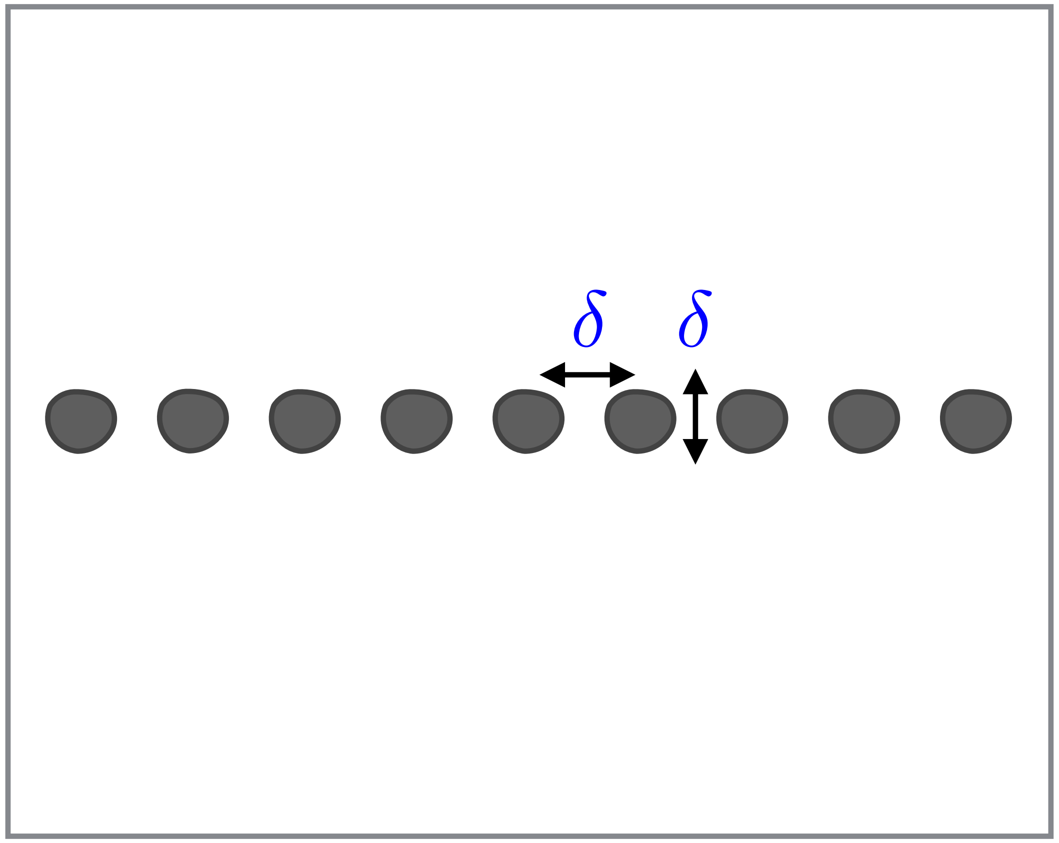



$$
\begin{cases}
-\Delta u_1^{\pm} = 0 \quad \text{in } \Omega^{\pm} \\
[u_1](x_1) = 2\,\mathcal{N}_\infty \,\langle \partial_{x_2} u_0 \rangle (x_1) \\
[\partial_{x_2} u_1] = 2\, C_{20}^{\infty}\, \partial_{x_1}^2 \langle u_0 \rangle + 2\, C_{21}^{\infty}\, \partial_{x_1} \langle \partial_{x_2} u_0 \rangle \\
\; + \text{ B.C}
\end{cases}
$$

✓ By induction, we can construct the far field terms and near field terms up to any order

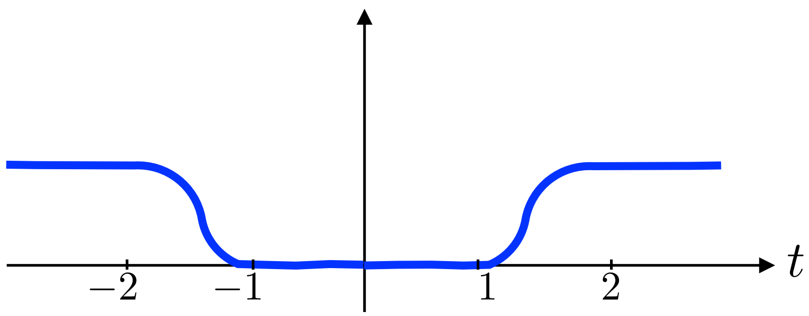
# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: justification




$\chi$ is a smooth cut-off function

$$\chi(t) : \begin{cases} 1 & |t| > 2 \\ 0 & |t| < 1 \end{cases}$$




$$\lim_{\delta \to 0} \frac{\eta(\delta)}{\delta} = 0 \qquad \lim_{\delta \to 0} \eta(\delta) = 0$$

$$\chi_\eta(\mathbf{x}) = \chi\left(\frac{\mathbf{x_2}}{\eta}\right)$$

Global approximation:

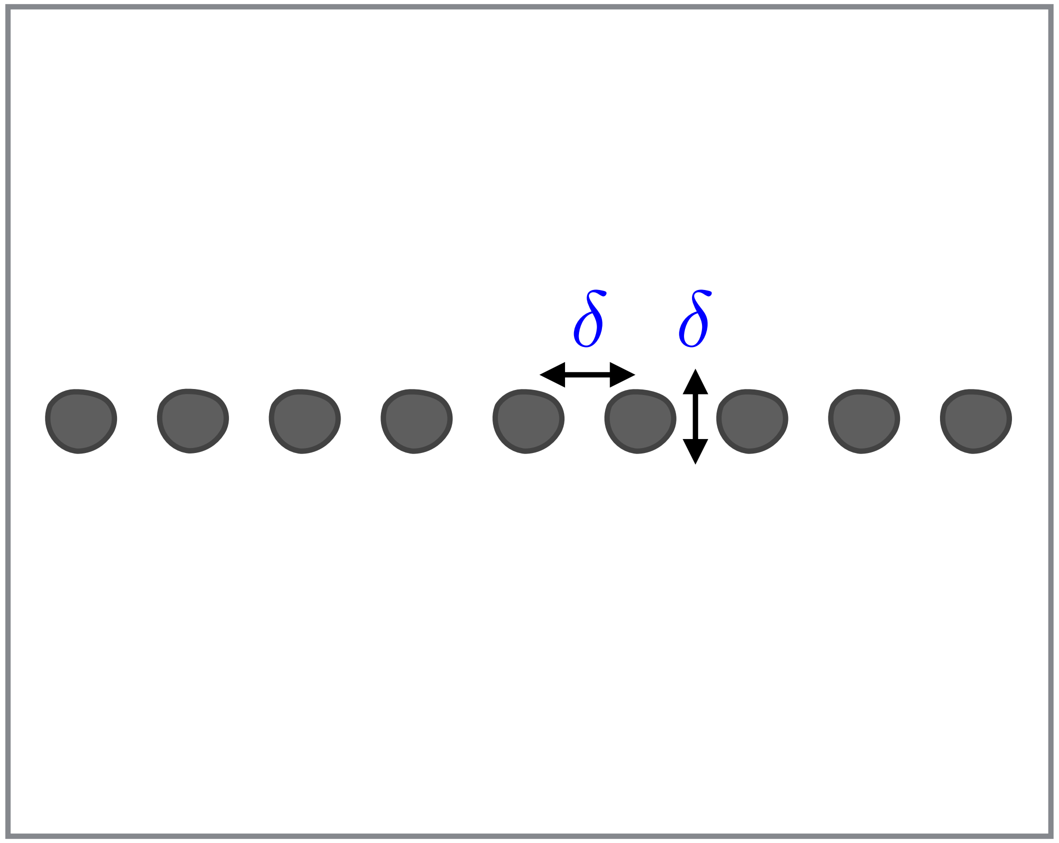
$$u^\delta_{n,\eta} = \chi_\eta(\mathbf{x}) \sum_{k=1}^{n} \delta^k u_k(\mathbf{x}) + (1 - \chi_\eta(\mathbf{x})) \sum_{k=1}^{n} \delta^k U_k(x_1, \frac{\mathbf{x}}{\delta})$$

Global error:

$$e^\delta_{n,\eta} = u^\delta - u^\delta_{n,\eta}$$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: justification




$$\begin{cases} -\Delta e^{\delta}_{n,\eta} = \mathcal{E}^{\delta}_{m,\eta} + \mathcal{E}^{\delta}_{c,\eta} \ \text{ in } \Omega^{\delta} \\ \partial_n e^{\delta}_{n,\eta} = 0 \ \text{ on } \partial\Omega^{\delta}_{\text{hole}} \\ + \text{B.C} \end{cases}$$

$\mathcal{E}^{\delta}_{m,\eta}$ matching error: measure the mismatch between far field and near field equations in the matching zones

$\mathcal{E}^{\delta}_{c,\eta}$ consistency error: measure how the near field expansion fails to solve the Laplace equation

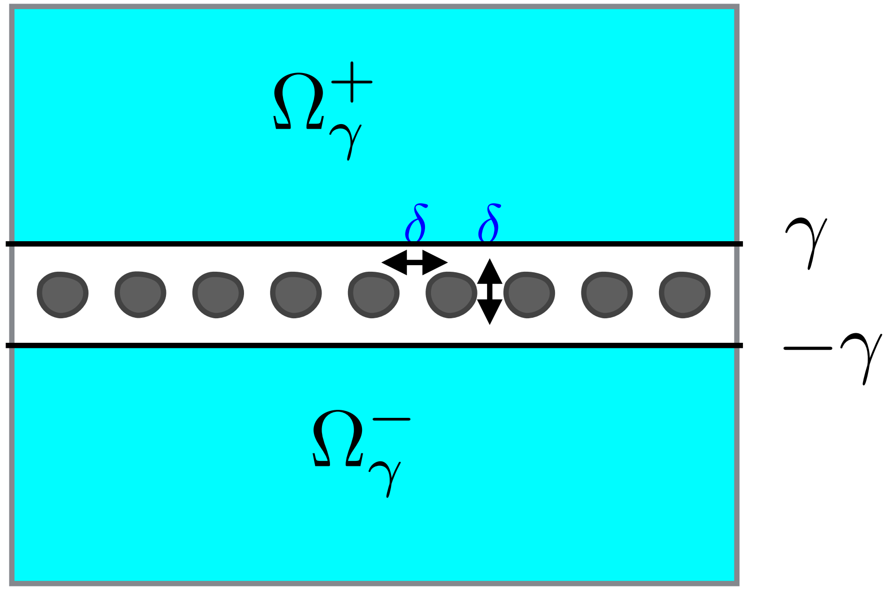
# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions: justification

$$\|\mathcal{E}^{\delta}_{m,\eta}\|_{L^2(\Omega^{\delta})} + \|\mathcal{E}^{\delta}_{c,\eta}\|_{L^2(\Omega^{\delta})} \leq C\,\eta^{n-1}\,\|f\|_{L^2(\Omega^{\delta})}$$

+ stability estimate

**Proposition:** $\|u^{\delta} - u^{\delta}_{n,\eta}\|_{H^1(\Omega^{\delta})} \leq C\eta^{n-1}\|f\|_{L^2(\Omega^{\delta})}$




$$\Omega_\gamma = \Omega_\gamma^+ \cup \Omega_\gamma^-$$

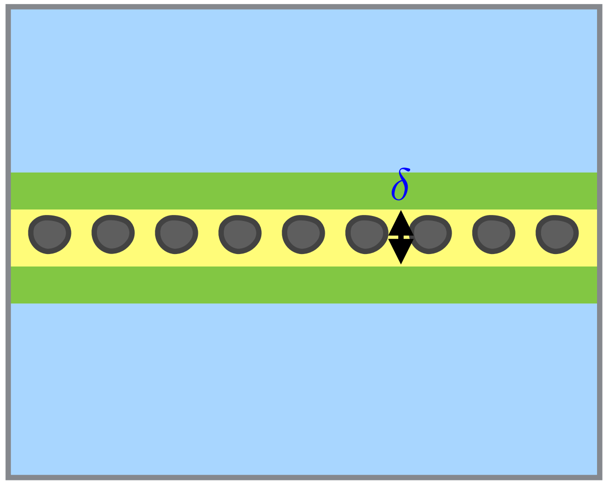
$\eta(\delta) = \delta^{1/2}$ + triangular inequality

**Proposition:** $\|u^{\delta} - (u_0 + \delta u_1)\|_{H^1(\Omega_\gamma)} \leq C\delta^2\|f\|_{L^2(\Omega^{\delta})}$

# I- Investigation of a 2D-model problem

## Method of matched asymptotic expansions:

*Matched asympototic expansion*




Far field zones

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, u_q(\mathbf{x})$$

Near field zone

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, U_q(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta})$$

Matching zones

*Compound method (boundary layer or multiscale approach)*

$$u^\delta = \sum_{q \in \mathbb{N}} \delta^q \left( \chi(\frac{x_2}{\delta}) v_q(\mathbf{x}) + \Pi_q(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta}) \right)$$

macroscopic term

boundary layer corrector
periodic w.r.t. $X_1$
exponentially decaying w.r.t $X_2$

*Nazarov 81, Sanchez-Palencia 83, Artola Cessenat 91, Abboud-Ammari 96, Achdou 92, Achdou-Pironneau-Valentin 98, Poirier-Bendali-Borderies 06, Madureira-Valentin 06, Bonnetier-Bresch-Milisic 10…*

*Link between the two types of expansions*

$$v_q = u_q \qquad \Pi_q = U_q - \sum_{\pm} \chi^{\pm}(X_2) \sum_{k=1}^{q} \frac{(X_2)^k}{k!} \partial^k_{x_2} u^{\pm}_{q-k}(x_1, 0)$$

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

**Objective:** replacement of the periodic layer with an approximate transmission condition posed on the limit interface

We construct an approximate problem whose the solution is close to

$$u_{1,\delta} = u_0 + \delta u_1$$

$$-\Delta u_{1,\delta} = f$$

$$[u_0](x_1) = 0$$

$$\times\delta \qquad [u_1](x_1) = 2\,\mathcal{N}_\infty\,\langle \partial_{x_2} u_0 \rangle (x_1)$$

---

$$\boldsymbol{[u_{1,\delta}](x_1) = 2\,\delta\,\mathcal{N}_\infty\,\langle \partial_{x_2} u_0 \rangle (x_1)}$$

$$[u_{1,\delta}](x_1) = 2\,\delta\,\mathcal{N}_\infty\,\langle \partial_{x_2} u_{1,\delta} \rangle (x_1) + O(\delta^2)$$

$$\approx \langle \partial_{x_2} u_{1,\delta} \rangle$$

Similarly

$$[\partial_{x_2} u_{1,\delta}](x_1) = 2\,\delta\,C_{20}^\infty\,\partial_{x_1}^2 \langle u_{1,\delta} \rangle (x_1) + 2\,\delta\,C_{21}^\infty \partial_{x_1} \langle \partial_{x_2} u_{1,\delta} \rangle (x_1) + O(\delta^2)$$

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

**Objective:** replacement of the periodic layer with an approximate transmission condition posed on the limit interface

$$
\begin{cases}
-\Delta \tilde{u}_{1,\delta} = f \quad \text{in } \Omega^{\pm} \\
[\tilde{u}_{1,\delta}](x_1) = 2\,\delta\,\mathcal{N}_\infty \,\langle \partial_{x_2} \tilde{u}_{1,\delta} \rangle (x_1) \\
\boldsymbol{[\partial_{x_2} \tilde{u}_{1,\delta}](x_1) = 2\,\delta\,C_{20}^{\infty}\,\partial_{x_1}^2 \langle \tilde{u}_{1,\delta} \rangle (x_1)} + 2\,\delta\,C_{21}^{\infty} \partial_{x_1} \langle \partial_{x_2} \tilde{u}_{1,\delta} \rangle (x_1) \\
\quad + \text{B.C}
\end{cases}
$$

Investigation in the symmetric case: $C_{21}^{\infty} = 0$

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

Variational formulation: $\forall v \in V = \{v \in H^1(\Omega^+ \cup \Omega^-), v\text{-periodic}, v = 0 \text{ on } \Gamma_D\}$

$$\int_{\Omega^+ \cup \Omega^-} \nabla \tilde{u}_{1,\delta} \cdot \nabla v - 2\delta C_{02}^{\infty} \int_{\Gamma} \langle \partial_{x_1} \tilde{u}_{1,\delta} \rangle \langle \partial_{x_1} v \rangle + \frac{1}{2\mathcal{N}_{\infty}} \int_{\Gamma} [\tilde{u}_{1,\delta}][v] = \int_{\Omega^+ \cup \Omega^-} f v$$

coercive term

compact term

coercive if $C_{02}^{\infty} < 0$

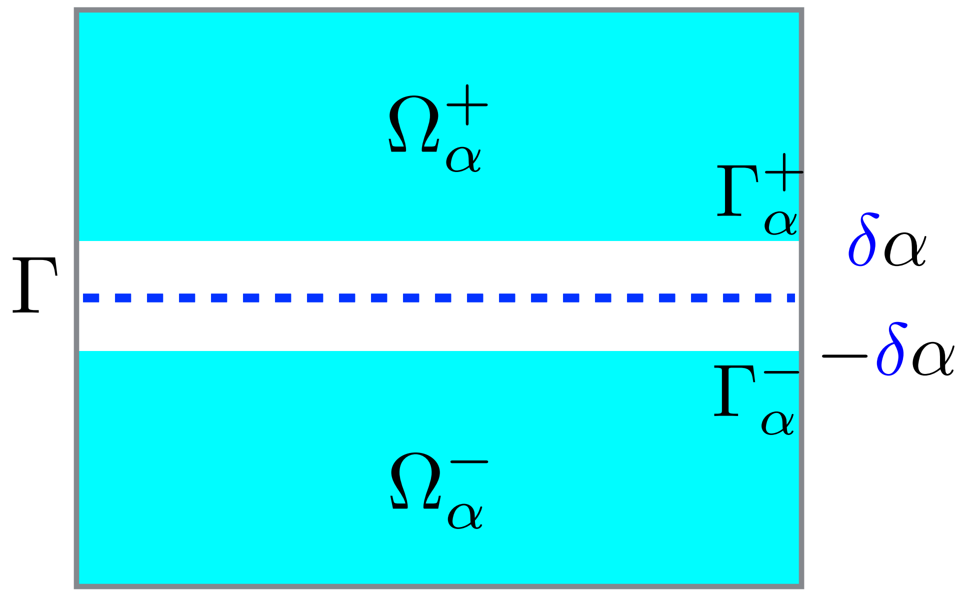
stable if $\mathcal{N}_{\infty} > 0$

Problem: it might be that $C_{02}^{\infty} > 0$ or $\mathcal{N}_{\infty} < 0$

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

A possible remedy: shift of the transmission condition: $\alpha > 0$



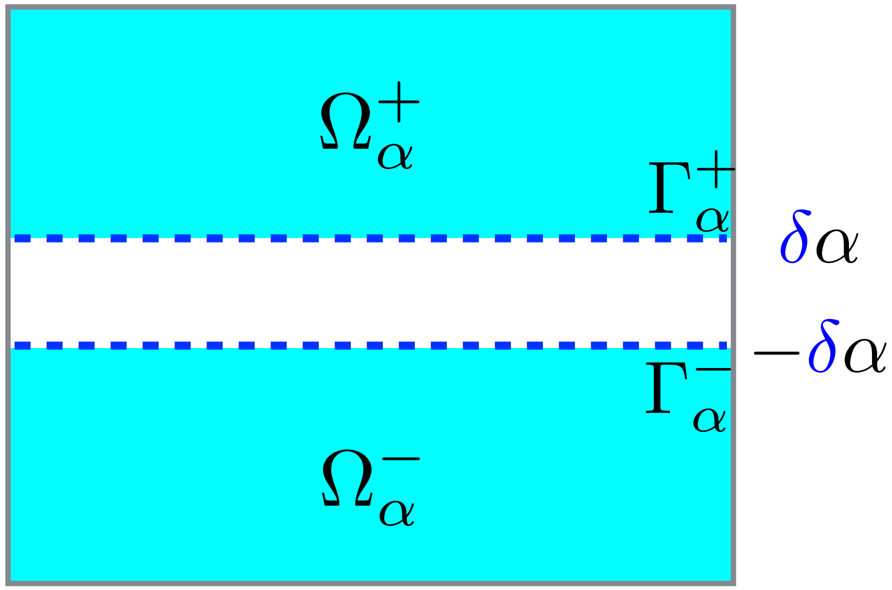

$\alpha$ is a parameter to be ajusted

$$[v]_\alpha := \Big[ v(x_1, \alpha\delta) - v(x_1, -\alpha\delta) \Big]$$

$$\langle v \rangle_\alpha := \frac{1}{2} \Big[ v(x_1, \alpha\delta) + v(x_1, -\alpha\delta) \Big]$$

## Approximate transmission conditions:

A possible remedy: shift of the transmission condition: $\alpha > 0$




$\alpha$ is a parameter to be ajusted

$$[u]_\alpha = [u] + 2\alpha\delta \, \langle \partial_{x_2} u \rangle_\alpha + \cancel{\mathcal{O}(\delta^3)}$$

$$[\partial_{x_2} u]_\alpha = [\partial_{x_2} u] - 2\alpha\delta \, \langle \partial_{x_1}^2 u \rangle_\alpha + + \cancel{\mathcal{O}(\delta^3)}$$

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

A possible remedy: shift of the transmission condition: $\alpha > 0$

$$\begin{cases} -\Delta \tilde{u}_{1,\delta} = f \quad \text{in } \Omega^{\pm}_{\alpha} \\ [\tilde{u}_{1,\delta}]_{\alpha}(x_1) = 2\,\delta\,\mathcal{N}^{\alpha}_{\infty}\,\langle \partial_{x_2}\tilde{u}_{1,\delta}\rangle_{\alpha}(x_1) \\ [\partial_{x_2}\tilde{u}_{1,\delta}]_{\alpha}(x_1) = 2\,\delta\,C^{\infty,\alpha}_{20}\,\partial^2_{x_1}\langle \tilde{u}_{1,\delta}\rangle_{\alpha}(x_1) \\ + \text{B.C} \end{cases}$$

$$\mathcal{N}^{\alpha}_{\infty} = \mathcal{N}_{\infty} + 2\alpha \quad > 0$$

$$C^{\infty,\alpha}_{20} = C^{\infty}_{20} - 2\alpha \quad < 0$$

Remark: important for the stability of time-domain problems

# I- Investigation of a 2D-model problem

## Approximate transmission conditions:

assumption: $\mathcal{N}_\infty^\alpha > 0$ et $C_{2,0}^{\infty,\alpha} < 0$

**Proposition:**

$$\|\tilde{u}_{1,\delta} - \tilde{u}^\delta\|_{H^1(\Omega_\gamma^\pm)} \leq C\ \delta^2\ \|f\|_{L^2(\Omega^\pm)}$$

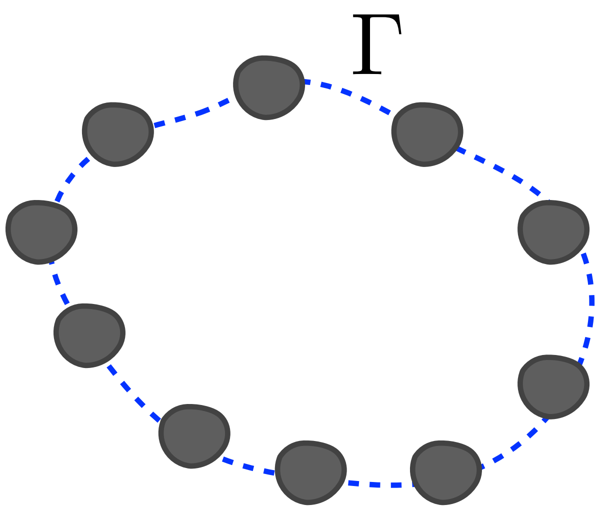
Well-posedness and stability of the approximate problem

Asymptotic expansion of $\tilde{u}_{1,\delta}$

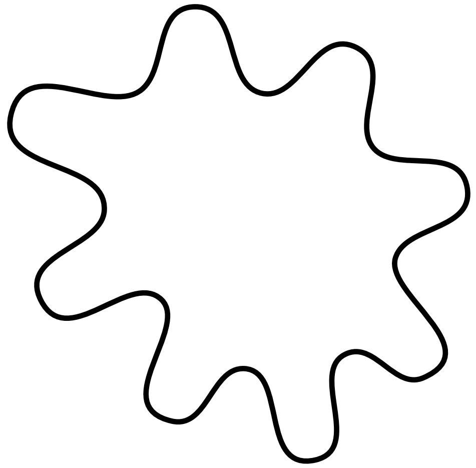
# 2- Extensions and numerical illustrations

## Applications and extensions

- Curve geometries




- Oscillatory boundary (wall-laws)

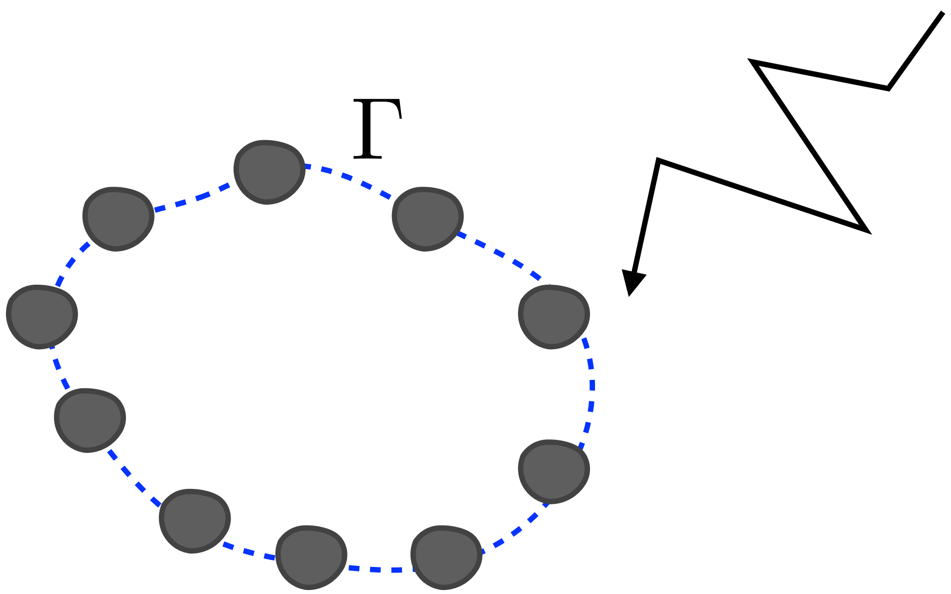


*Nazarov 81, Sanchez-Palencia 83, Conca 87, Artola Cessenat 91, Abboud-Ammari 96, Achdou 92, Achdou-Pironneau-Valentin 98, Poirier-Bendali-Borderies 06, Madureira-Valentin 06, Mikelic 09, Bonnetier-Bresch-Milisic 10…*

# 2- Extensions and numerical illustrations

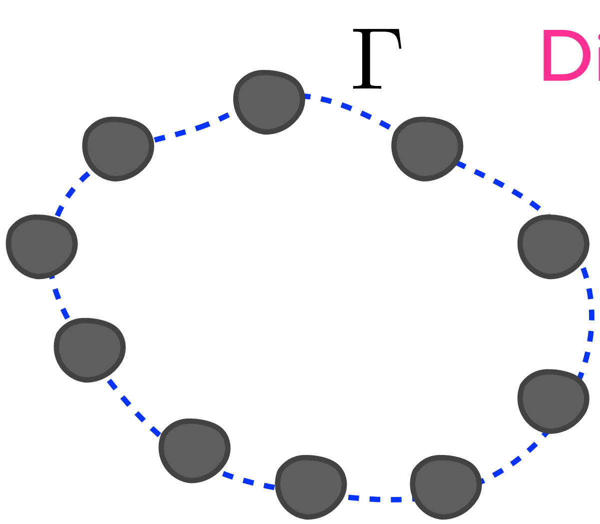
## Applications and extensions

- Application to other linear equations (Helmholtz)
- Time domain problems




*Joly-Semin 10, Lombard-Maurel-Marigo 17,Maurel-Marigo-Mercier-Pham 18, Maurel-Pham-Marigo 19,*

- other types of boundary conditions/dielectrical material




Dirichlet/Robin

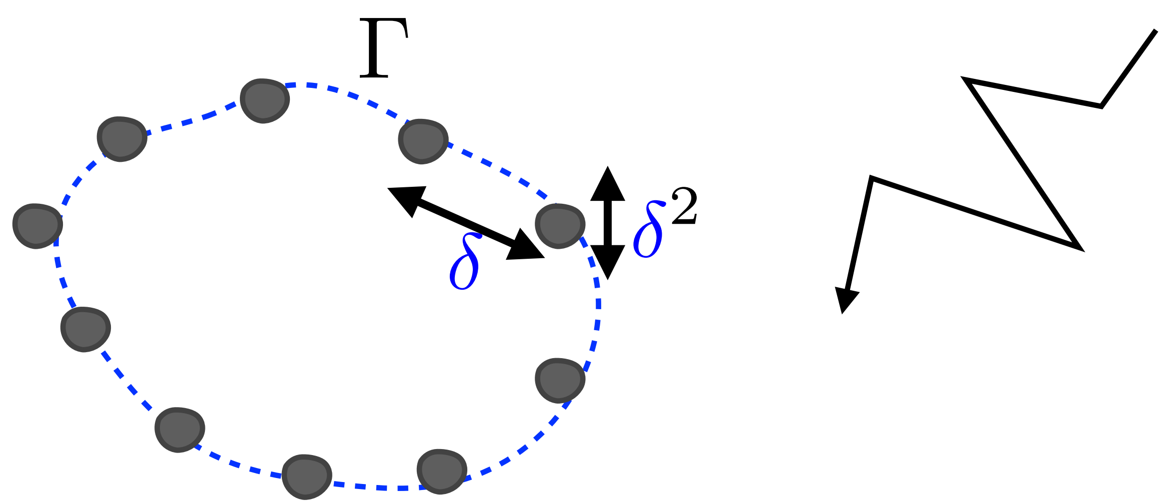
dielectrical obstacles

*Hewitt-Hewett*

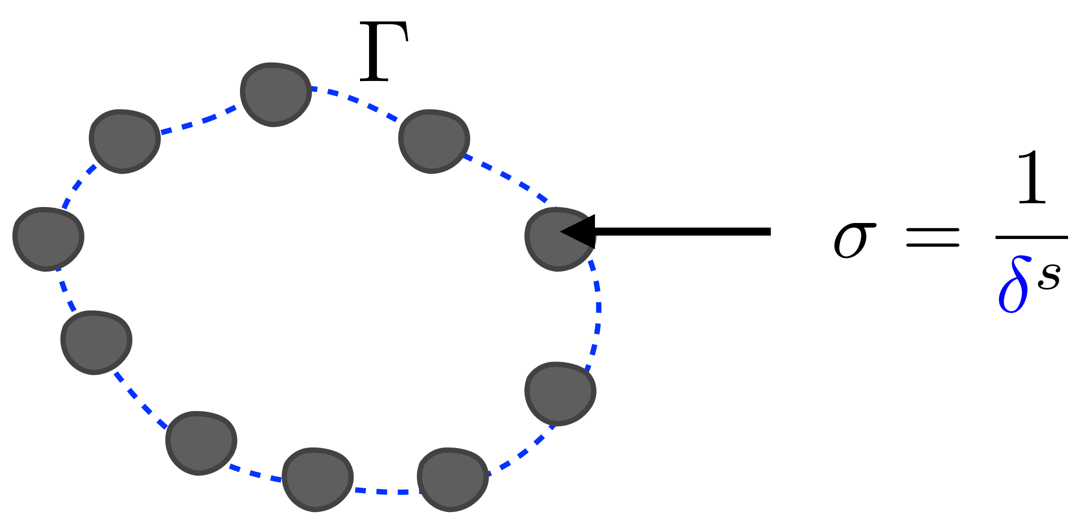
# 2- Extensions and numerical illustrations

## Applications and extensions

- Three scale problems *Hewett-Hewitt 16…*



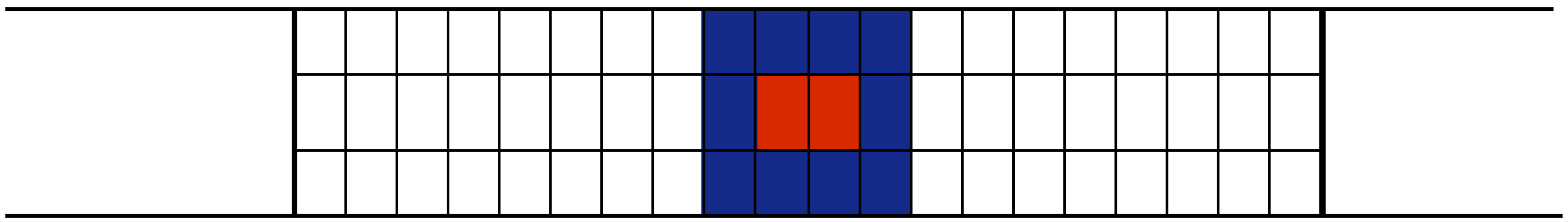

- high contrast meta-surfaces



# 2- Extensions and numerical illustrations

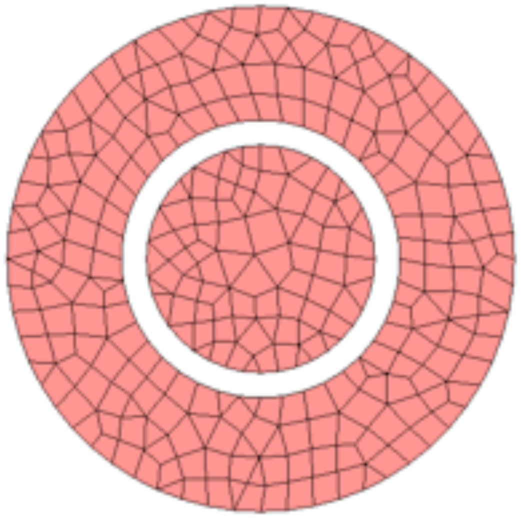
## Numerical results for the Helmholtz equation: algorithm

1- Computations of the 'profile' functions in the periodicity cell



2- Computation of the constants $\alpha\,,\mathcal{N}_{\infty}^{\alpha}\,,C_{20}^{\infty,\alpha}$ .

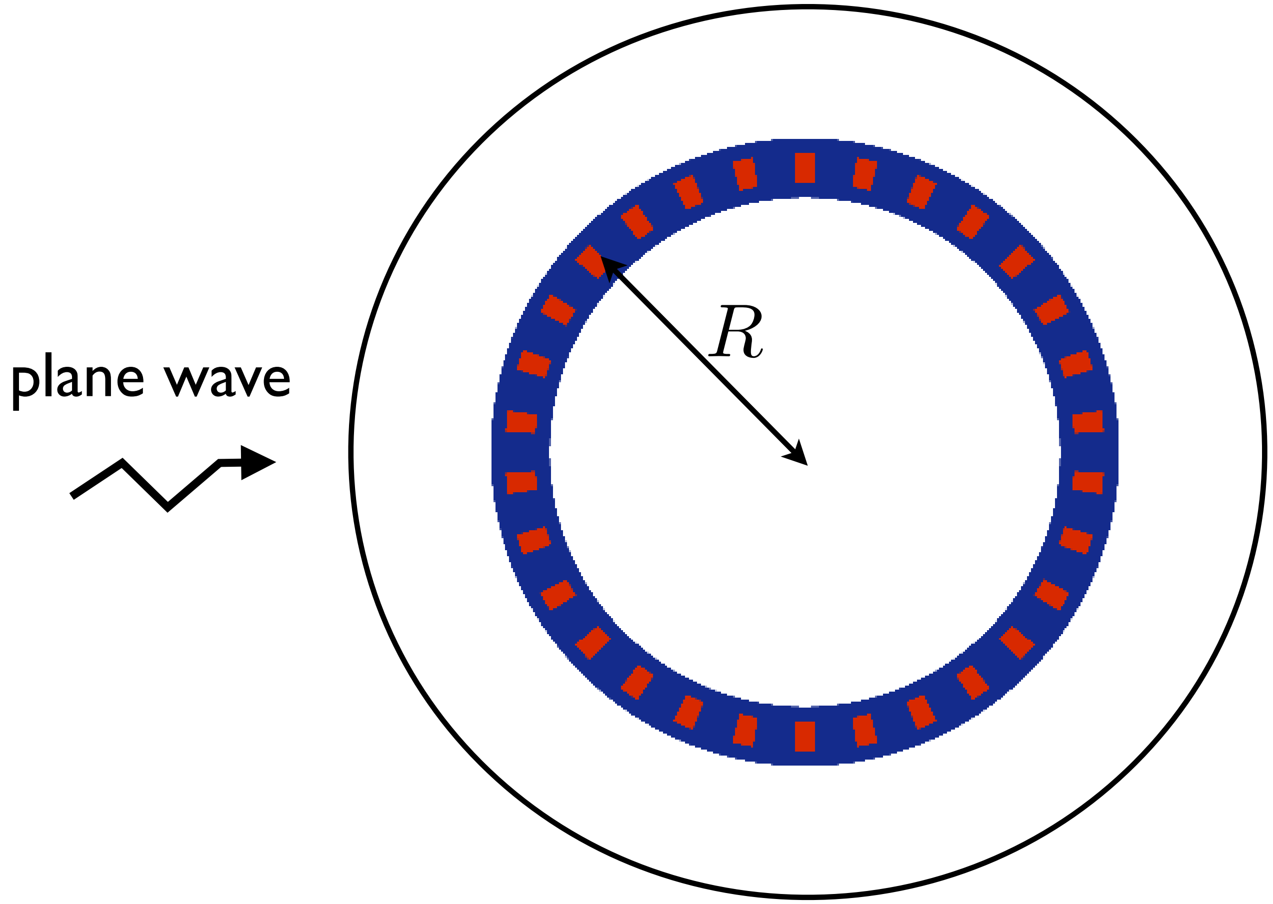
3- Computation of the approximate solution (coarse mesh).



4- A posteriori construction of the near field (optional).

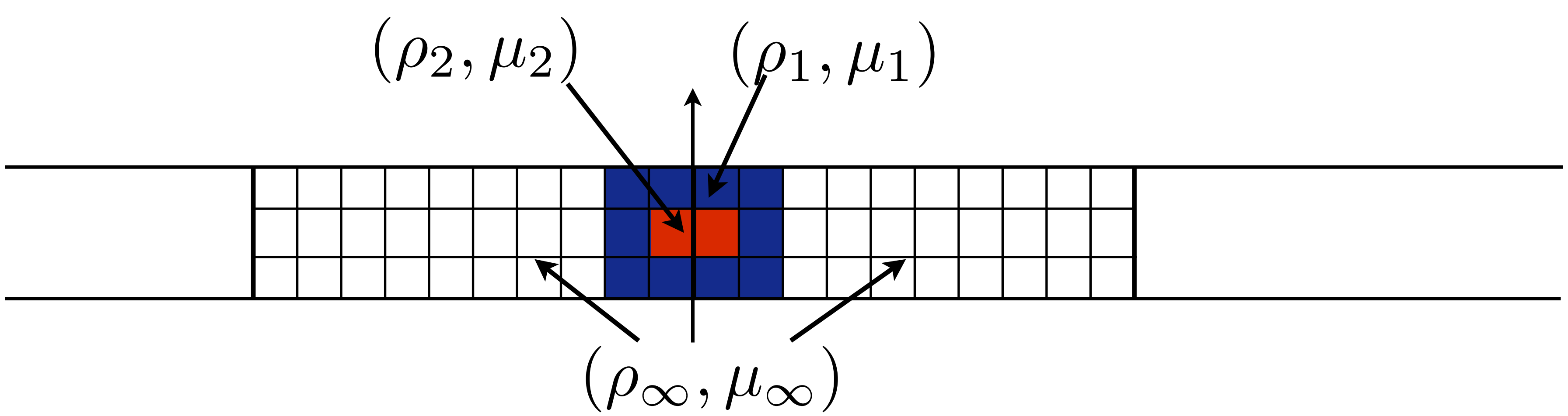
# 2- Extensions and numerical illustrations

## Numerical results for the Helmholtz equation: algorithm



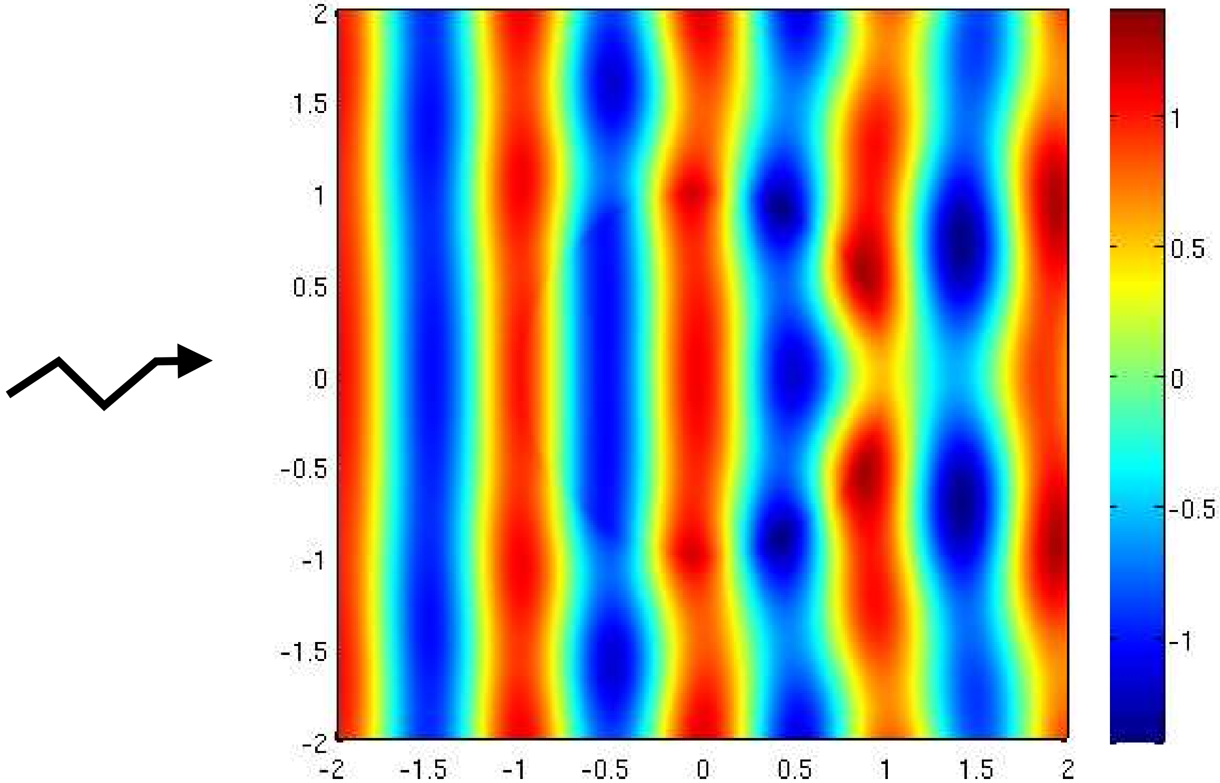

$$\omega = 2\pi, \; R = 1$$
$$\rho_1 = 2, \rho_2 = 4, \rho_\infty = 1$$
$$\mu_1 = 0.5, \mu_2 = 2, \mu_\infty = 1$$

$$N = \frac{2\pi R}{\delta}$$ number of cell problems

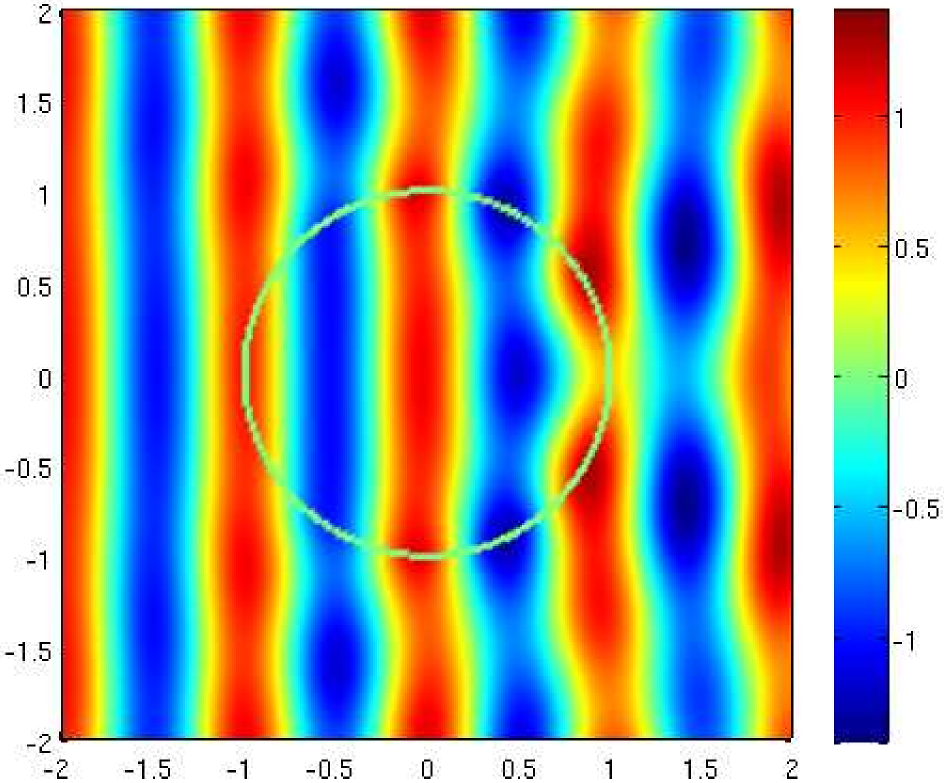


## Numerical results for the Helmholtz equation: algorithm

Total field, N =160



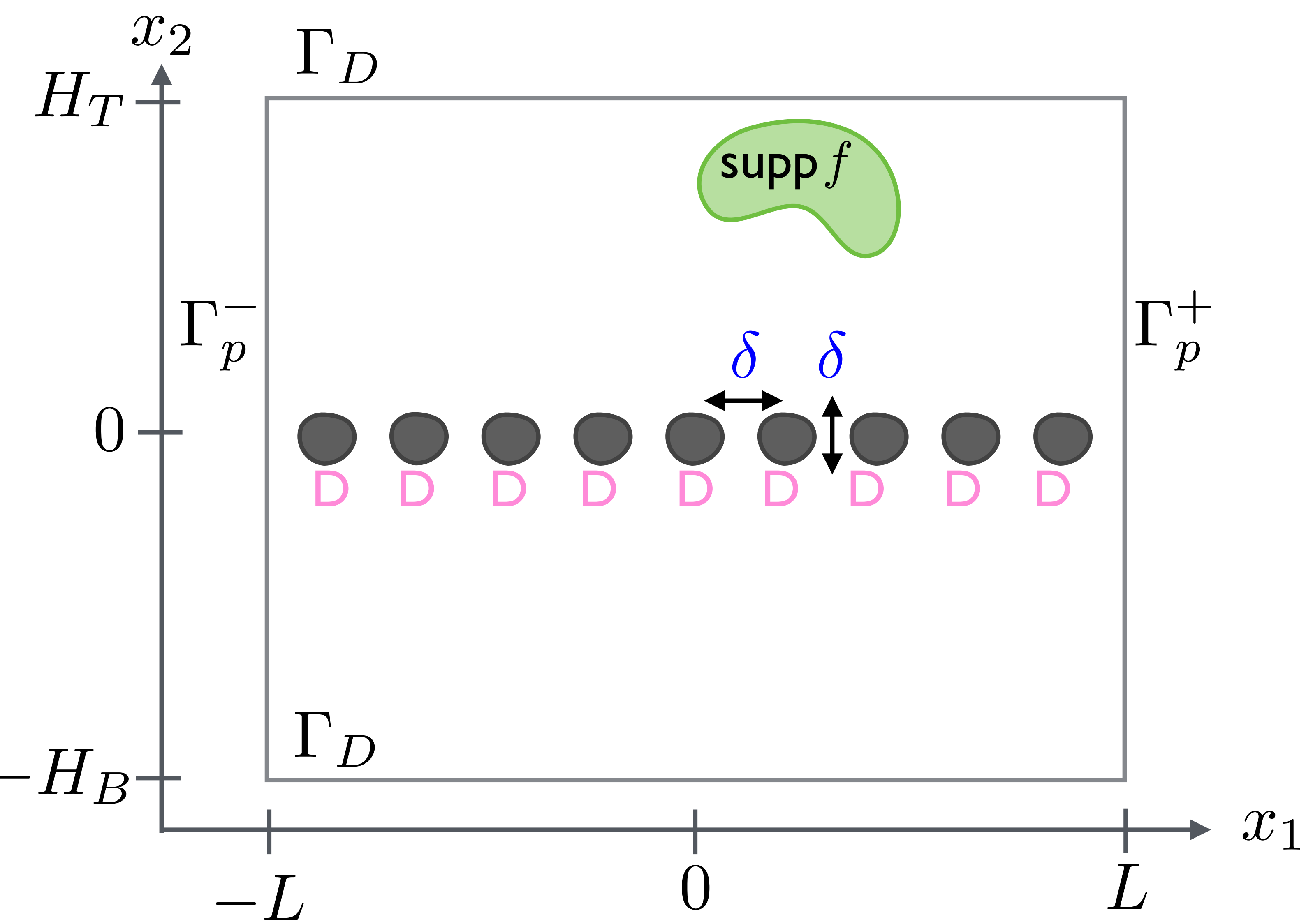

Exact solution




Approximate solution

# 2- Extensions and numerical illustrations

## The Dirichlet case




$$(\mathcal{P})\begin{cases} -\Delta u^\delta = f \text{ in } \Omega_p^\delta \\ u^\delta = 0 \text{ on } \Gamma_D \\ u^\delta = 0 \quad \text{on } \partial\Omega^\delta_{\text{hole}} \\ u^\delta \ 2L\text{-periodic} \end{cases}$$

$$\Omega_p = (-L, L) \times (-H_B, H_T)$$

$$\Omega_p^\delta = \Omega_p \setminus \overline{\Omega^\delta_{\text{hole}}}$$

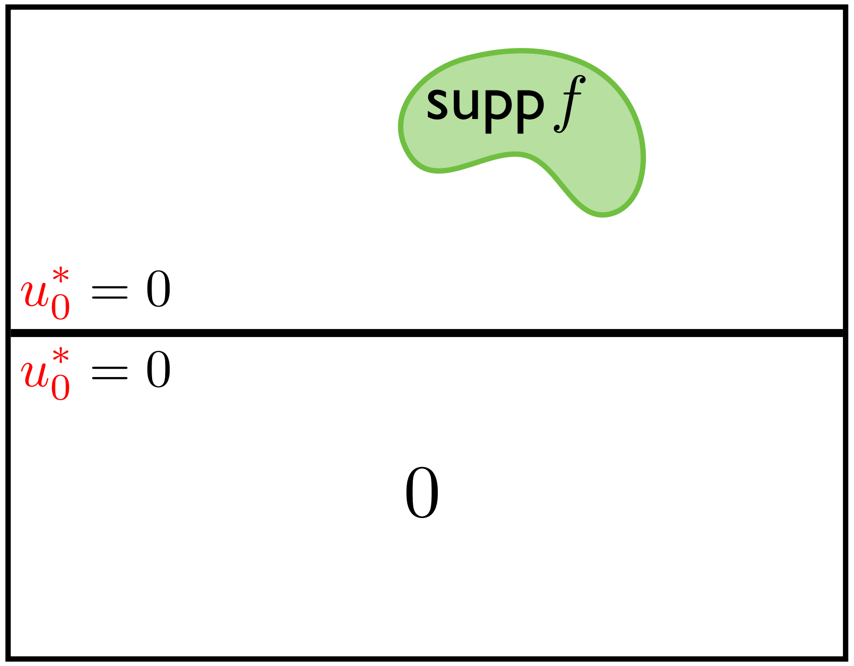
$$\partial\Omega_p = \Gamma_p^- \cup \Gamma_p^+ \cup \Gamma_D$$

# I- Investigation of a 2D-model problem

## The Dirichlet case

Theorem: the limit of $u^\delta$ as $\delta$ tends to $0$ is the function $u_0^* \in H^1(\Omega)$ unique solution to the problem

$$\begin{cases} -\Delta u_0^* = f \quad \text{in } \Omega^+ \\ u_0^* = 0 \quad \text{on } \Gamma \\ \text{B.C. on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u_0^* = f \quad \text{in } \Omega^- \\ u_0^* = 0 \quad \text{on } \Gamma \\ \text{B.C. on } \partial\Omega \end{cases}$$




At the limit, the two problems are uncoupled : shielding effect.

# I- Investigation of a 2D-model problem

## The Dirichlet case: idea of proof

Far field expansion: $u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, u_q^\pm(\mathbf{x})$

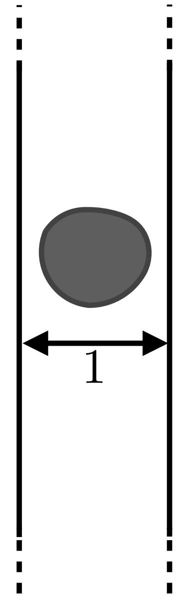
Near field expansion: $u^\delta = \sum_{q \in \mathbb{N}} \delta^q \, U_q(x_1, \frac{x_1}{\delta}, \frac{x_2}{\delta})$

Matching (order 0): $U_0 \sim u_0^\pm(x_1, 0)$ as $X_2 \to +\infty$

## The Dirichlet case: idea of proof

Near field equation of order 0:

$$(\mathcal{D})\begin{cases} -\Delta_{\mathbf{X}} U_0(x_1,\mathbf{X}) = 0 \quad \text{in } \mathcal{B} \\ U_0 = 0 \text{ on } \partial\widehat{\Omega}_{\text{hole}} \\ U_0 \ 1\text{-periodic} \\ U_0 \sim u_0^{\pm}(x_1,0) \ \text{ as } X_2 \to +\infty \end{cases}$$




The near field term $U_0$ is in the kernel $\mathcal{K}_d$ of the Laplacian operator with homogeneous Dirichlet boundary conditions.

$$\mathcal{K}_d = \operatorname{span}\{\mathcal{D}_1, \mathcal{D}_2\}$$

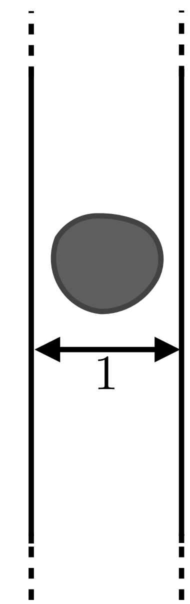
$$\mathcal{D}_1 \sim X_2 \qquad \mathcal{D}_2 \sim |X_2| \qquad \text{as} \quad X_2 \to \pm\infty$$

# I- Investigation of a 2D-model problem

## The Dirichlet case: idea of proof

Near field equation of order 0:

$$(\mathcal{D})\begin{cases} -\Delta_{\mathbf{X}} U_0(x_1, \mathbf{X}) = 0 \quad \text{in } \mathcal{B} \\ U_0 = 0 \text{ on } \partial\widehat{\Omega}_{\mathrm{hole}} \\ U_0 \text{ } 1\text{-periodic} \\ U_0 \sim u_0^{\pm}(x_1, 0) \quad \text{as } X_2 \to +\infty \end{cases}$$




$$U_0(x_1, \mathbf{x}) = \alpha_1(x_1)\,\mathcal{D}_1(\mathbf{X}) + \alpha_2(x_1)\mathcal{D}_2(\mathbf{X})$$

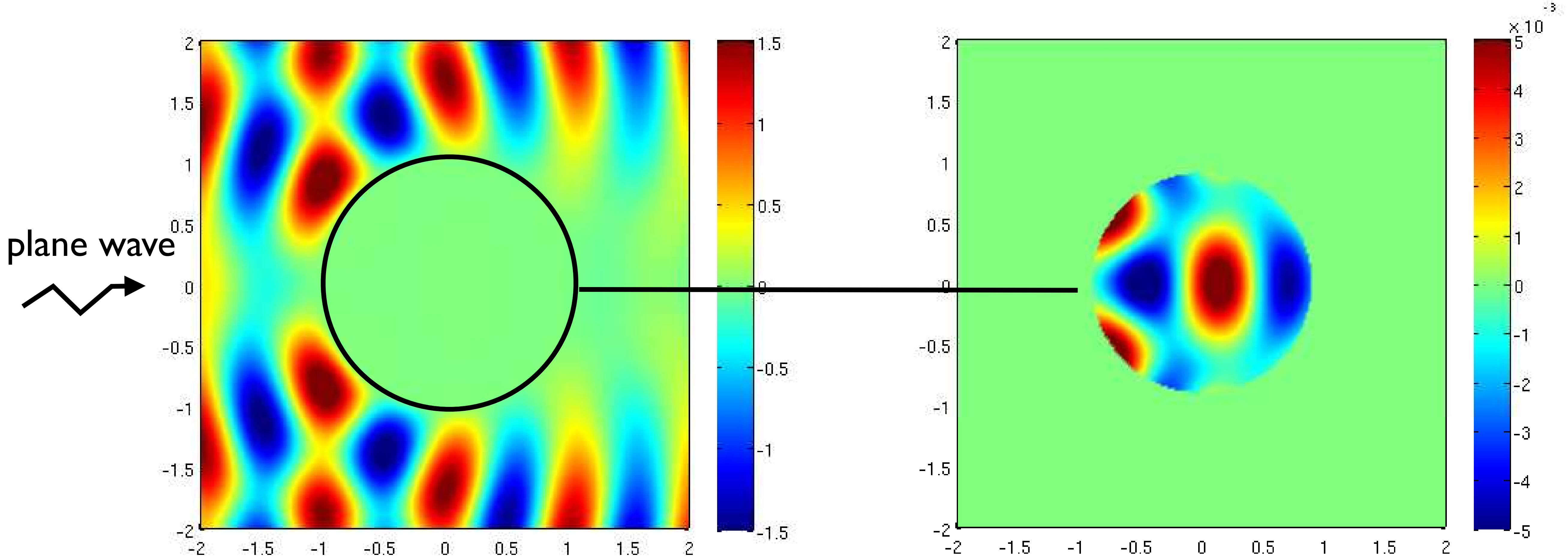
$$U_0(x_1, \mathbf{x}) \sim u_0(x_1, 0^{\pm}) \qquad X_2 \to \pm\infty$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = 0$$

$$\boxed{u_0(x_1, 0^{\pm}) = 0}$$

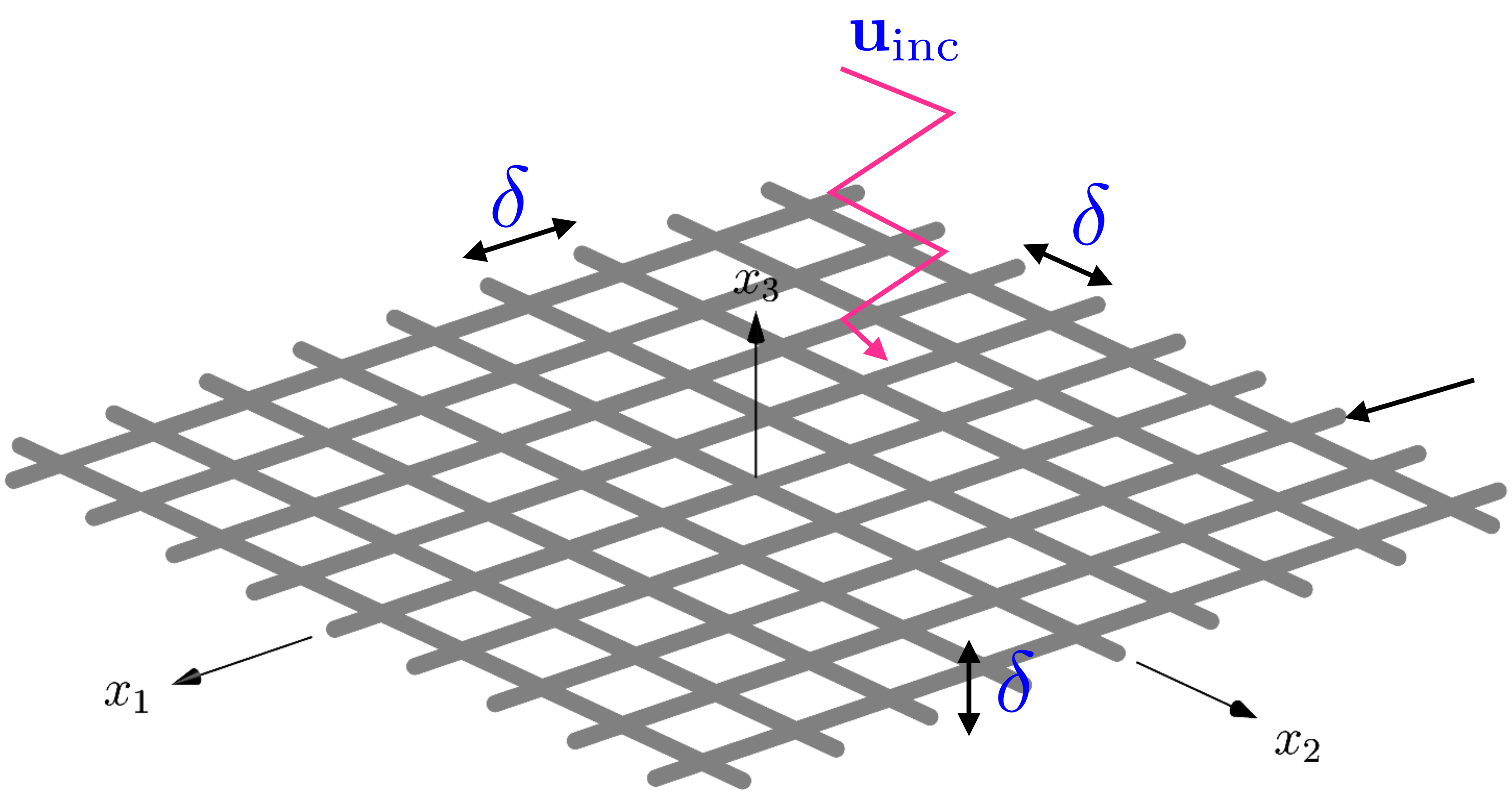
# I- Investigation of a 2D-model problem

## The Dirichlet case: numerical illustration



# 3- 3D time-harmonic Maxwell's equations

## Presentation of the problem




thin and periodic interface (meta-surface)

Main features of the meta-surface:

- set of equi-spaced metal obstacles
- periodic of period $\delta$ w.r.t $x_1$ and $x_2$
- thickness $\delta$

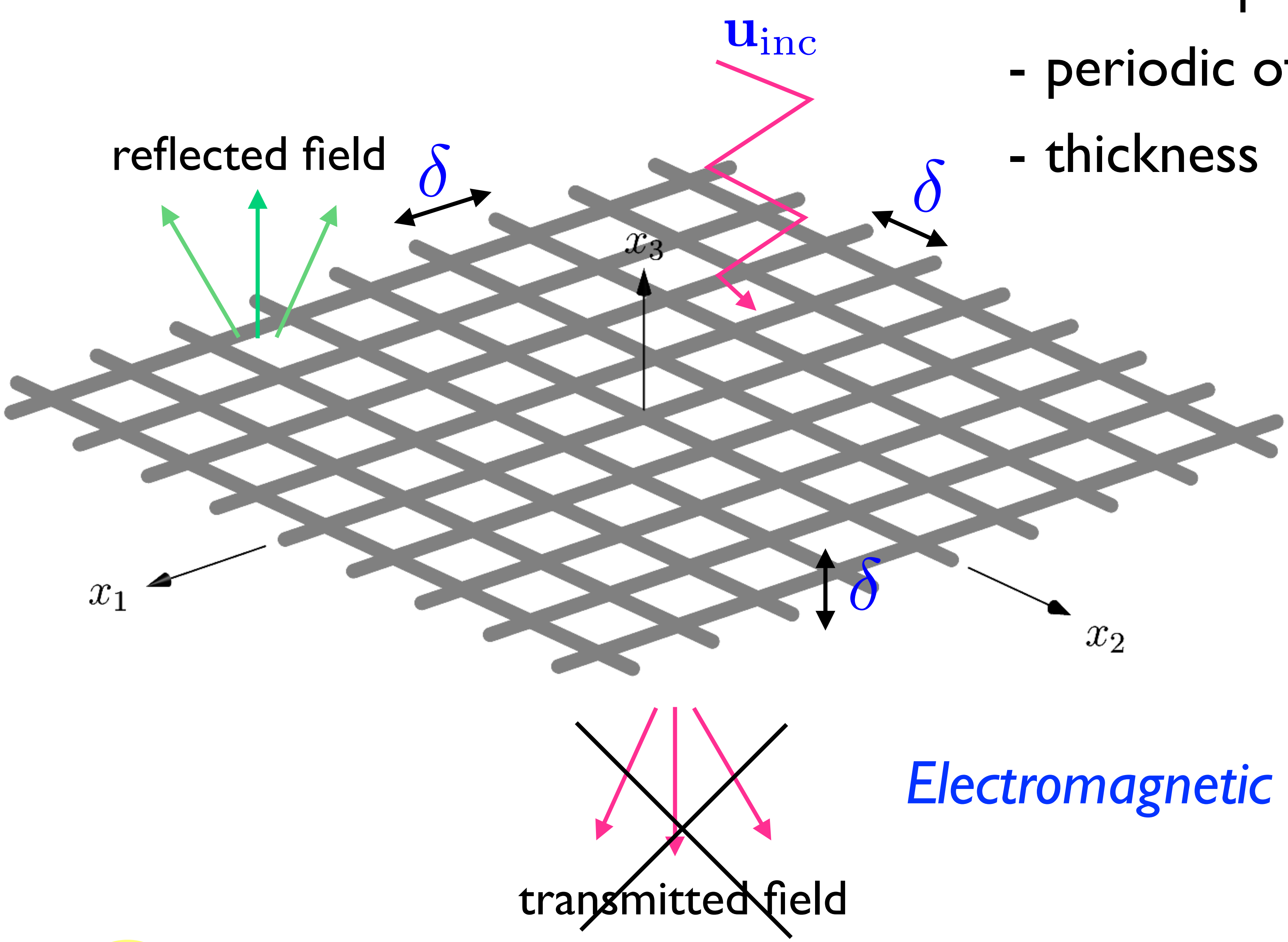
$\delta$ small

# 3- 3D time-harmonic Maxwell's equations

## Presentation of the problem

Main features of the meta-surface:

- set of equi-spaced metal obstacles
- periodic of period $\delta$ w.r.t $x_1$ and $x_2$
- thickness $\delta$



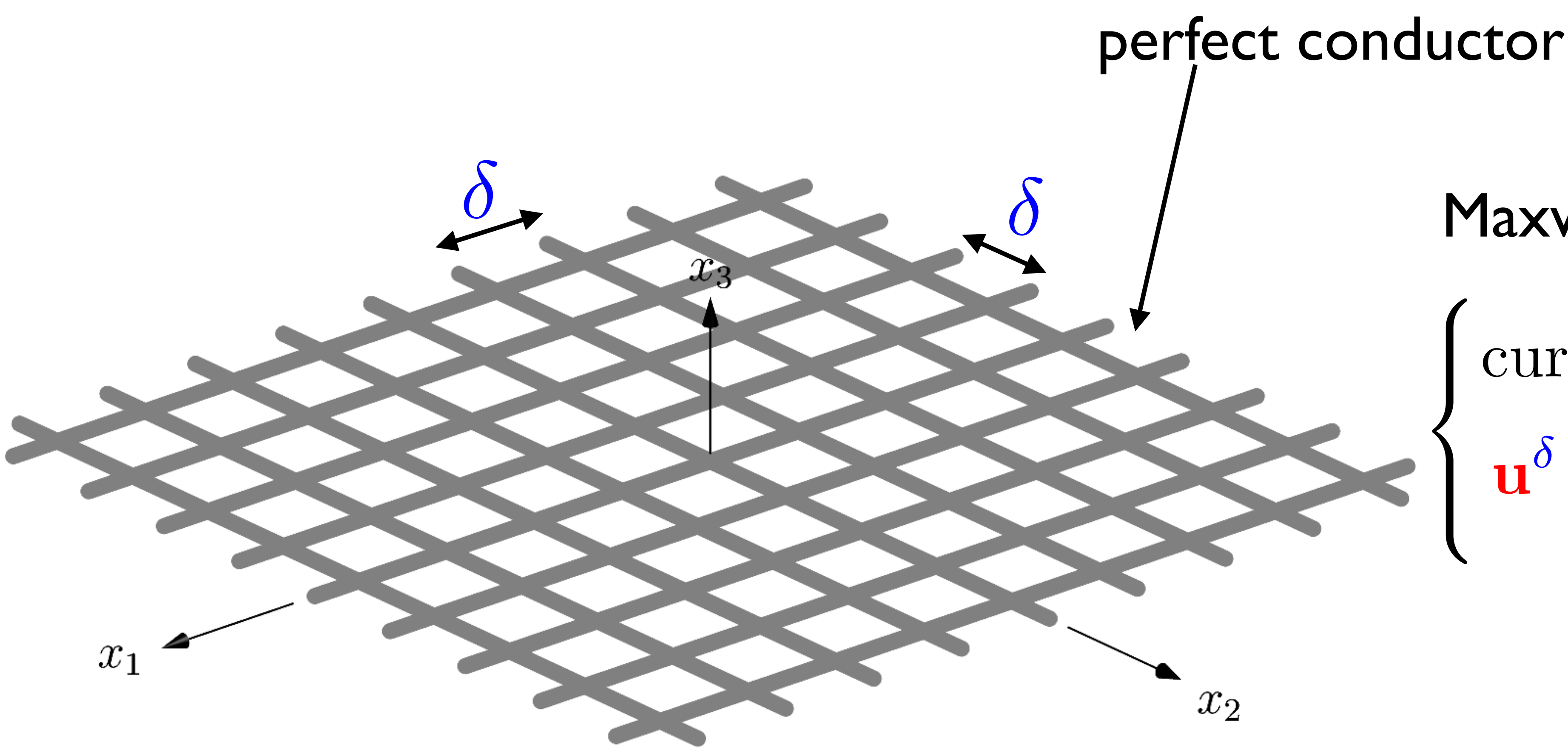

*Electromagnetic shielding*

?

Behavior of the electromagnetic field as $\delta$ tend to $0$.

# 3- 3D time-harmonic Maxwell's equations

## Presentation of the problem




Maxwell's equations

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u}^\delta - \omega^2\varepsilon\mathbf{u}^\delta = \mathbf{f} \\ \mathbf{u}^\delta \times \mathbf{n} = 0 \end{cases}$$

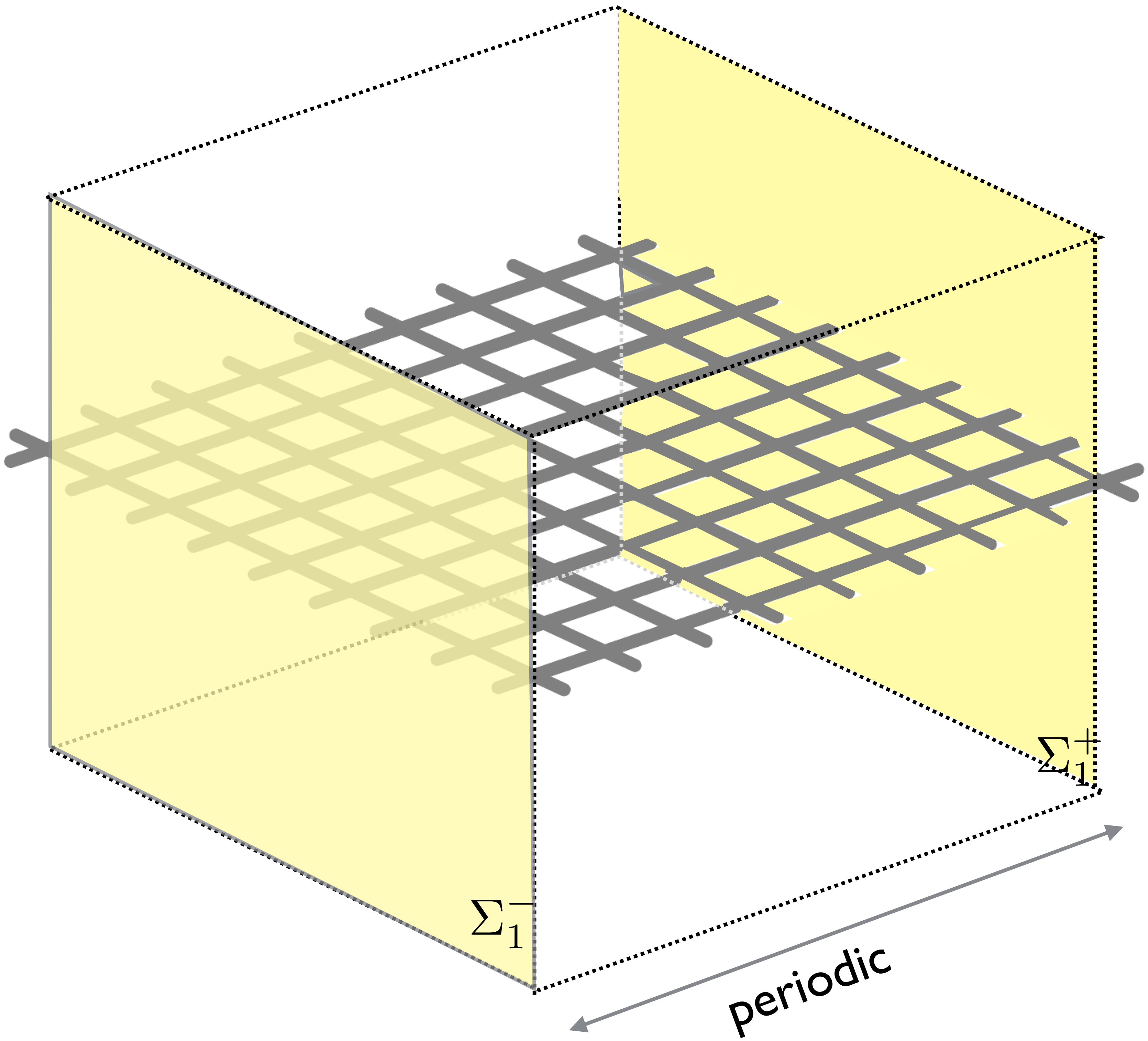
Assumptions:

the source term is supported away from the thin periodic interface

$\operatorname{Im}\varepsilon > 0 \quad \operatorname{Re}\varepsilon > 0$

# 3- 3D time-harmonic Maxwell's equations

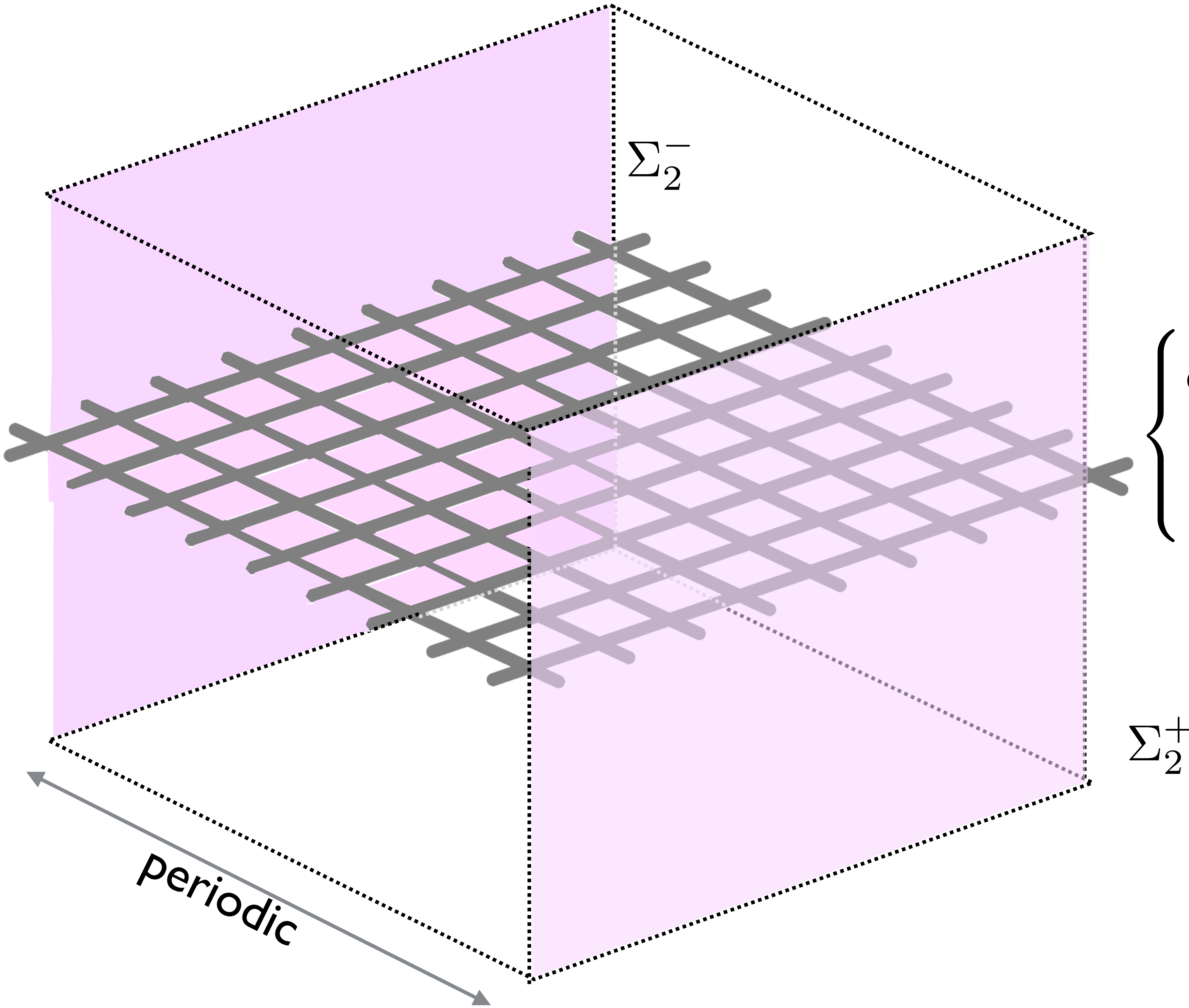
## Presentation of the problem




$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u}^\delta - \omega^2 \varepsilon \mathbf{u}^\delta = \mathbf{f} \\ \mathbf{u}^\delta \times \mathbf{n} = 0 \quad \text{on } \partial\Omega^\delta \end{cases}$$

# 3- 3D time-harmonic Maxwell's equations

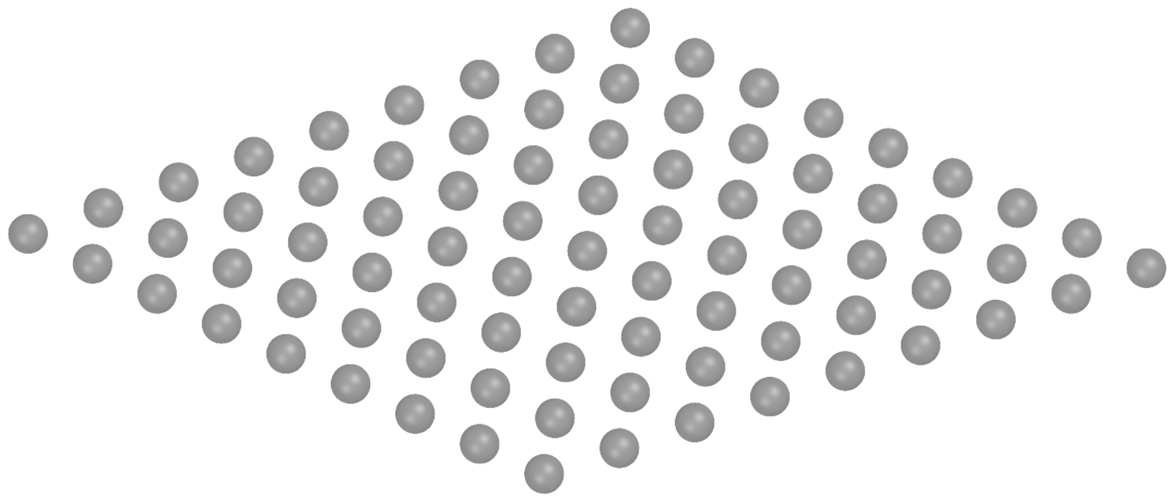
## Presentation of the problem



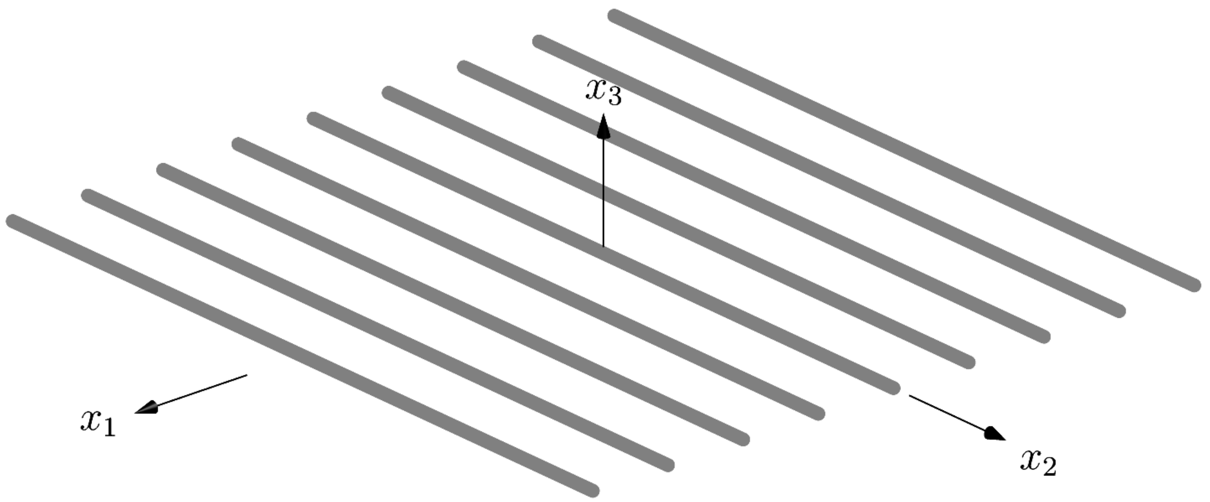

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u}^\delta - \omega^2 \varepsilon \mathbf{u}^\delta = \mathbf{f} \text{ in} \\ \mathbf{u}^\delta \times \mathbf{n} = 0 \quad \text{on } \partial\Omega^\delta \end{cases}$$

# 3- 3D time-harmonic Maxwell's equations

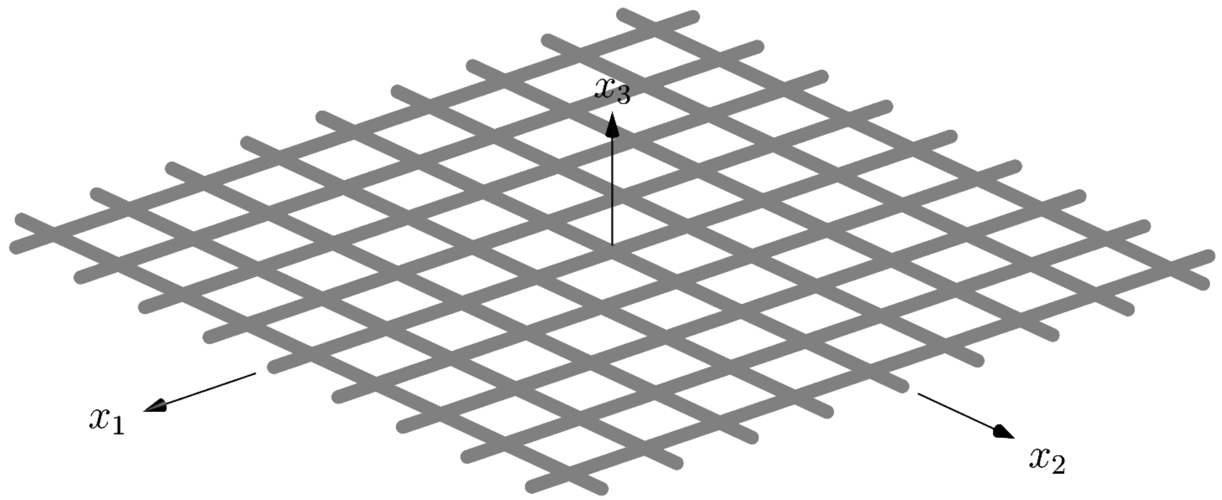
## Presentation of the problem



case 1



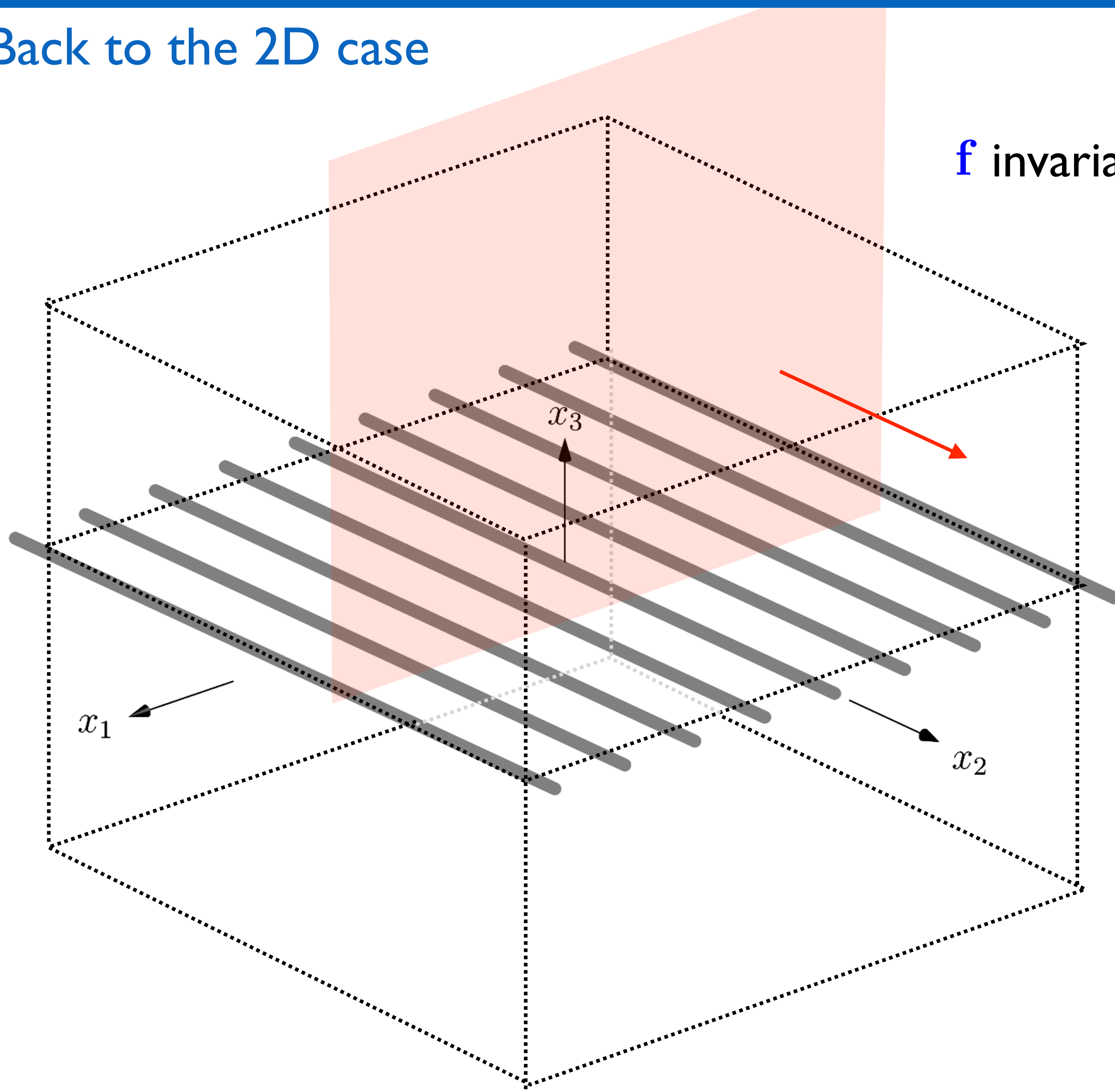

case 2




case 3

# 3- 3D time-harmonic Maxwell's equations

## Back to the 2D case




$\mathbf{f}$ invariant with respect to $x_2$

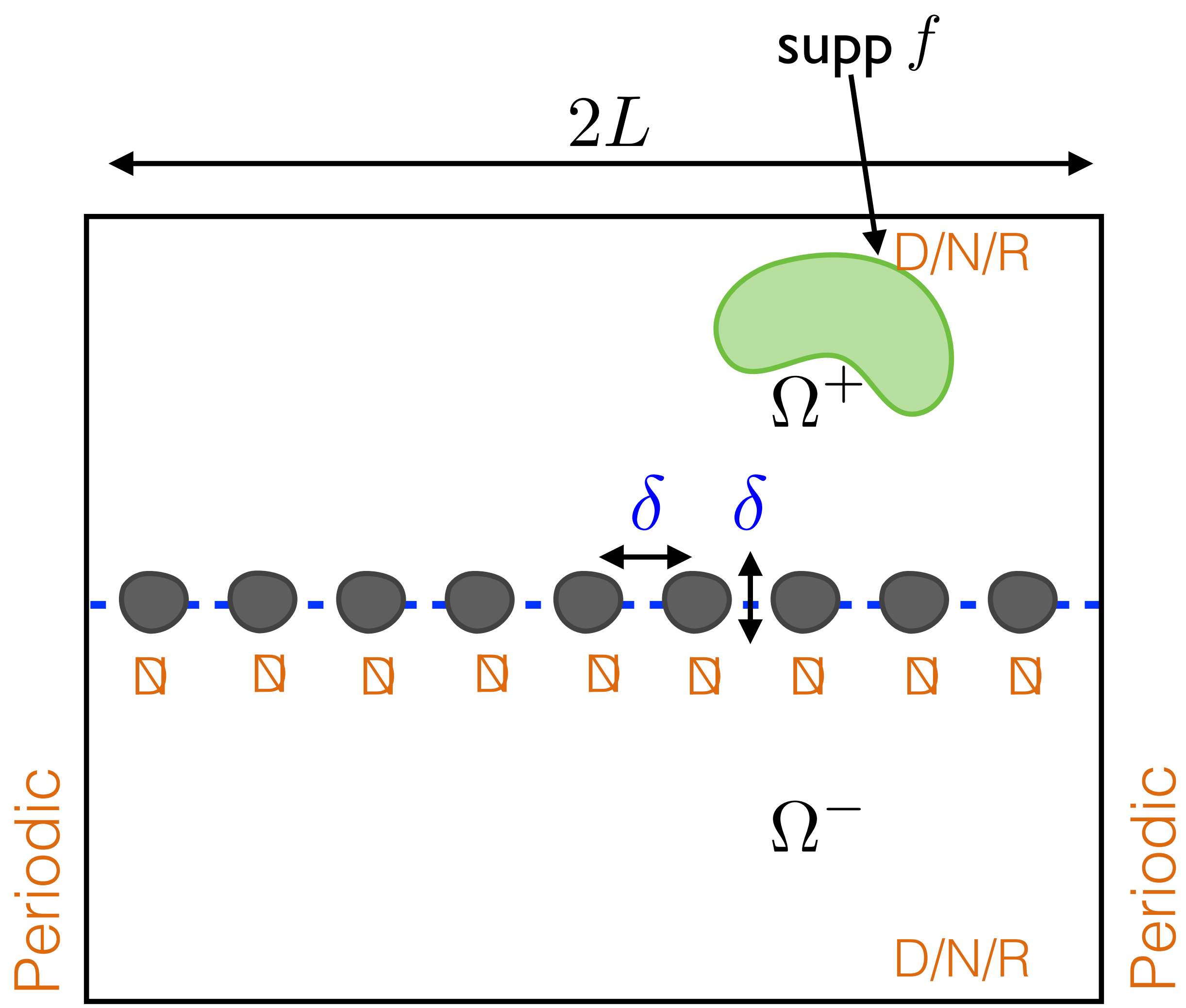
$$\mathbf{h}^\delta = \frac{1}{\mathrm{i}\omega}\,\mathbf{curl}\,\mathbf{u}^\delta$$

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}^\delta - \omega^2 \varepsilon \mathbf{u}^\delta = \mathbf{f} \text{ in } \Omega^\delta \\ \mathbf{u}^\delta \times \mathbf{n} = 0 \quad \text{on } \partial\Omega^\delta \end{cases}$$

$$\begin{cases} (\mathbf{u}^\delta)_2 = 0 \quad \text{on } \partial\Omega^\delta \\ \partial_{\mathbf{n}}(\mathbf{h}^\delta)_2 = 0 \quad \text{on } \partial\Omega^\delta \end{cases}$$

→ 2 uncoupled bi-dimensional Helmholtz problems for $(\mathbf{u}^\delta)_2$ and $(h^\delta)_2$ .

## Back to the 2D case




$$\begin{cases} -\Delta u^\delta - \omega^2 u^\delta = f \text{ in } \Omega^\delta \\ u^\delta = 0 \quad \text{or } \partial_n u^\delta = 0 \quad \text{on } \partial\Omega^\delta \text{ in } \Omega^\delta \end{cases}$$

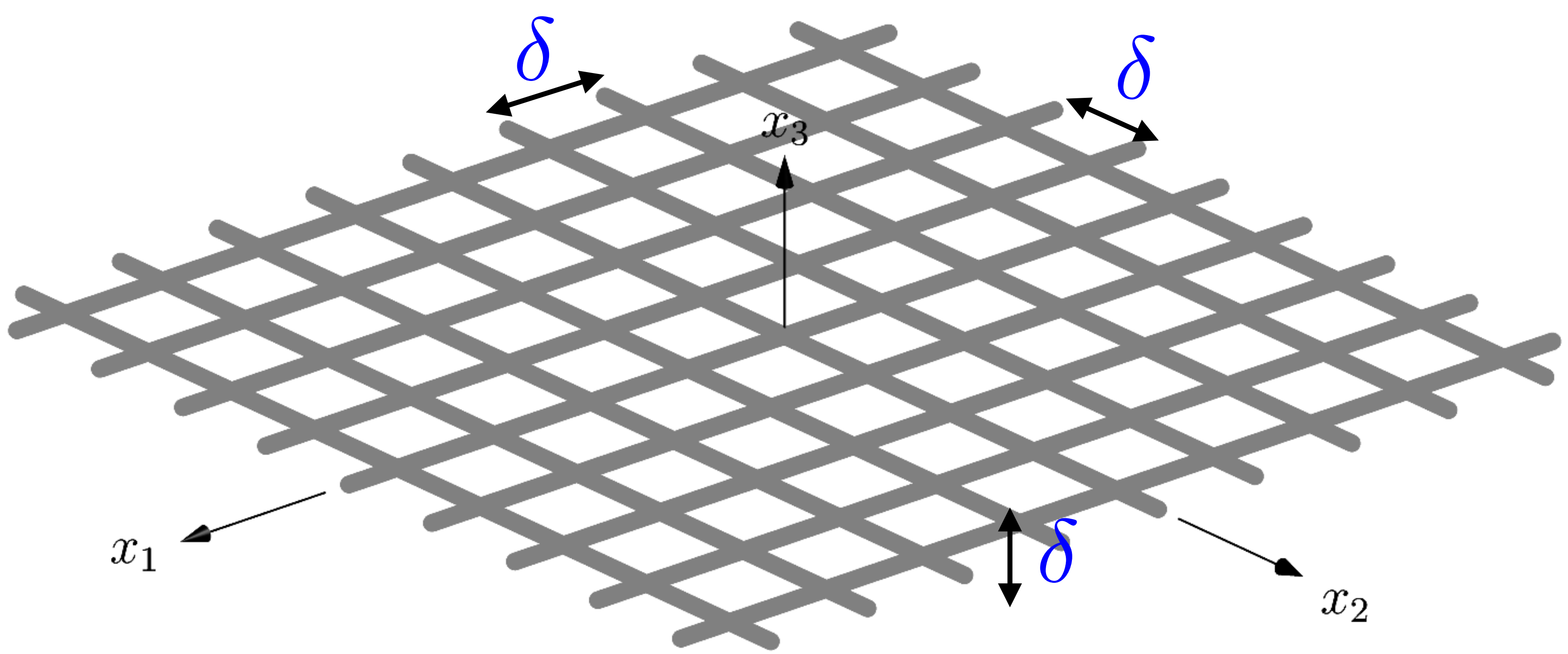
$$\Omega^\delta = \Omega \setminus \overline{\Omega^\delta_{\text{hole}}}$$

✓ The limit depends on the geometry of the meta-surface

# 3- 3D time-harmonic Maxwell's equations

## Asymptotic expansion

$$\mathbf{h}^\delta = \frac{1}{\mathrm{i}\omega}\,\mathbf{curl}\,\mathbf{u}^\delta$$

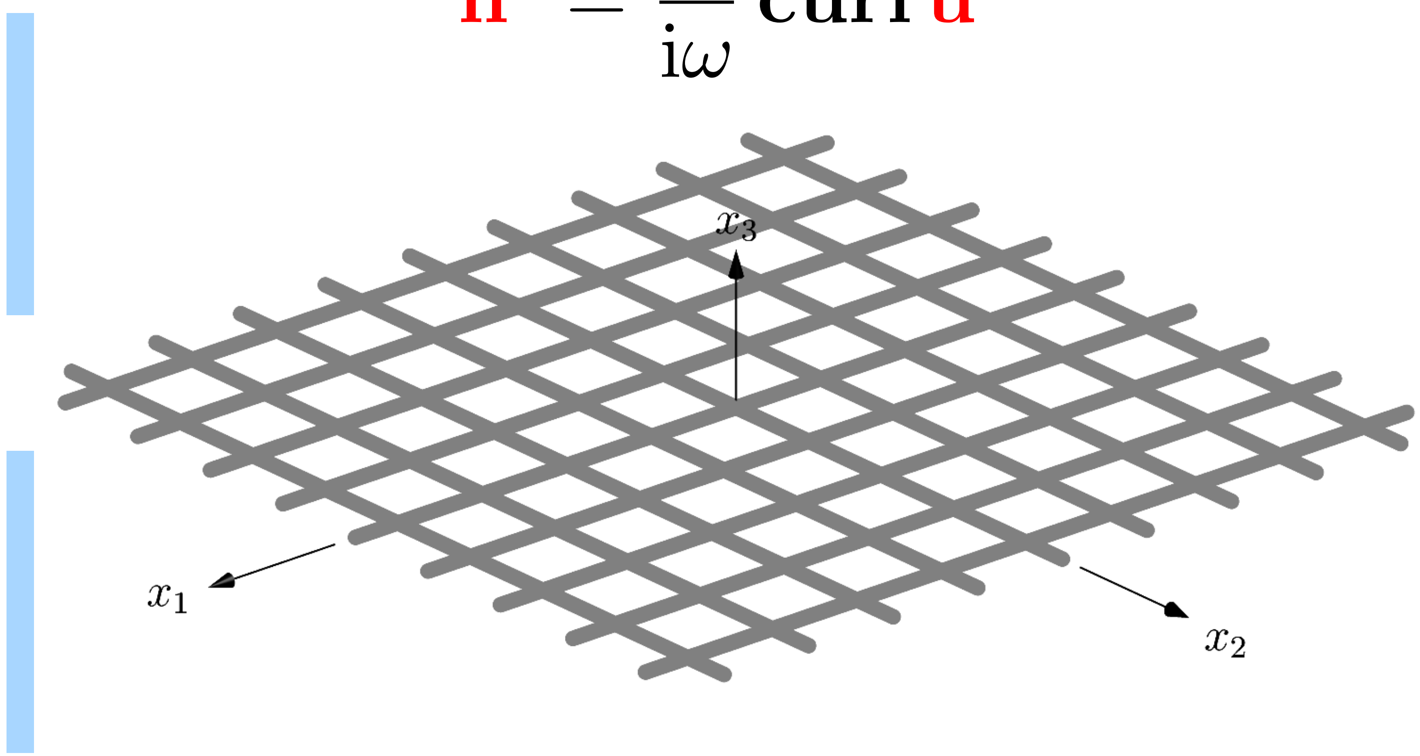



$$\begin{cases} -\mathrm{i}\omega\mathbf{h}^\delta + \mathbf{curl}\,\mathbf{u}^\delta = 0 & \text{in } \Omega^\delta, \\ -\mathrm{i}\omega\mathbf{u}^\delta - \mathbf{curl}\,\mathbf{h}^\delta = -\dfrac{1}{\mathrm{i}\omega}\mathbf{f} & \text{in } \Omega^\delta, \end{cases} \qquad \mathbf{u}^\delta \times \mathbf{n} = 0 \text{ and } \mathbf{h}^\delta \cdot \mathbf{n} = 0 \text{ on } \Gamma^\delta.$$

# 3- 3D time-harmonic Maxwell's equations

## Asymptotic expansion

$$\mathbf{h}^{\delta} = \frac{1}{\mathrm{i}\omega}\,\mathbf{curl}\,\mathbf{u}^{\delta}$$

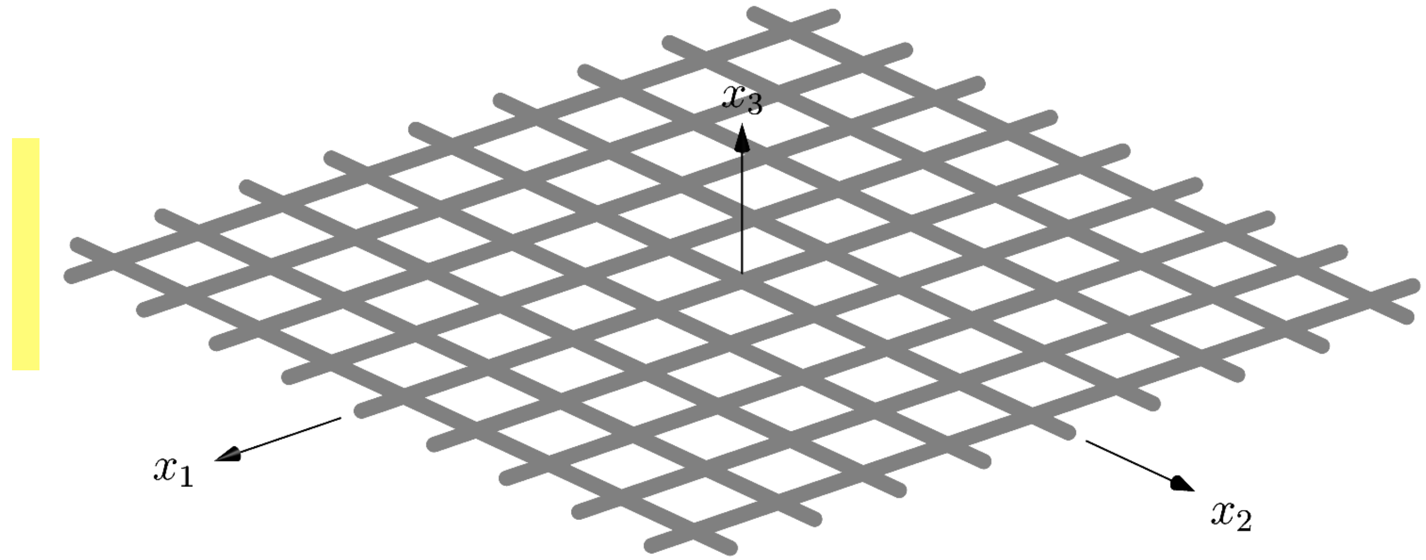



✓ Far from the meta-surface (above and below the grating)

$$\mathbf{u}^{\delta} = \sum_{q\,\in\,\mathbb{N}} \delta^{q}\ \mathbf{u}_{q}(\mathbf{x}) \qquad \mathbf{h}^{\delta} = \sum_{q\,\in\,\mathbb{N}} \delta^{q}\ \mathbf{h}_{q}(\mathbf{x})$$

# 3- 3D time-harmonic Maxwell's equations

## Asymptotic expansion



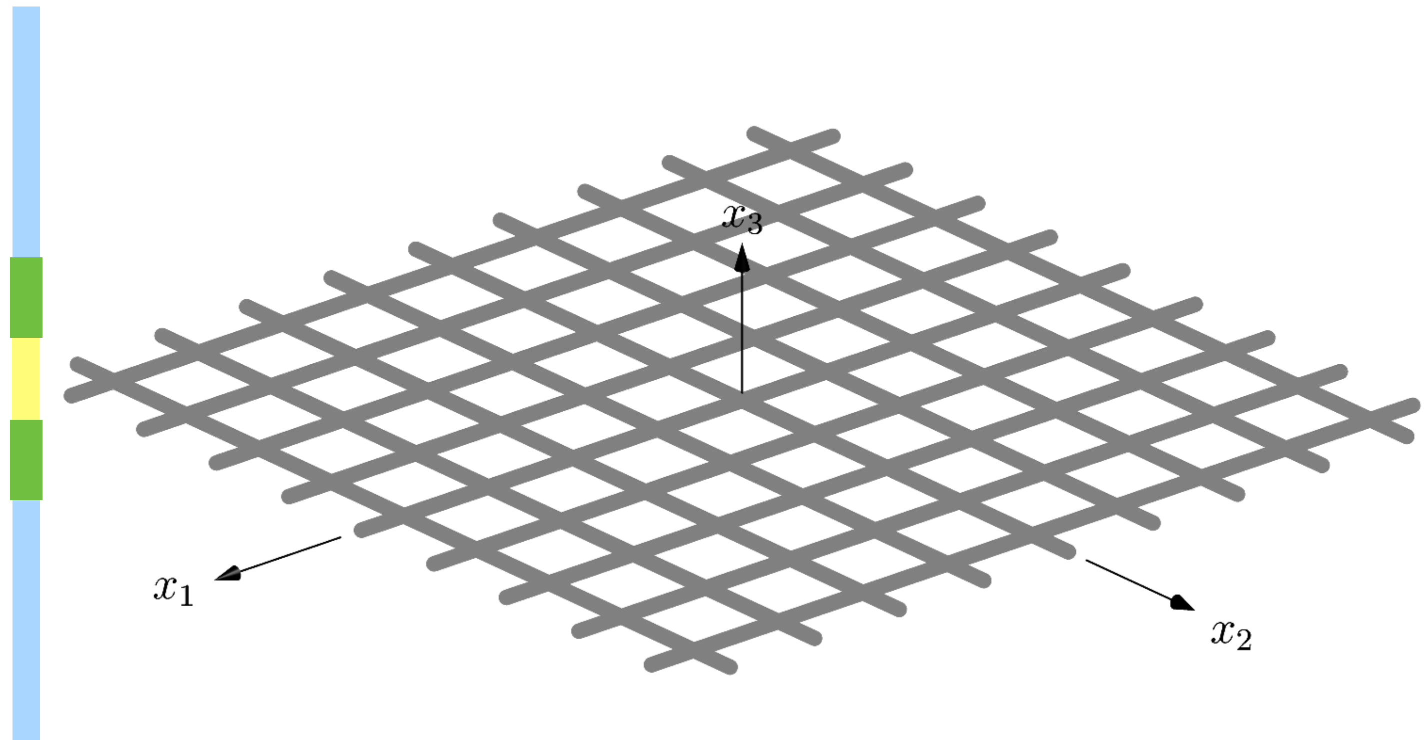

✓ In the neighborhood of the meta-surface

$$\mathbf{u}^\delta = \sum_{q \in \mathbb{N}} \delta^q \, \mathbf{U}_q(\overbrace{x_1, x_2}^{\text{slow variations}}, \underbrace{\frac{x_1}{\delta}, \frac{x_2}{\delta}}_{\text{1-periodic w.r.t } X_1 \text{ and } X_2}, \frac{x_3}{\delta}) \qquad \mathbf{h}^\delta = \sum_{q \in \mathbb{N}} \delta^q \, \mathbf{H}_q(x_1, x_2, \frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta})$$

# 3- 3D time-harmonic Maxwell's equations

## Asymptotic expansion

Matching zones




✓ Matching zones: far and near field expansions coincide in some intermediate areas

$$\lim_{x_3 \to 0^\pm} \mathbf{u}_0 = \lim_{X_3 \to \pm\infty} \mathbf{U}_0$$

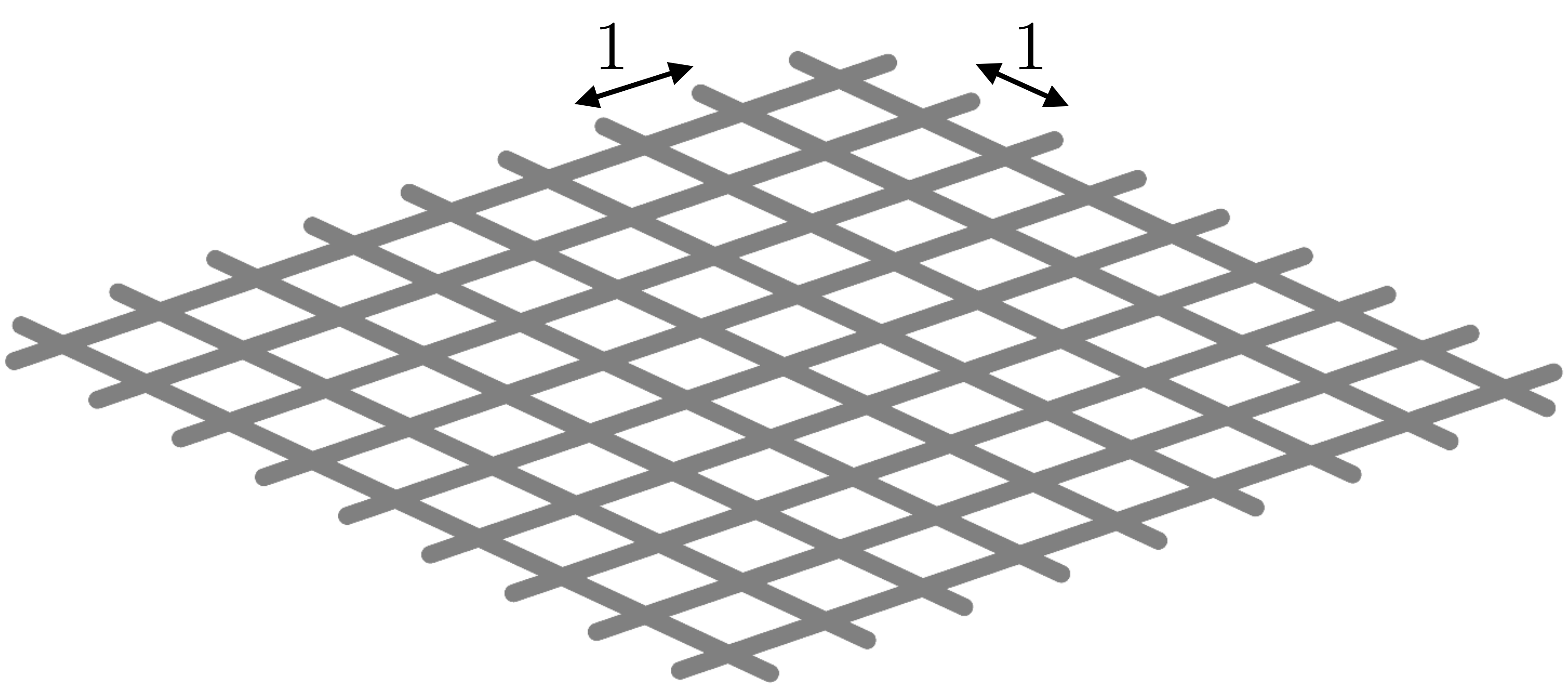
$$\lim_{x_3 \to 0^\pm} \mathbf{h}_0 = \lim_{X_3 \to \pm\infty} \mathbf{H}_0$$

# 3- 3D time-harmonic Maxwell's equations

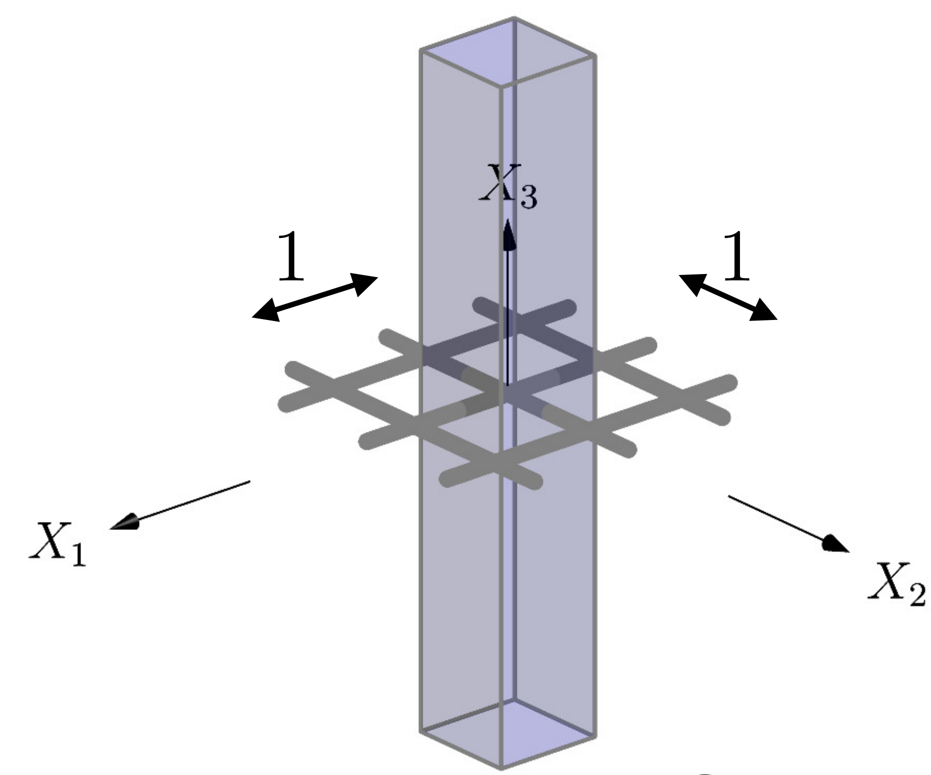
## Asymptotic expansion: limit near field problem

$$\begin{cases} \mathbf{curl}_X\, \mathbf{U}_0 = 0 & \text{in } \mathscr{B}_\infty, \\ \mathrm{div}_X\, \mathbf{U}_0 = 0 & \text{in } \mathscr{B}_\infty, \\ \mathbf{U}_0 \times \mathbf{n} = 0 & \text{on } \partial\mathscr{B}_\infty, \end{cases} \qquad \begin{cases} \mathbf{curl}_X\, \mathbf{H}_0 = 0 & \text{in } \mathscr{B}_\infty, \\ \mathrm{div}_X\, \mathbf{H}_0 = 0 & \text{in } \mathscr{B}_\infty, \\ \mathbf{H}_0 \cdot \mathbf{n} = 0 & \text{on } \partial\mathscr{B}_\infty, \end{cases}$$

→ Electrostatic kind problem *(Ciarlet 04)*




Normalized infinite domain $\mathscr{B}_\infty$




Periodicity cell $\mathscr{B}$

# 3- 3D time-harmonic Maxwell's equations

## Near field problem for $\mathbf{U}_0$ :

✓ functional space

$$\mathscr{H}_N(\mathscr{B}_\infty) = \{\mathbf{u} \in H_{\text{loc}}(\text{curl};\mathscr{B}_\infty) \cap H_{\text{loc}}(\text{div};\mathscr{B}_\infty) : \boxed{\mathbf{u} \text{ is 1-periodic in } X_1 \text{ and } X_2,} \\ \frac{\mathbf{u}_{|\mathscr{B}}}{\sqrt{1+(X_3)^2}} \in (L^2(\mathscr{B}))^3, \quad \text{curl}\mathbf{u}_{|\mathscr{B}} \in (L^2(\mathscr{B}))^3, \quad \text{div}\,\mathbf{u}_{|\mathscr{B}} \in L^2(\mathscr{B}), \quad \boxed{\mathbf{u}\times\mathbf{n} = 0} \text{ on } \partial\mathscr{B}_\infty\}$$

✓ Investigation of the space $K_N$

$$K_N = \{\mathbf{u} \in \mathscr{H}_N(\mathscr{B}_\infty), \boxed{\text{curl}\mathbf{u} = 0}, \boxed{\text{div}\mathbf{u} = 0}\}\,.$$

*(Monk, Girault-Raviart, Amrouche-Bernardi-Dauge-Girault, Gramain)*

$$\boxed{\mathbf{U}_0 \in K_N}$$

# 3- 3D time-harmonic Maxwell's equations

## Near field problem for $\mathbf{H}_0$ :

- functional space

$$\mathscr{H}_T(\mathscr{B}_\infty) = \{\mathbf{h} \in H_{\text{loc}}(\text{curl};\mathscr{B}_\infty) \cap H_{\text{loc}}(\text{div};\mathscr{B}_\infty) : \boxed{\mathbf{h} \text{ is 1-periodic in } X_1 \text{ and } X_2},$$

$$\frac{\mathbf{h}_{|\mathscr{B}}}{\sqrt{1+(X_3)^2}} \in (L^2(\mathscr{B}))^3, \quad \text{curl}\mathbf{h}_{|\mathscr{B}} \in (L^2(\mathscr{B}))^3, \quad \text{div}\,\mathbf{h}_{|\mathscr{B}} \in L^2(\mathscr{B}), \quad \boxed{\mathbf{h}\cdot\mathbf{n} = 0} \text{ on } \partial\mathscr{B}_\infty\},$$

- Investigation of the space $K_T$

$$K_T = \{\mathbf{h} \in \mathscr{H}_T(\mathscr{B}_\infty), \boxed{\text{curl}\mathbf{h} = 0}, \boxed{\text{div}\mathbf{h} = 0}\}$$

*(Monk, Girault-Raviart, Amrouche-Bernardi-Dauge-Girault, Gramain)*

$$\boxed{\mathbf{H}_0 \in K_T}$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$

$$K_N = \left\{u \in V_{0,per}(\mathscr{B}_\infty), \mathrm{curl}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty)^3, \mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty)\right\}$$

$\mathrm{curl}\,\mathbf{u} = 0$

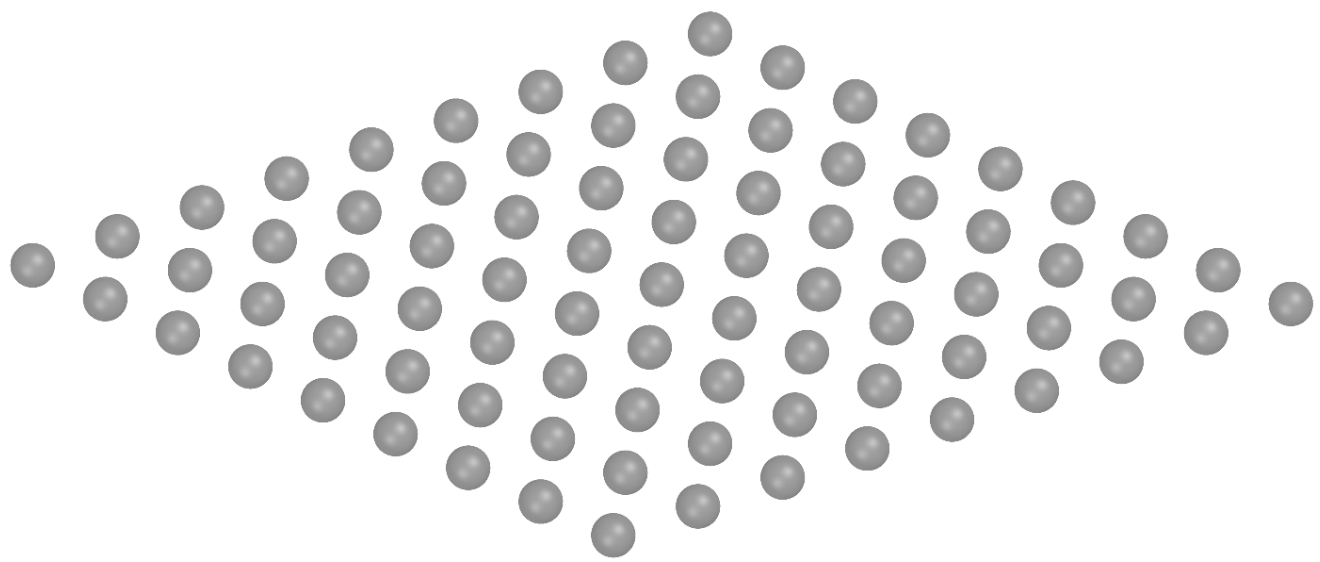
$\mathbf{u} \times \mathbf{n} = 0$ on $\partial\mathscr{B}_\infty$

$\Rightarrow$

$\mathbf{u} = \nabla p \qquad \mathrm{div}\,\mathbf{u} = 0 \Rightarrow -\Delta p = 0$

$p = c_e$ on each connected component of $\partial\mathscr{B}_\infty$

*(Monk,Amrouche-Bernardi-Dauge-Girault)*



case 1: one constant $c_{ij}$ for each ball

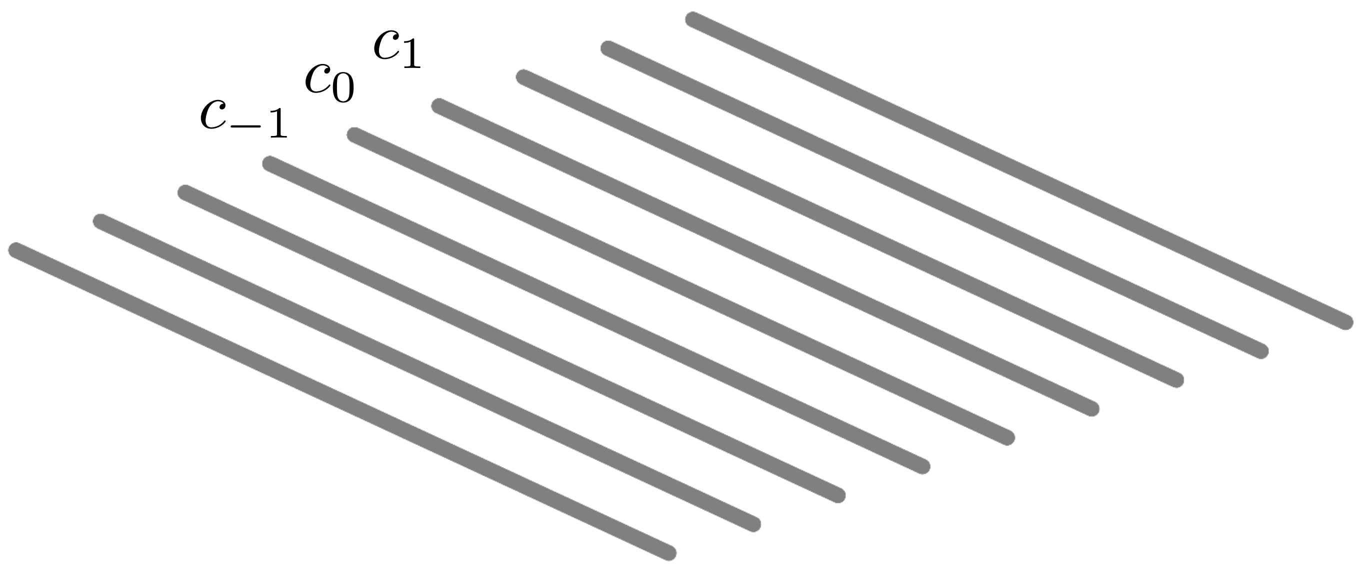
# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$

$$K_N = \left\{ u \in V_{0,per}(\mathscr{B}_\infty), \mathrm{curl}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty)^3, \mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty) \right\}$$

$$\left.\begin{array}{l} \mathrm{curl}\,\mathbf{u} = 0 \\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\mathscr{B}_\infty \end{array}\right| \Rightarrow \begin{array}{l} \mathbf{u} = \nabla p \qquad \mathrm{div}\,\mathbf{u} = 0 \Rightarrow -\Delta p = 0 \\ p = c_e \text{ on each connected component of } \partial\mathscr{B}_\infty \end{array}$$

*(Monk,Amrouche-Bernardi-Dauge-Girault)*




case 2: one constant $c_i$ on each 'line'

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$

$$K_N = \{u \in V_{0,per}(\mathscr{B}_\infty), \mathrm{curl}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty)^3, \mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathscr{B}_\infty)\}$$

$\mathrm{curl}\,\mathbf{u} = 0$

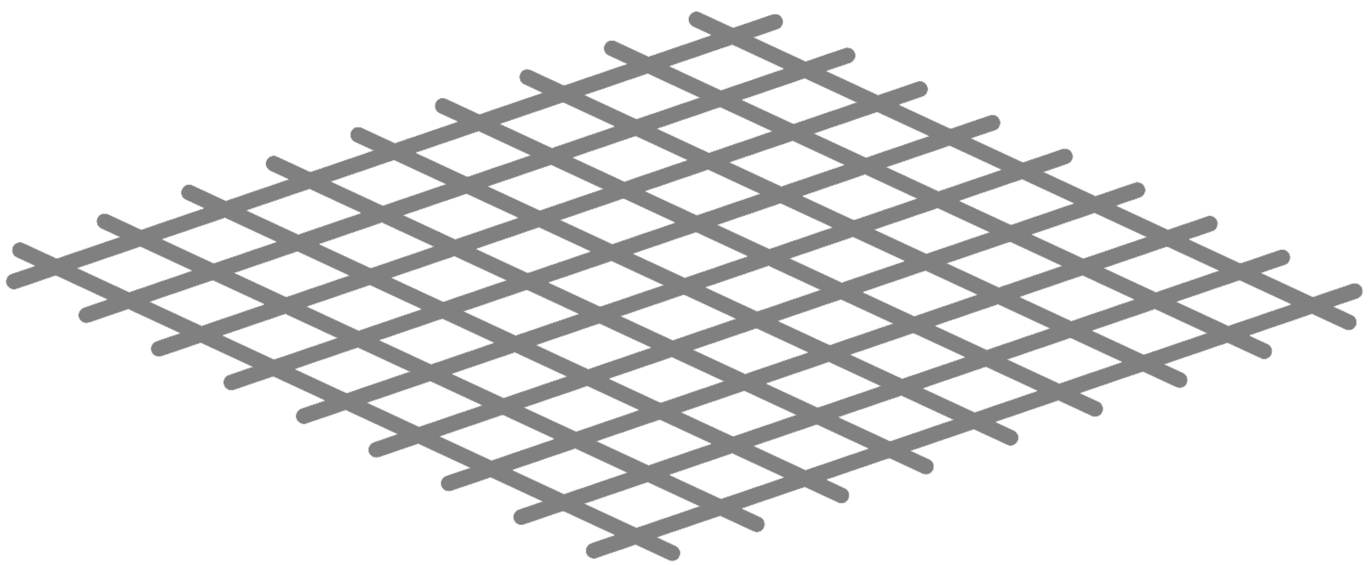
$\mathbf{u} \times \mathbf{n} = 0$ on $\partial\mathscr{B}_\infty$

$\Rightarrow$

$\mathbf{u} = \nabla p \qquad \mathrm{div}\,\mathbf{u} = 0 \;\Rightarrow\; -\Delta p = 0$

$p = c_e$ on each connected component of $\partial\mathscr{B}_\infty$

*(Monk,Amrouche-Bernardi-Dauge-Girault)*



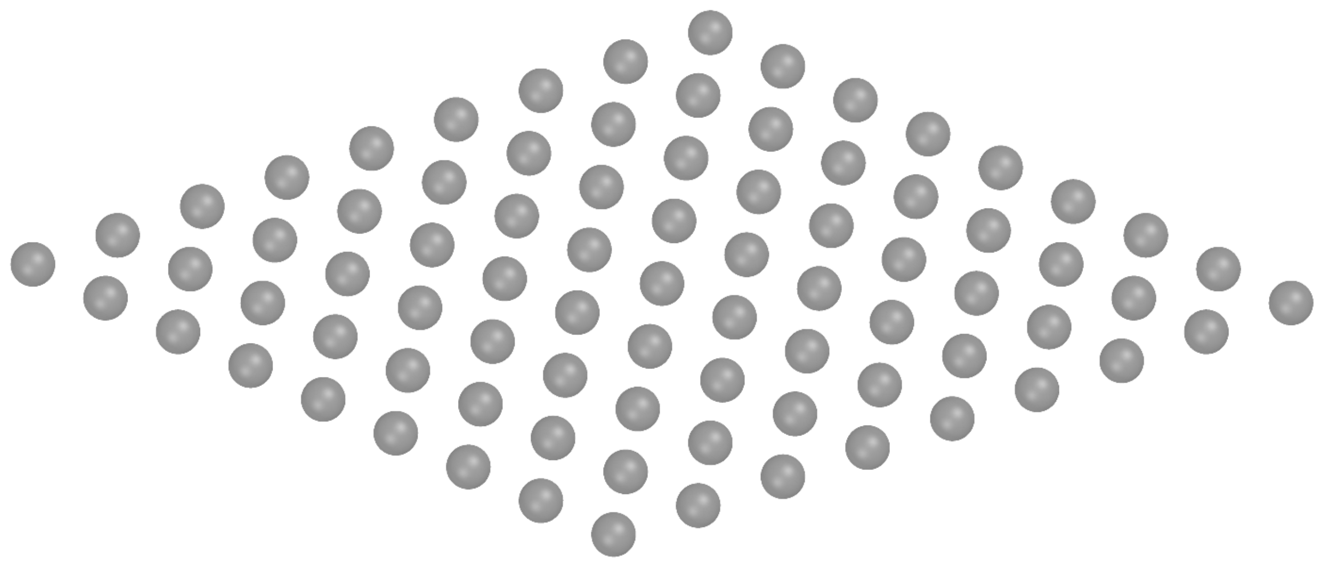
case 3: one constant $c$ for the whole structure

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$ : case I

$\mathbf{u} = \nabla p$ , $\nabla p$ periodic

There are two constants $\alpha_1$ and $\alpha_2$ s.t $\tilde{p} = p - \alpha_1 X_1 - \alpha_2 X_2$ is periodic



case I: one constant $c_{ij}$ for each ball

$\tilde{p}$ periodic $\Rightarrow$

$$\left| \begin{array}{l} c_{(i+1)j} - \alpha_1(X_1+1) - \alpha_2 X_2 = c_{ij} - \alpha_1 X_1 - \alpha_2 X_2 \\ c_{i(j+1)} - \alpha_1 X_1 - \alpha_2(X_2+1) = c_{ij} - \alpha_1 X_1 - \alpha_2 X_2 \end{array} \right.$$

$$\Rightarrow \quad c_{ij} = c_{00} + \alpha_1 i + \alpha_2 j$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$ : case I

$\mathbf{u} = \nabla p$ , $\nabla p$ periodic

$\Rightarrow$ There are two constants $\alpha_1$ and $\alpha_2$ s.t $\tilde{p} = p - \alpha_1 X_1 - \alpha_2 X_2$ is periodic

$$-\Delta p = 0 \quad \Rightarrow \quad \tilde{p}_{|\mathscr{B}} = \alpha_1 \widetilde{\mathscr{D}_{X_1}} + \alpha_2 \widetilde{\mathscr{D}_{X_2}} + \beta_1 \mathscr{D}_1 + \beta_2 \mathscr{D}_2 + \beta_3$$

$$\begin{cases} -\Delta \widetilde{\mathscr{D}_{X_i}} = 0 & \text{in } \mathscr{B} \\ \widetilde{\mathscr{D}_{X_i}} = -X_i & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \widetilde{\mathscr{D}_{X_i}} \sim c^{\pm}_{X_i} & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \mathscr{D}_1 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_1 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_1 \sim X_3 & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \mathscr{D}_2 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_2 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_2 \sim |X_3| & \text{as } X_3 \to +\infty \end{cases}$$

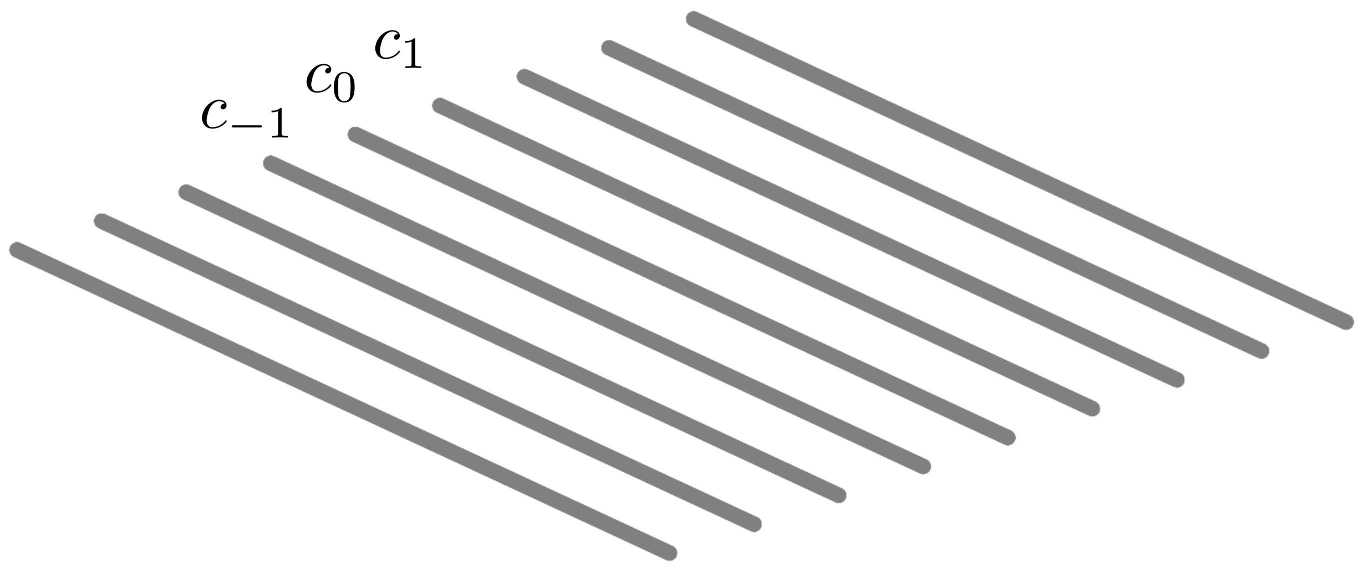
$$\Rightarrow \mathbf{u} = \nabla p \sim \left| \begin{array}{l} \alpha_1 \\ \alpha_2 \\ \beta_1 \pm \beta_2 \end{array} \right. \qquad \text{as } X_3 \to \pm\infty$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$: case 2

$\mathbf{u} = \nabla p$ , $\nabla p$ periodic

$\Rightarrow$ There are two constants $\alpha_1$ and $\alpha_2$ s.t $\tilde{p} = p - \alpha_1 X_1 - \alpha_2 X_2$ is periodic




case 2: one constant $c_i$ on each 'line'

$\tilde{p}$ periodic $\Rightarrow$
$$\left| \begin{array}{l} c_i - \alpha_1 X_1 - \alpha_2 (X_2 + 1) = c_i - \alpha_1 X_1 - \alpha_2 X_2 \\ c_{i+1} - \alpha_1 (X_1 + 1) - \alpha_2 X_2 = c_i - \alpha_1 X_1 - \alpha_2 X_2 \end{array} \right.$$

$$\Rightarrow \quad \alpha_2 = 0 \qquad c_i = c_0 + \alpha_1 i$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$ : case 2

$\mathbf{u} = \nabla p$ , $\nabla p$ periodic

$\Rightarrow$ There is a constant $\alpha_1$ s.t $\tilde{p} = p - \alpha_1 X_1$ is periodic

$$-\Delta p = 0 \quad \Rightarrow \quad \tilde{p}_{|\mathscr{B}} = \alpha_1 \widetilde{\mathscr{D}_{X_1}} + \beta_1 \mathscr{D}_1 + \beta_2 \mathscr{D}_2 + \beta_3$$

$$\begin{cases} -\Delta \widetilde{\mathscr{D}_{X_i}} = 0 & \text{in } \mathscr{B} \\ \widetilde{\mathscr{D}_{X_i}} = -X_i & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \widetilde{\mathscr{D}_{X_i}} \sim c^{\pm}_{X_i} & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \mathscr{D}_1 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_1 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_1 \sim X_3 & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \mathscr{D}_2 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_2 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_2 \sim |X_3| & \text{as } X_3 \to +\infty \end{cases}$$

exponentially decaying in $X_3$

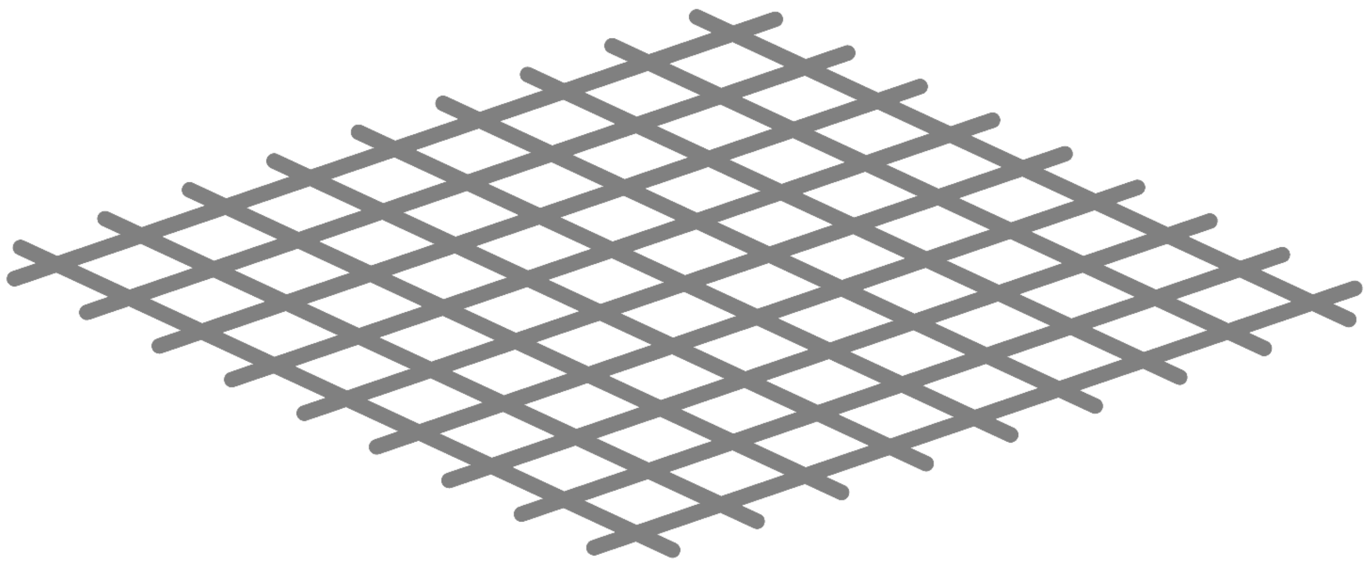
$$\Rightarrow \quad \mathbf{u} = \nabla p \sim \left| \begin{matrix} \alpha_1 \\ 0 \\ \beta_1 \pm \beta_2 \end{matrix} \right. \qquad \text{as } X_3 \to \pm\infty$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_N$ : case 3

$\mathbf{u} = \nabla p$ , $\nabla p$ periodic

There are two constants $\alpha_1$ and $\alpha_2$ s.t $\tilde{p} = p - \alpha_1 X_1 - \alpha_2 X_2$ is periodic



case 3: one constant $c$ for the whole structure

$\tilde{p}$ periodic $\Rightarrow$ $\alpha_2 = 0$ and $\alpha_1 = 0$

$-\Delta p = 0$ $\Rightarrow$ $\tilde{p}_{|\mathscr{B}} = \beta_1 \mathscr{D}_1 + \beta_2 \mathscr{D}_2 + \beta_3$ exponentially decaying in $X_3$

$$\Rightarrow \quad \mathbf{u} = \nabla p \sim \left| \begin{array}{l} 0 \\ 0 \\ \beta_1 \pm \beta_2 \end{array} \right. \quad \text{as } X_3 \to \pm\infty$$

## Identification of $K_N$ :

**Theorem:**

Case 1: $K_N$ is the space of dimension 4 given by $K_N = span\left\{\nabla \mathscr{D}_{X_1}, \nabla \mathscr{D}_{X_2}, \nabla \mathscr{D}_1, \nabla \mathscr{D}_2\right\}$.

Case 2: $K_N$ is the space of dimension 3 given by $K_N = span\left\{\nabla \mathscr{D}_{X_1}, \nabla \mathscr{D}_1, \nabla \mathscr{D}_2\right\}$.

Case 3: $K_N$ is the space of dimension 2 given by $K_N = span\left\{\nabla \mathscr{D}_1, \nabla \mathscr{D}_2\right\}$.

$$\mathscr{D}_{X_i} = \widetilde{\mathscr{D}_{X_i}} + X_i$$

$$\begin{cases} -\Delta \mathscr{D}_1 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_1 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_1 \sim X_3 & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \widetilde{\mathscr{D}_{X_i}} = 0 & \text{in } \mathscr{B} \\ \widetilde{\mathscr{D}_{X_i}} = -X_i & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \widetilde{\mathscr{D}_{X_i}} \sim c^{\pm}_{X_i} & \text{as } X_3 \to +\infty \end{cases}$$

$$\begin{cases} -\Delta \mathscr{D}_2 = 0 & \text{in } \mathscr{B} \\ \mathscr{D}_2 = 0 & \text{on } \partial\mathscr{B} \cap \partial\mathscr{B}_\infty \\ \mathscr{D}_2 \sim |X_3| & \text{as } X_3 \to +\infty \end{cases}$$

# 3- 3D time-harmonic Maxwell's equations

## Identification of $K_T$

**Theorem:**

Case 1: $K_T$ is the space of dimension 3 given by $K_T = span\left\{\nabla \mathscr{N}_1, \nabla \mathscr{N}_2, \nabla \mathscr{N}_3\right\}$.

Case 2: $K_T$ is the space of dimension 4 given by $K_T = span\left\{\nabla \mathscr{N}_1, \nabla \mathscr{N}_2^{\pm}, \nabla \mathscr{N}_3\right\}$.

Case 3: $K_T$ is the space of dimension 5 given by $K_T = span\left\{\nabla \mathscr{N}_1^{\pm}, \nabla \mathscr{N}_2^{\pm}, \nabla \mathscr{N}_3\right\}$.

case 1 :

$i \in \{1,2\}$

$$\mathscr{N}_i = \widetilde{\mathscr{N}_i} + X_i \quad \begin{cases} \widetilde{\mathscr{N}_i} \text{ periodic} \\ -\Delta \widetilde{\mathscr{N}_i} = 0 & \text{in } \mathscr{B}_\infty, \\ \partial_{\mathbf{n}} \widetilde{\mathscr{N}_i} = -\mathbf{e}_i \cdot \mathbf{n} & \text{on } \partial \mathscr{B}_\infty, \end{cases} \quad \lim_{X_3 \to \pm\infty} \nabla \widetilde{\mathscr{N}_i} = \mathbf{0}, \quad \lim_{X_3 \to +\infty} \widetilde{\mathscr{N}_i} = 0$$

$$\begin{cases} \mathscr{N}_3 \text{ periodic} \\ -\Delta \mathscr{N}_3 = 0 & \text{in } \mathscr{B}_\infty, \\ \partial_{\mathbf{n}} \mathscr{N}_3 = 0 & \text{on } \partial \mathscr{B}_\infty, \end{cases} \quad \lim_{X_3 \to \pm\infty} \nabla \mathscr{N}_3 = \mathbf{e}_3, \quad \lim_{X_3 \to +\infty} \mathscr{N}_3 - y_3 = 0.$$

$$\nabla \mathcal{N}_i \sim \mathbf{e}_i \quad \text{as } X_3 \to \pm\infty$$

## Identification of $K_T$

**Theorem:**

Case 1: $K_T$ is the space of dimension 3 given by $K_T = span\left\{\nabla\mathcal{N}_1, \nabla\mathcal{N}_2, \nabla\mathcal{N}_3\right\}.$

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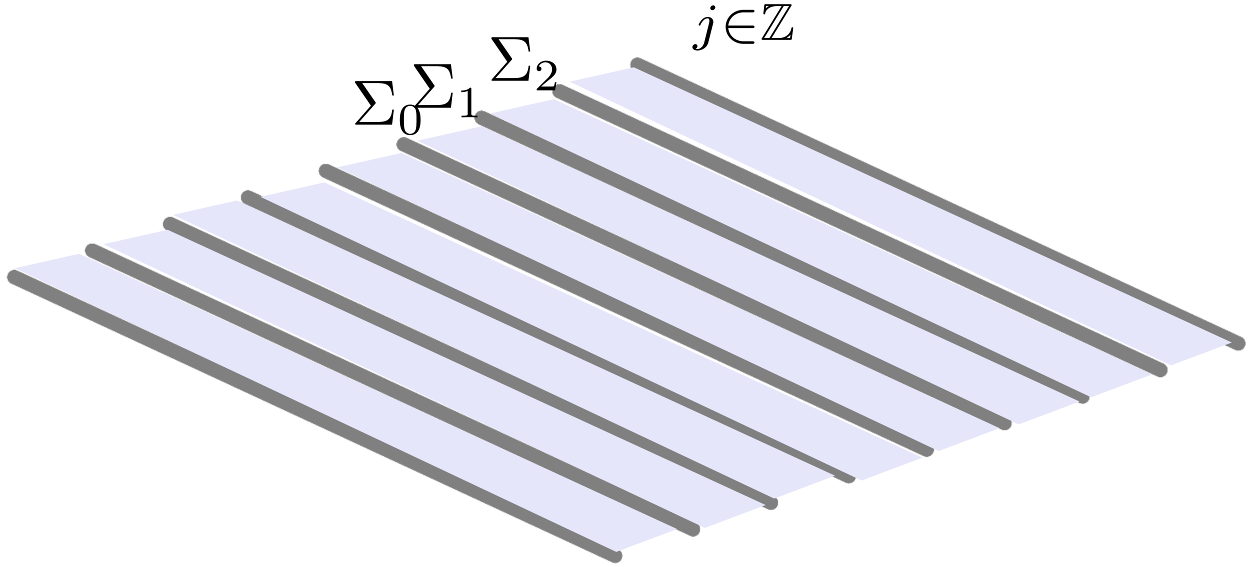
case 2: set of 'cuts' $\Sigma = \bigcup_{j\in\mathbb{Z}} \Sigma_j$




$$\mathscr{B}_\infty^{\pm} = (\mathscr{B}_\infty \setminus \Sigma) \cap \{\pm X_3 > 0\}$$

(simply connected domains)

$$\mathcal{N}_2^{\pm} = \widetilde{\mathcal{N}_2^{\pm}} + X_2 1_{\mathscr{B}_\infty^{\pm}} \qquad \begin{cases} \widetilde{\mathcal{N}_2^{\pm}} \text{ periodic} \\ -\Delta\widetilde{\mathcal{N}_2^{\pm}} = 0 & \text{in } \mathscr{B}_\infty \setminus \Sigma, \\ \partial_{\mathbf{n}}\widetilde{\mathcal{N}_2^{\pm}} = -\mathbf{e}_2\cdot\mathbf{n} & \text{on } \partial\mathscr{B}_\infty^{\pm} \cap \partial\mathscr{B}_\infty, \\ \partial_{\mathbf{n}}\widetilde{\mathcal{N}_2^{\pm}} = 0 & \text{on } \partial\mathscr{B}_\infty^{\mp} \cap \partial\mathscr{B}_\infty, \end{cases} \qquad \begin{cases} [\widetilde{\mathcal{N}_2^{\pm}}]_{\Sigma_j} = \pm(j - X_2), \\ [\partial_{X_3}\widetilde{\mathcal{N}_2^{\pm}}]_{\Sigma_j} = 0, \\ \lim\limits_{X_3\to+\infty} \widetilde{\mathcal{N}_2^{\pm}} = 0, \end{cases}$$

## Identification of $K_T$

**Theorem:**

Case 1: $K_T$ is the space of dimension 3 given by $K_T = span\left\{\nabla\mathscr{N}_1, \nabla\mathscr{N}_2, \nabla\mathscr{N}_3\right\}.$

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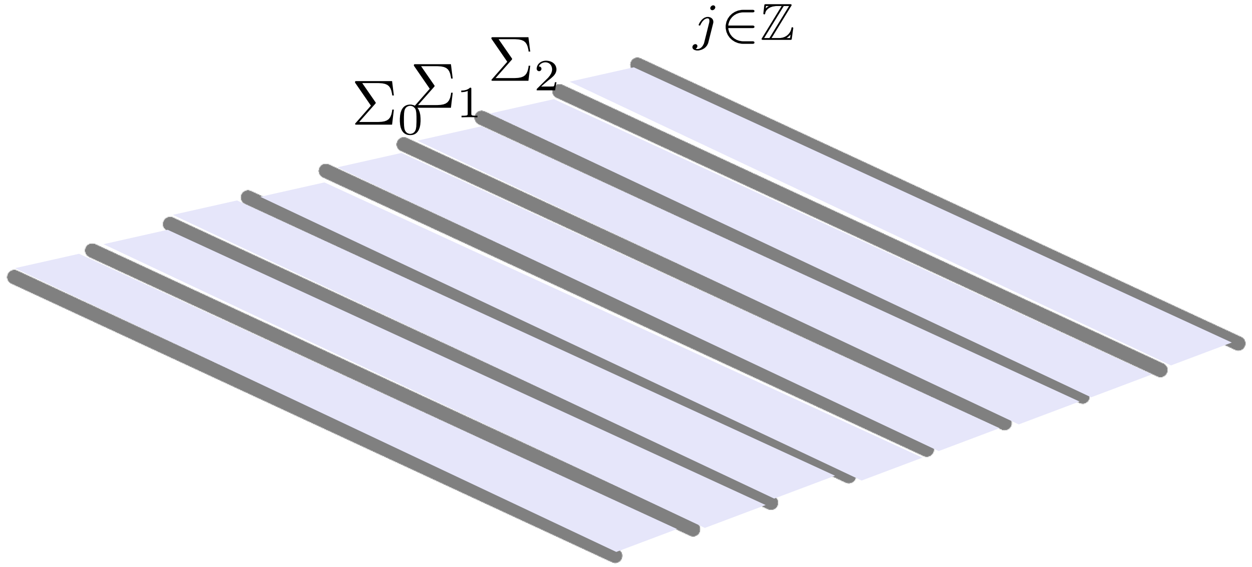
case 2: set of 'cuts' $\Sigma = \bigcup_{j\in\mathbb{Z}} \Sigma_j$




$$\mathscr{B}_\infty^{\pm} = (\mathscr{B}_\infty \setminus \Sigma) \cap \{\pm X_3 > 0\}$$

(simply connected domains)

$$\nabla\mathcal{N}_2^{+} \sim \begin{cases} \mathbf{e}_2 & \text{as } X_3 \to +\infty \\ 0 & \text{as } X_3 \to -\infty \end{cases} \qquad \nabla\mathcal{N}_2^{-} \sim \begin{cases} 0 & \text{as } X_3 \to +\infty \\ \mathbf{e}_2 & \text{as } X_3 \to -\infty \end{cases}$$

# Analysis of the simple 3D case

## Identification of $K_T$

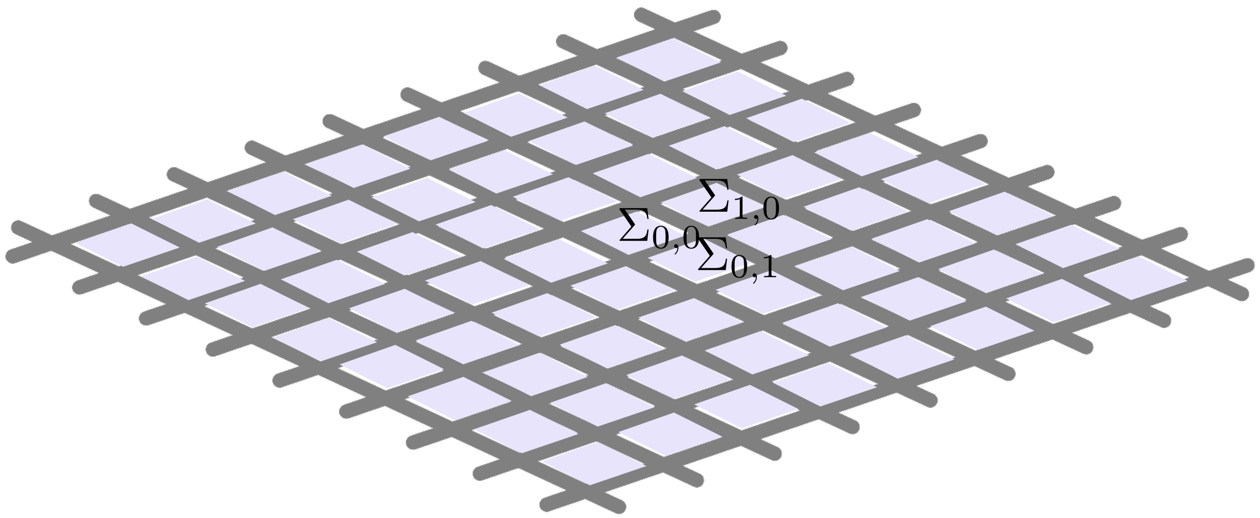
**Theorem:**

Case 1: $K_T$ is the space of dimension 3 given by $K_T = span\left\{\nabla\mathcal{N}_1, \nabla\mathcal{N}_2, \nabla\mathcal{N}_3\right\}$.

Case 2: $K_T$ is the space of dimension 4 given by $K_T = span\left\{\nabla\mathcal{N}_1, \nabla\mathcal{N}_2^{\pm}, \nabla\mathcal{N}_3\right\}$.

Case 3: $K_T$ is the space of dimension 5 given by $K_T = span\left\{\nabla\mathcal{N}_1^{\pm}, \nabla\mathcal{N}_2^{\pm}, \nabla\mathcal{N}_3\right\}$.

**case3: set of 'cuts'** $\Sigma = \bigcup_{(i,j)\in\mathbb{Z}^2} \Sigma_{i,j}$



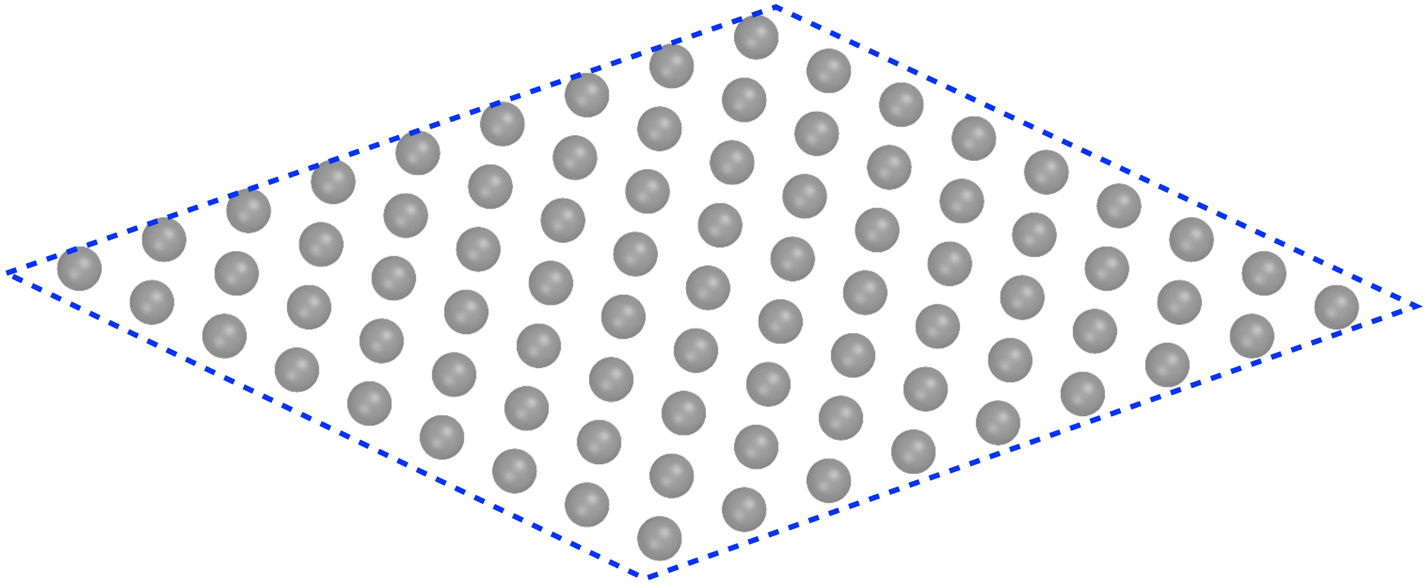

$$i \in \{1,2\} \quad \nabla\mathcal{N}_i^{+} \sim \begin{cases} \mathbf{e}_i & \text{as } X_3 \to +\infty \\ 0 & \text{as } X_3 \to -\infty \end{cases} \qquad \nabla\mathcal{N}_i^{-} \sim \begin{cases} 0 & \text{as } X_3 \to +\infty \\ \mathbf{e}_i & \text{as } X_3 \to -\infty \end{cases}$$

# 3- 3D time-harmonic Maxwell's equations

## Application to the asymptotic expansion

$$\mathbf{H}_0 \in K_T$$

$$\lim_{x_3 \to 0^\pm} \mathbf{u}_0 = \lim_{X_3 \to \pm\infty} \mathbf{U}_0$$
$$\lim_{x_3 \to 0^\pm} \mathbf{h}_0 = \lim_{X_3 \to \pm\infty} \mathbf{H}_0$$



case 1:

$$\mathbf{U}_0 = \alpha_1 \mathscr{D}_{X_1} + \alpha_2 \mathscr{D}_{X_2} + \alpha_3^+ \nabla \mathscr{D}_1 + \alpha_3^- \nabla \mathscr{D}_2$$

$$\mathbf{H}_0 = \beta_1 \mathscr{N}_1 + \beta_2 \mathscr{N}_2 + \beta_3 \nabla \mathscr{N}_3$$

$$\mathbf{U}_0 \sim \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + (\alpha_3^+ \pm \alpha_3^-) \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_3$$
$$\text{as } X_3 \to \pm\infty$$
$$\mathbf{H}_0 \sim \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3$$

$\Rightarrow$ $\mathbf{u}_0 \times \mathbf{n}$ and $\mathbf{h}_0 \times \mathbf{n}$ are continuous across the limit interface $x_3 = 0$

At the limit, the thin periodic interface disappears *(no shielding effect)*

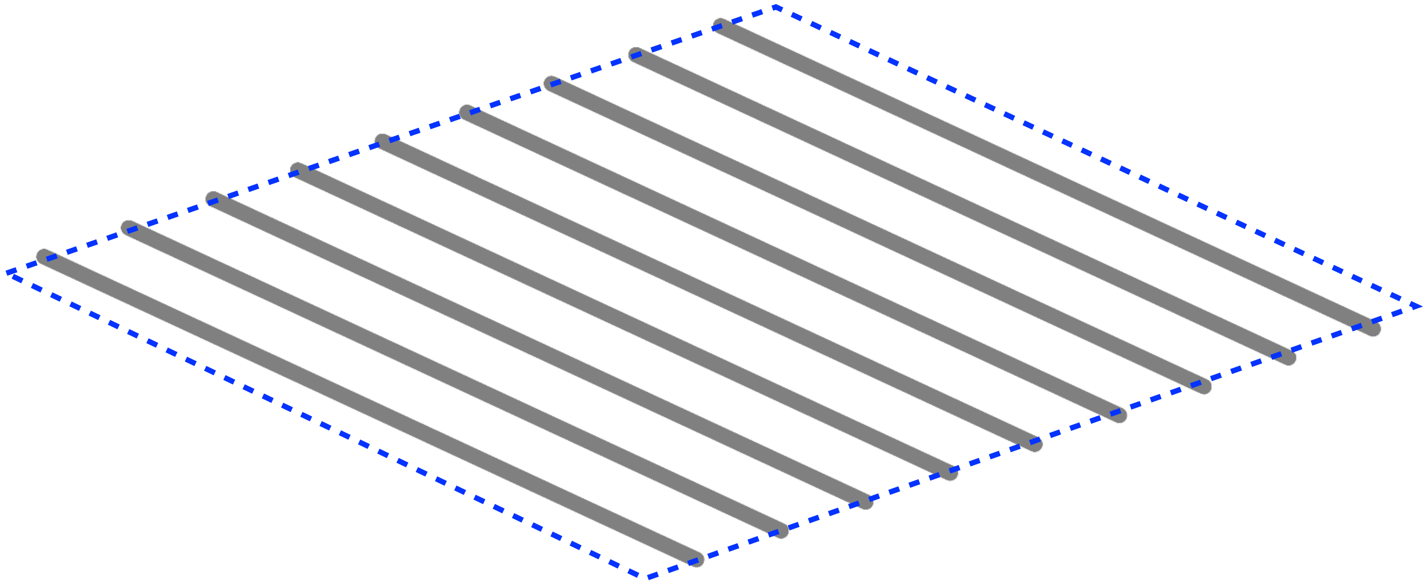
# 3- 3D time-harmonic Maxwell's equations

## Application to the asymptotic expansion

$\mathbf{U}_0 \in K_N \qquad \mathbf{H}_0 \in K_T$

$$\lim_{x_3 \to 0^\pm} \mathbf{u}_0 = \lim_{X_3 \to \pm\infty} \mathbf{U}_0$$

$$\lim_{x_3 \to 0^\pm} \mathbf{h}_0 = \lim_{X_3 \to \pm\infty} \mathbf{H}_0$$



case 2:

$$\mathbf{U}_0 = \alpha_1 \mathscr{D}_{X_1} + \alpha_3^+ \nabla \mathscr{D}_1 + \alpha_3^- \nabla \mathscr{D}_2$$

$$\mathbf{H}_0 = \beta_1 \mathscr{N}_1 + \beta_2^+ \mathscr{N}_2^+ + \beta_2^- \mathscr{N}_2^- + \beta_3 \nabla \mathscr{N}_3$$

$$\mathbf{U}_0 \sim \alpha_1 \mathbf{e}_1 + (\alpha_3^+ \pm \alpha_3^-) \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_3$$

$$\mathbf{H}_0 \sim \beta_1 \mathbf{e}_1 + \beta_2^\pm \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_2 + \beta_3 \mathbf{e}_3$$

as $X_3 \to \pm\infty$

$\Rightarrow$ $(\mathbf{u}_0)_2 = 0$ on the limit interface

$(\mathbf{u}_0)_1$ and $(\mathbf{h}_0)_1$ continuous across the limit interface

*partial shielding effect* for one component of the waves

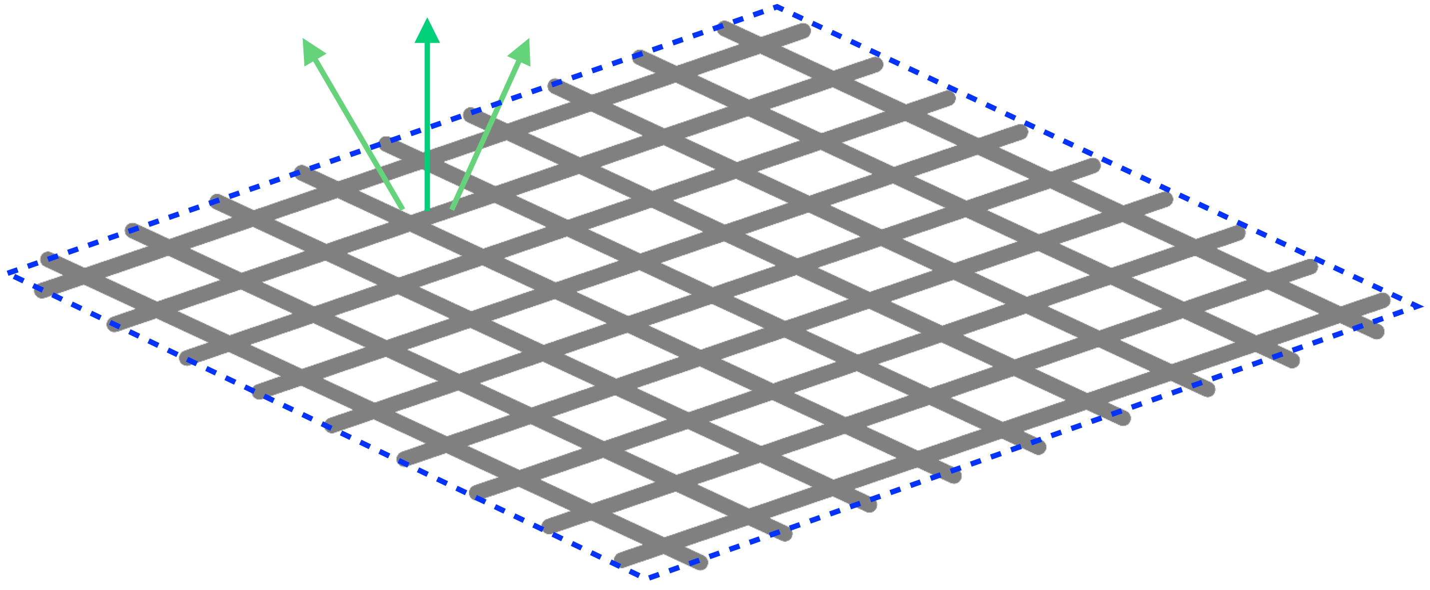
# 3- 3D time-harmonic Maxwell's equations

## Application to the asymptotic expansion

$\mathbf{U}_0 \in K_N \qquad \mathbf{H}_0 \in K_T$

$$\lim_{x_3 \to 0^\pm} \mathbf{u}_0 = \lim_{X_3 \to \pm\infty} \mathbf{U}_0$$

$$\lim_{x_3 \to 0^\pm} \mathbf{h}_0 = \lim_{X_3 \to \pm\infty} \mathbf{H}_0$$




case 3:

$$\mathbf{U}_0 \sim \alpha_3^\pm \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_3$$

$$\mathbf{H}_0 \sim \beta_1^\pm \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_1 + \beta_2^\pm \mathbf{1}_{\pm X_3 > 0} \mathbf{e}_2 + \beta_3 \mathbf{e}_3$$

$\Rightarrow$ $\mathbf{u}_0 \times \mathbf{n} = 0$ on the limit interface $x_3 = 0$

At the limit, no electromagnetic field below the meta-surface (total shielding effect)

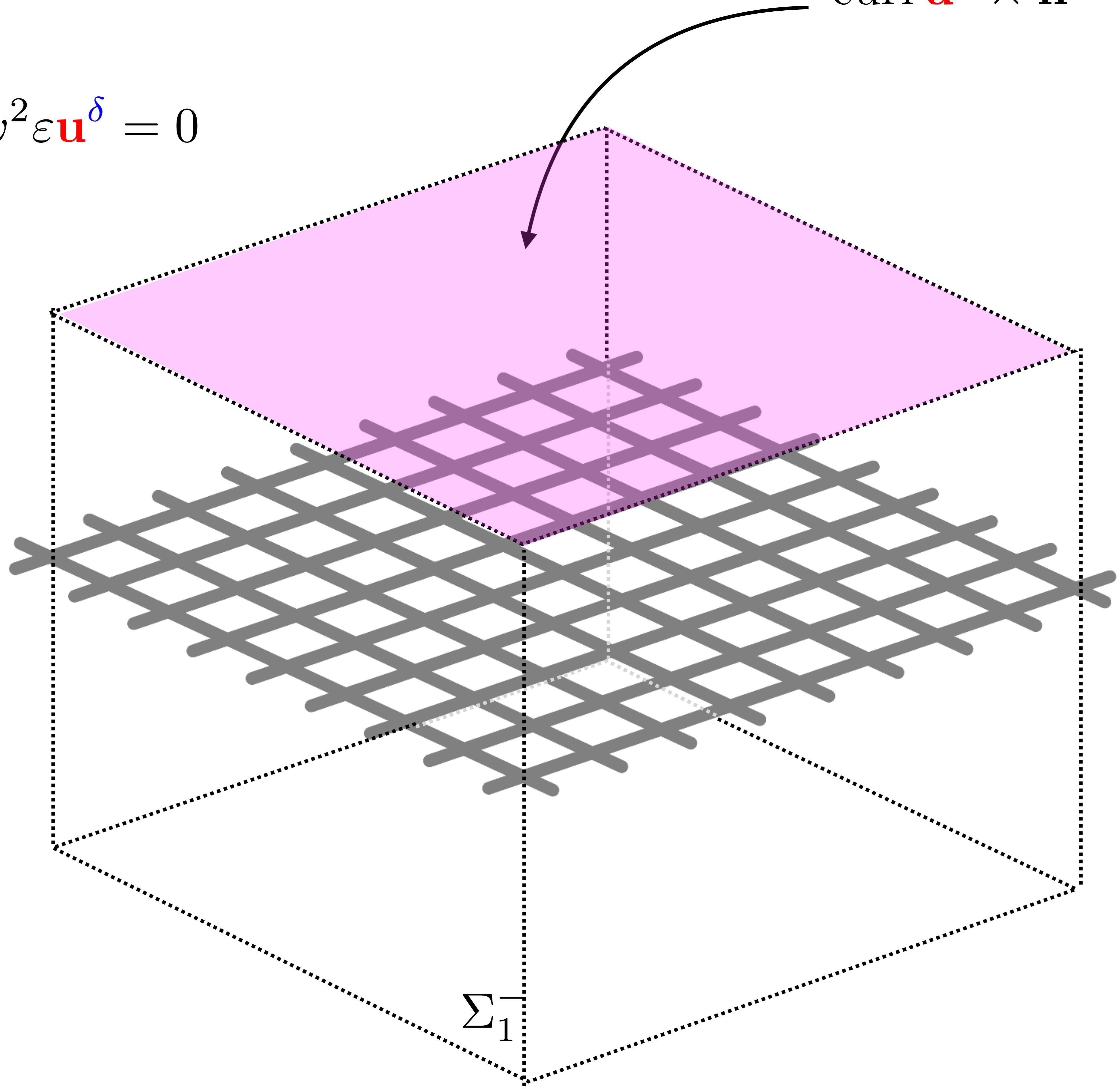
*(B. Schweizer 17, Holloway-Kuester 18)*

# 3- 3D time-harmonic Maxwell's equations

$$\delta = \frac{1}{40}$$

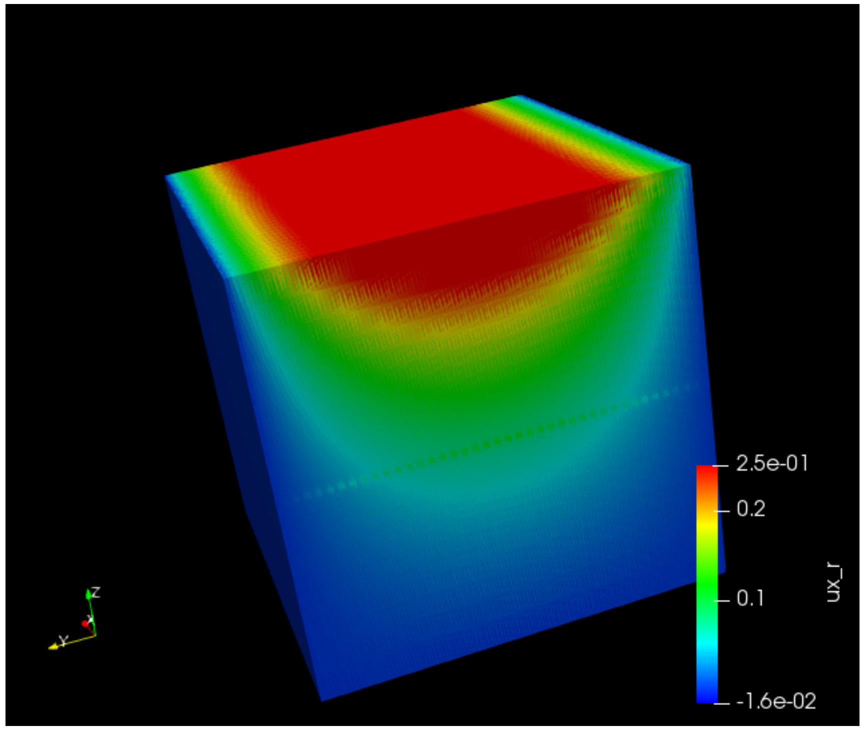
$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u}^{\delta} - \omega^2\varepsilon\mathbf{u}^{\delta} = 0 \\ \mathbf{u}^{\delta}\times\mathbf{n} = 0 \end{cases}$$

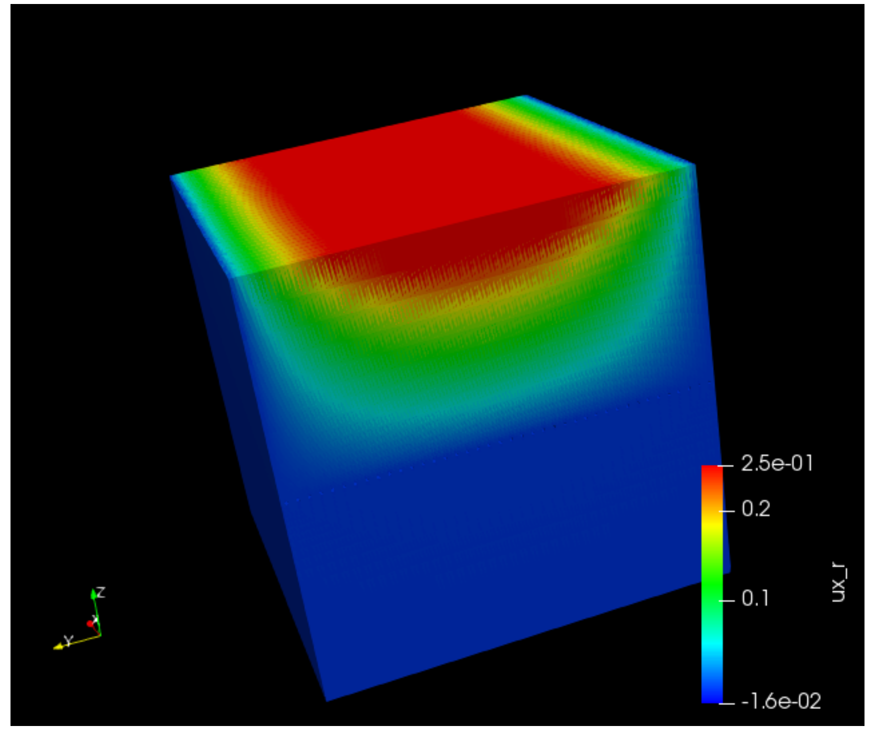
$$\operatorname{curl}\mathbf{u}^{\delta}\times\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2$$

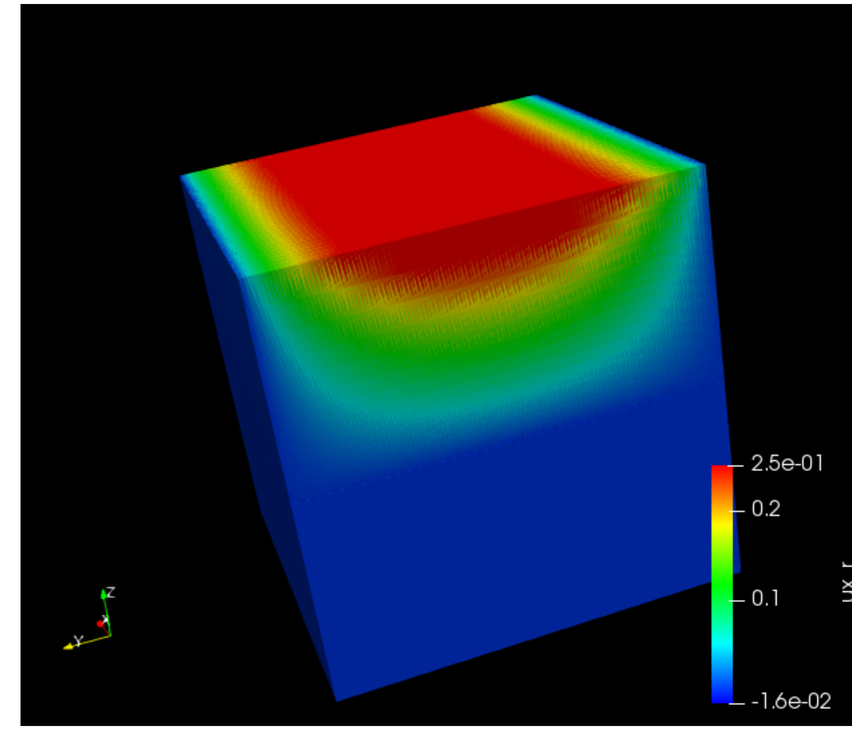


# 3- 3D time-harmonic Maxwell's equations

$\mathrm{Re}(u_1)$

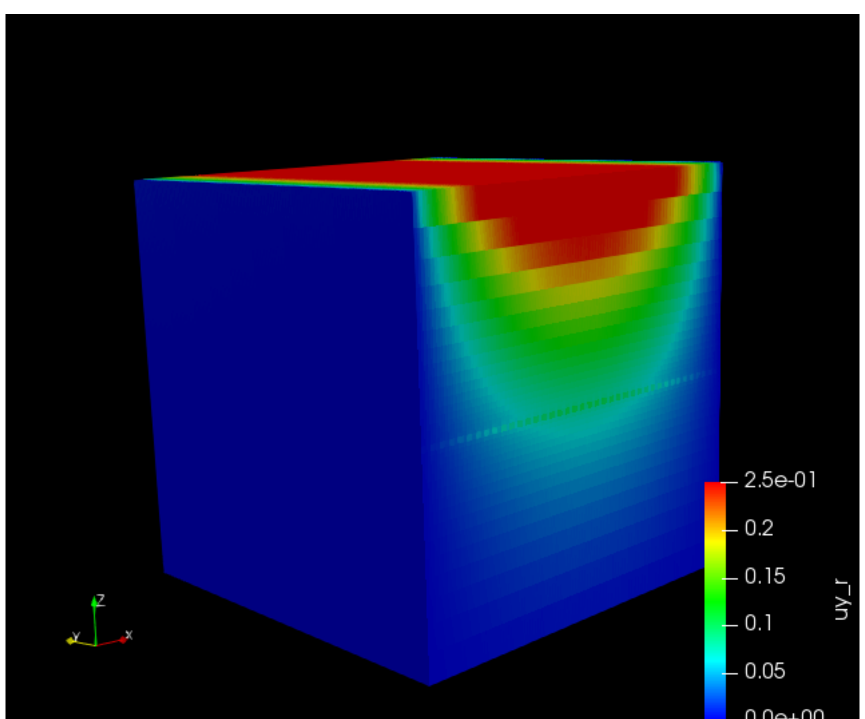


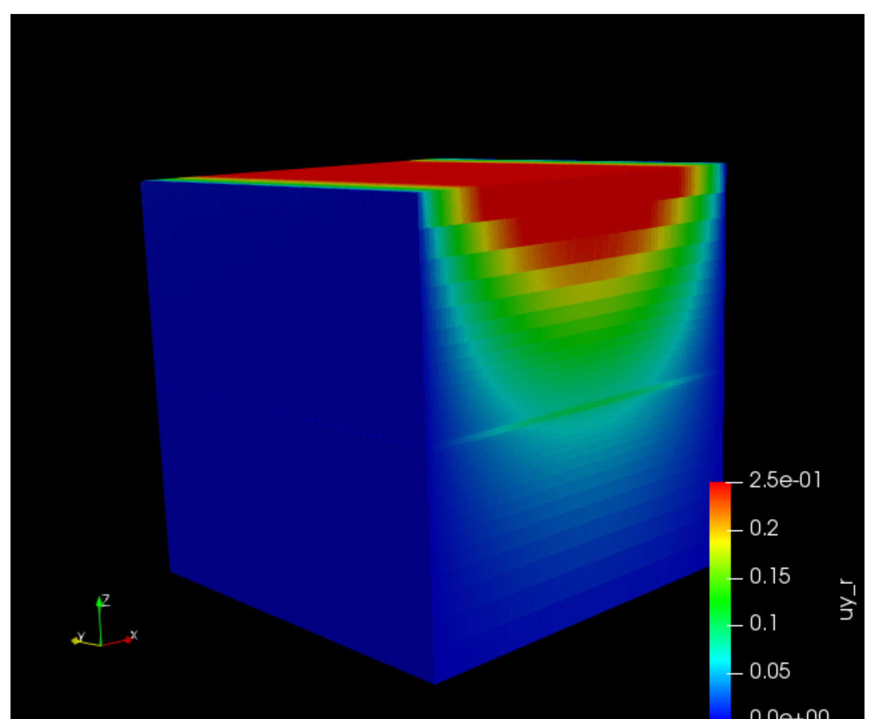



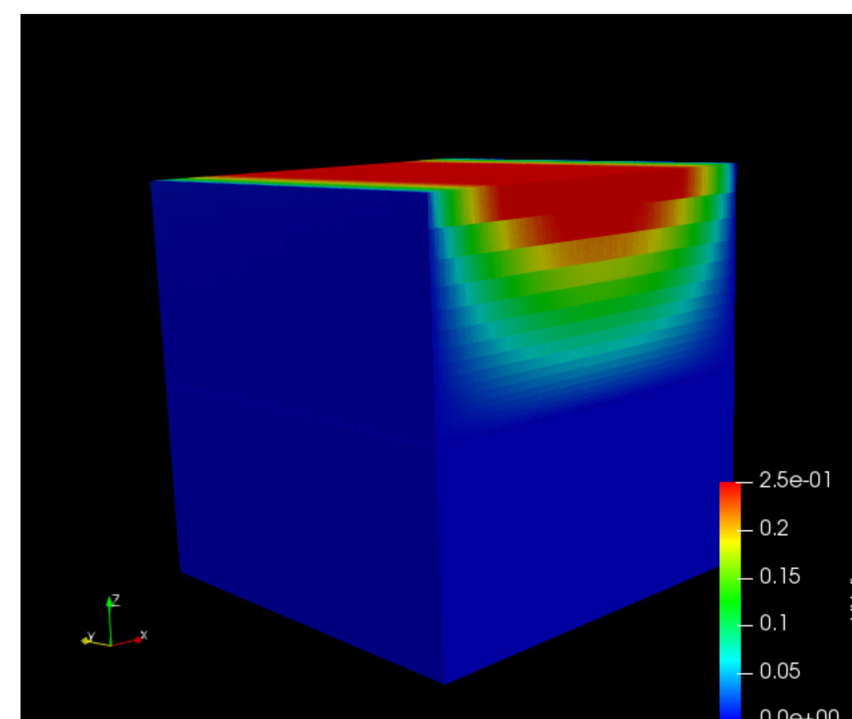




case 1 case 2 case 3

$\mathrm{Re}(u_2)$

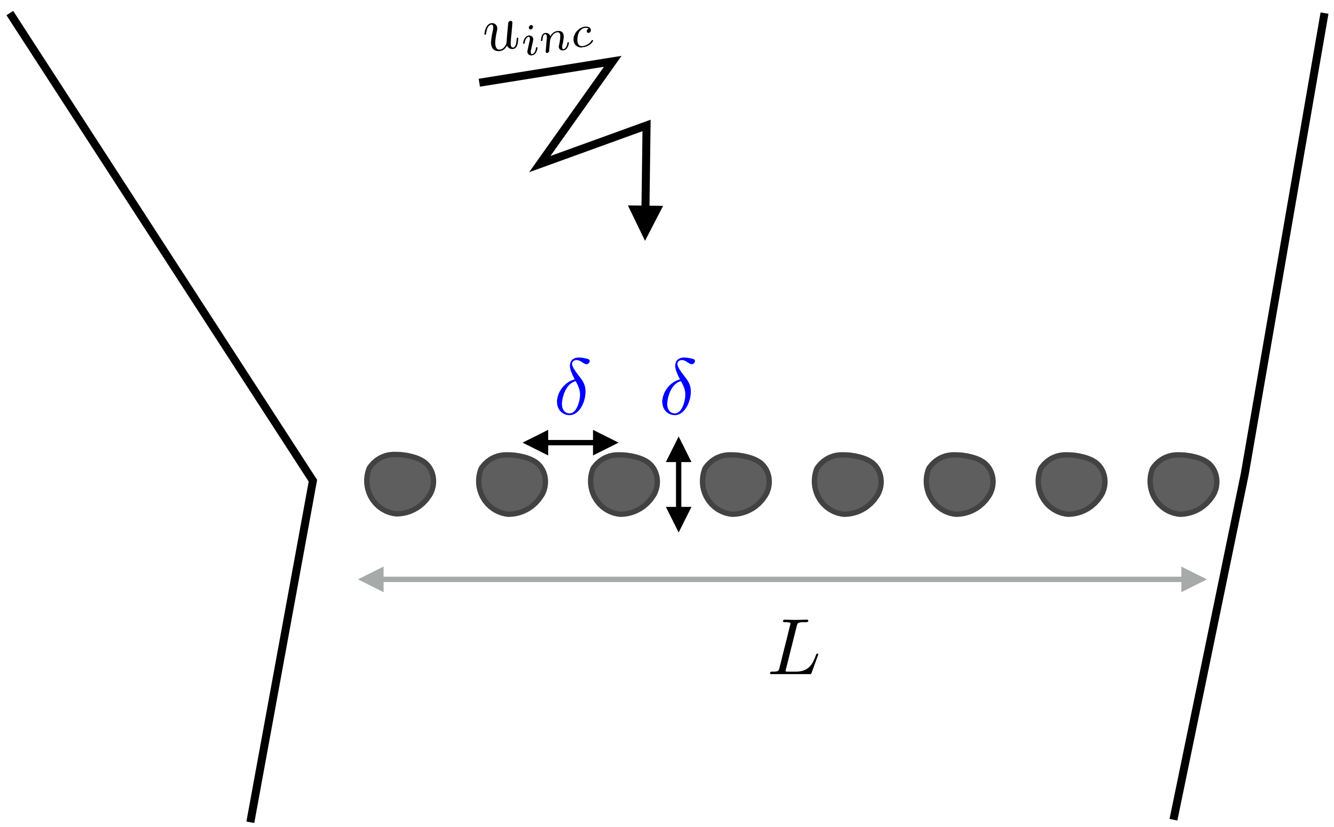


# 4- Homogenization in presence of corners

Diffraction by infinite line of equi-spaced obstacles



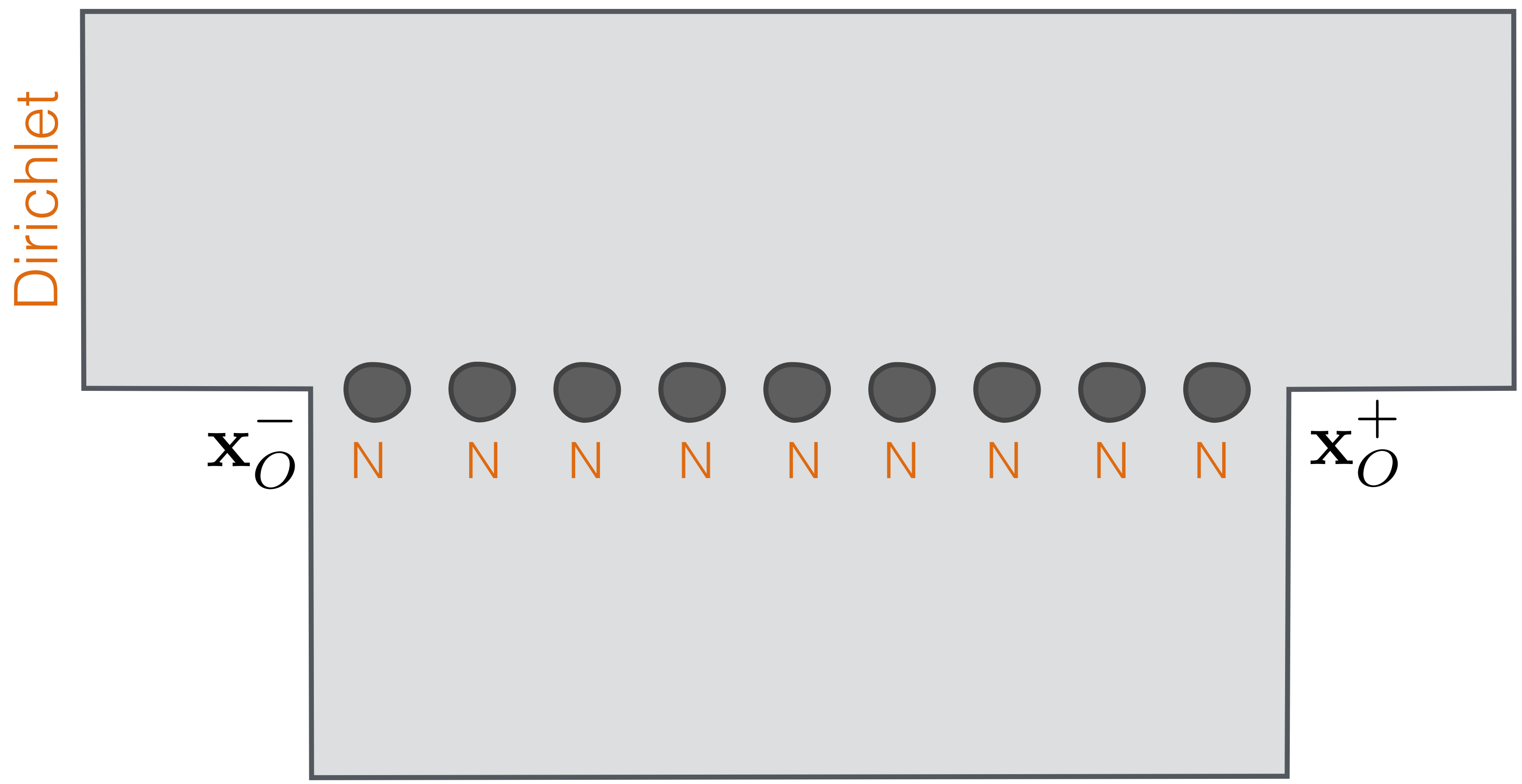

Analysis of the solution as $\delta$ goes to $0$: periodic homogenization

What can we do when the periodicity of the problem is lost ?

# 4- Homogenization in presence of corners

## A model problem:

$$\Omega^{\delta} = \Omega \setminus \overline{\Omega^{\delta}_{\text{hole}}}$$




## Description of the problem:

$$(\mathcal{P}) \begin{cases} -\Delta u^{\delta} = f \text{ in } \Omega^{\delta} \\ u^{\delta} = 0 \text{ on } \partial\Omega \\ \partial_n u^{\delta} = 0 \text{ on } \partial\Omega^{\delta}_{\text{hole}} \end{cases}$$

# 4- Homogenization in presence of corners

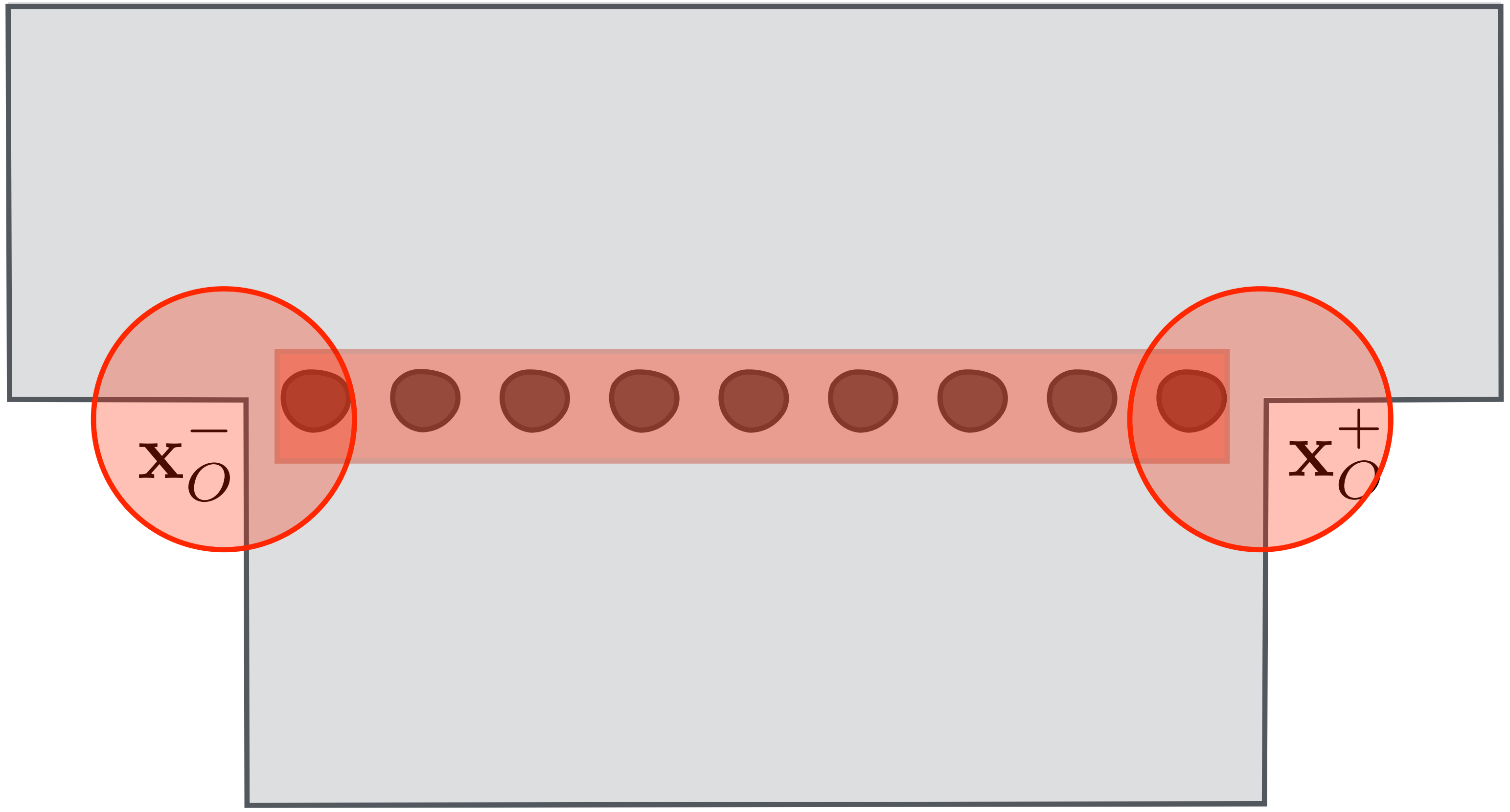
## Well-posedness and stability property

**Proposition:** let $f \in L^2(\Omega^\delta)$. Problem $(\mathcal{P})$ has a unique solution $u^\delta \in H^1(\Omega^\delta)$ that satisfies the following stability estimate: $\exists C > 0$,

$$\|u^\delta\|_{H^1(\Omega^\delta)} \leq C \, \|f\|_{L^2(\Omega^\delta)}$$

**Objective:** behavior of $u^\delta$ with respect to $\delta$ as $\delta$ tends to $0$
construction of an asymptotic expansion of $u^\delta$ w.r.t $\delta$

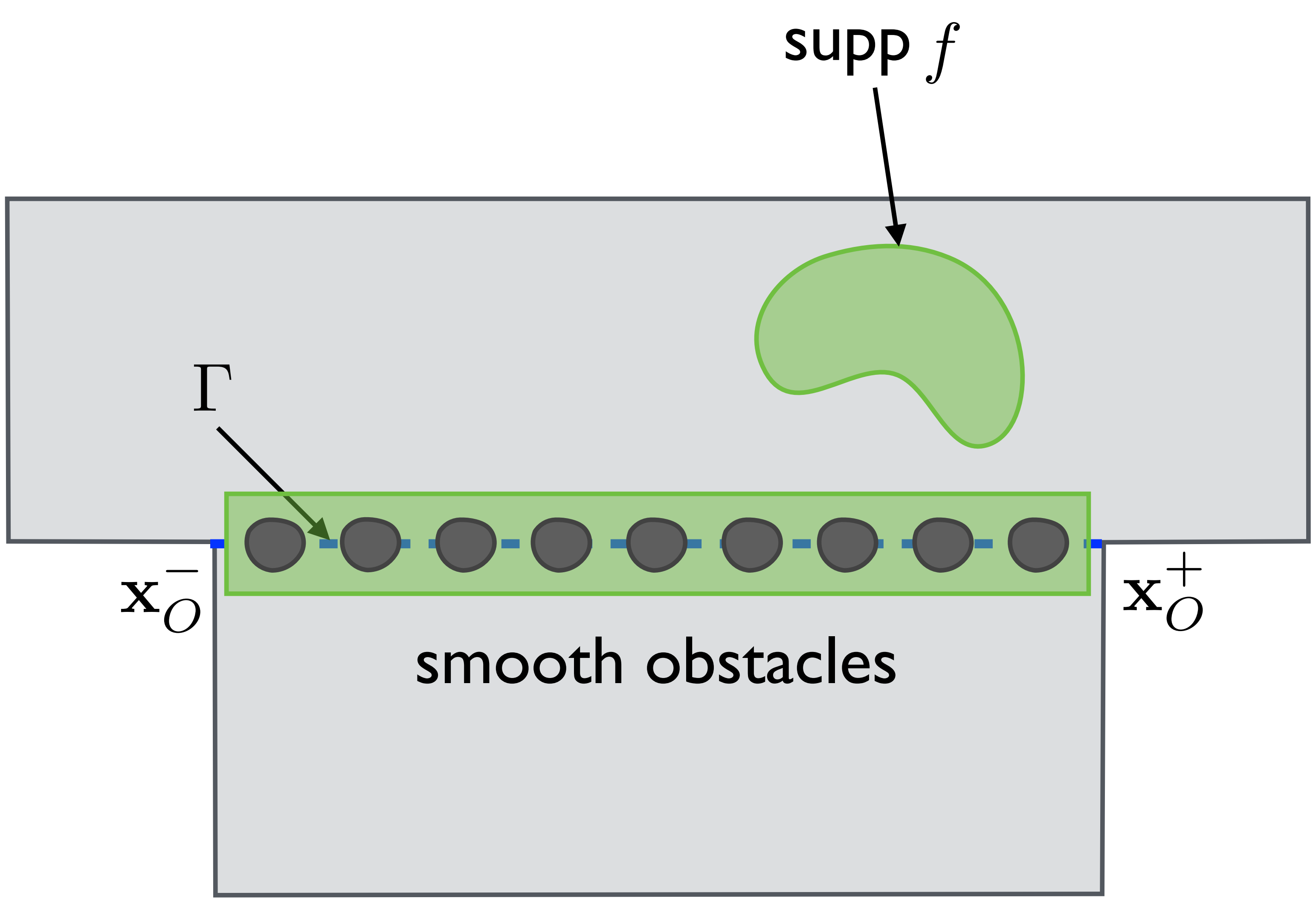
**Main difficulty:** presence of both corners and the periodic layer



# 4- Homogenization in presence of corners

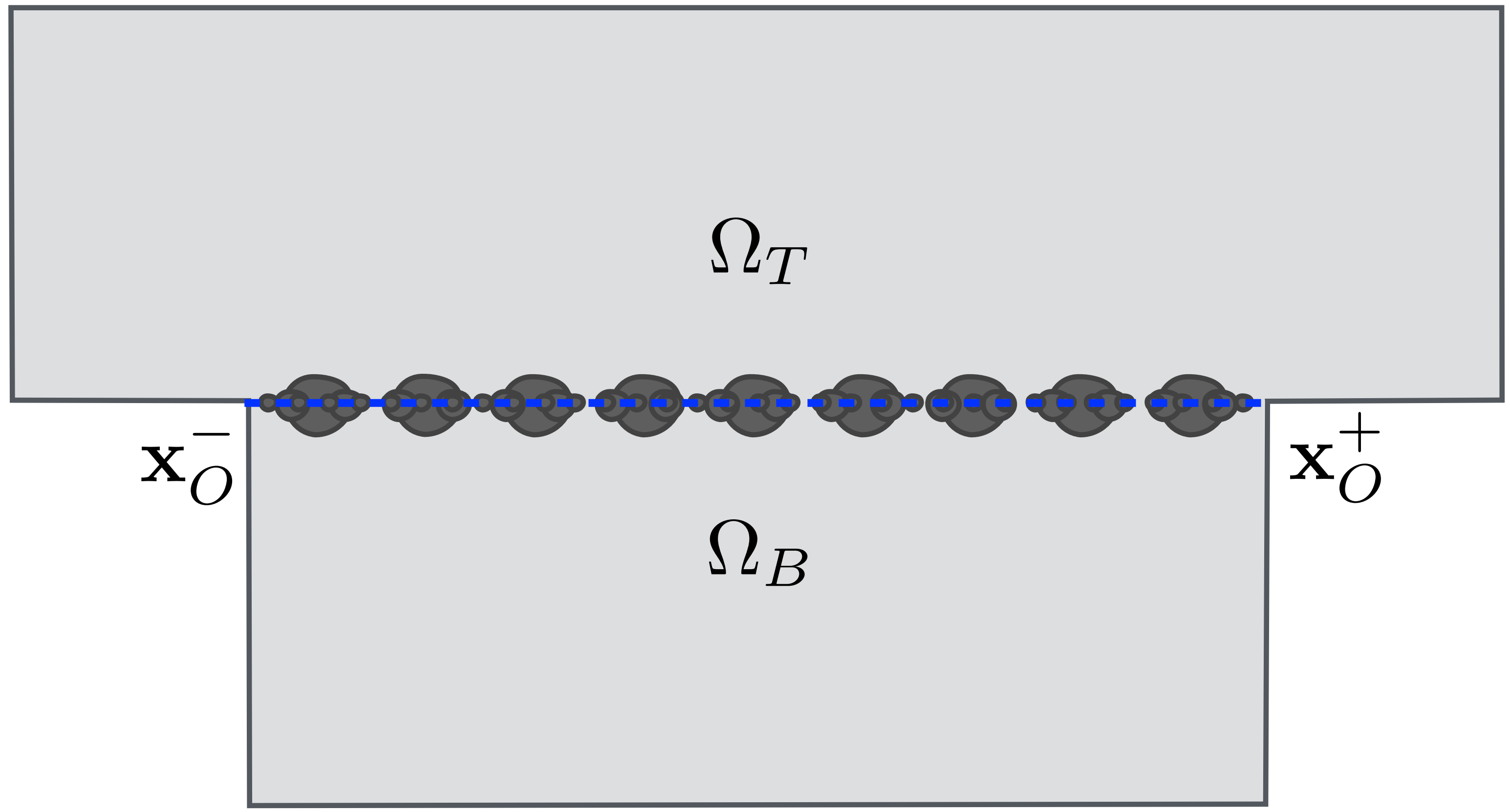
## Technical Assumptions:

- ✓ the support of $f$ does not intersect the interface $\Gamma$
- ✓ the canonical obstacle $\hat{\Omega}_{\mathrm{hole}}$ is smooth.



# 4- Homogenization in presence of corners

**Limit domain as $\delta \to 0$:**




**Limit problem:** $u_{0,0} \in H^1(\Omega)$

$$\begin{cases} -\Delta u_{0,0} = f \text{ in } \Omega \\ u_{0,0} = 0 \text{ on } \partial\Omega \end{cases}$$

$$\Omega = \Omega_T \cup \Omega_B \cup \Gamma$$

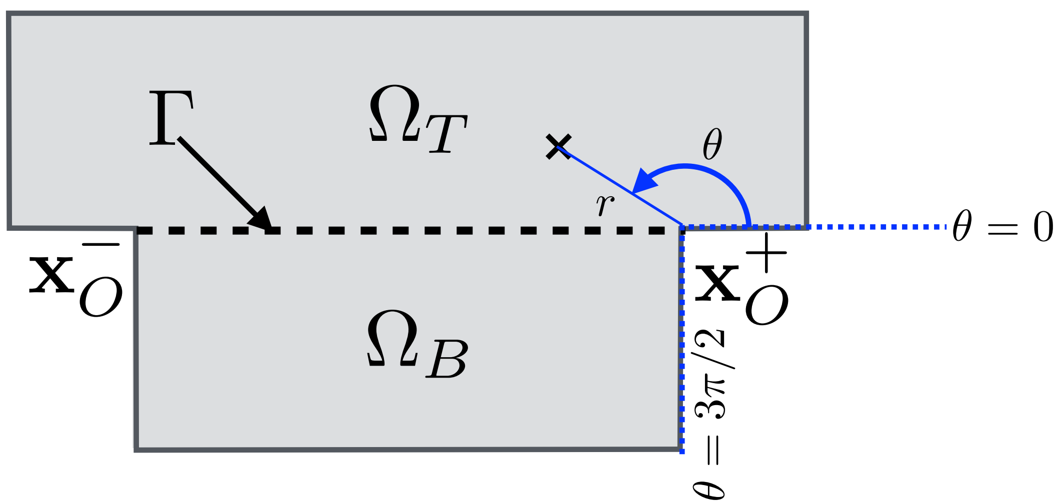
## 4- Homogenization in presence of corners

### The necessary introduction of singular macroscopic terms

*When using the classical Ansatz…* $u^\delta = \sum_{q \in \mathbb{N}} \delta^q \left( \chi(\frac{x_2}{\delta}) u_q(\mathbf{x}) + \Pi_q(x_1, \frac{\mathbf{x}}{\delta}) \right)$

**Limit (macroscopic) problem:** $(u_0 = u_{0,0})$

$$(\mathcal{P}_0) \begin{cases} -\Delta u_0 = f \text{ in } \Omega \\ u_0 = 0 \text{ on } \partial\Omega \end{cases}$$




$\Omega = \Omega_T \cup \Omega_B \cup \Gamma$

→ $(\mathcal{P}_0)$ has a unique regular solution $u_0 \in H^1(\Omega)$

*In the vicinity of the corners:* $u_0 = \sum_{n \in \mathbb{N}^*} c_n \, r^{\frac{2}{3}n} \sin(\frac{2}{3} n \, \theta)$

singular exponents (depend on the angle)

# 4- Homogenization in presence of corners

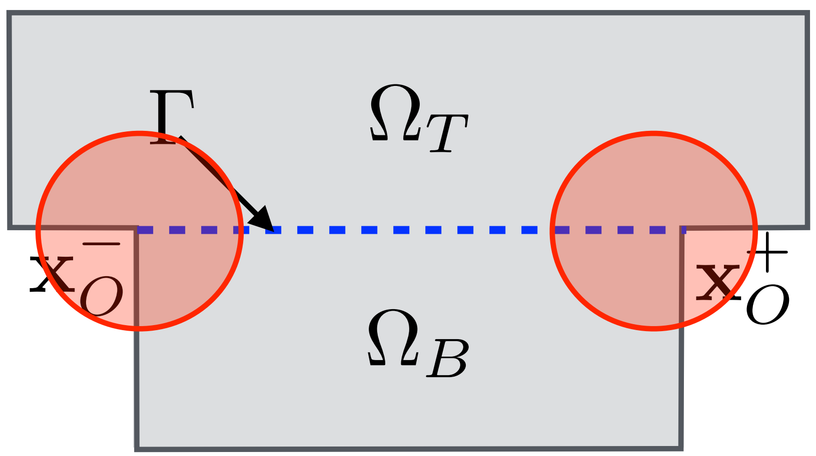
## The necessary introduction of singular macroscopic terms

**Problem for $u_1$**

$$(\mathcal{P}_1)\begin{cases} -\Delta u_1 = 0 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \\ [u_1]_\Gamma = \mathcal{D}_1^{\mathrm{t}}\,\partial_{x_1}\langle u_0\rangle_\Gamma + \mathcal{D}_1^{\mathrm{n}}\,\langle \partial_{x_2} u_0\rangle_\Gamma \\ [\partial_n u_1]_\Gamma = \mathcal{N}_2^{\mathrm{t}}\,\partial^2_{x_1}\langle u_0\rangle_\Gamma + \mathcal{N}_2^{\mathrm{n}}\,\partial_{x_1}\langle \partial_{x_2} u_0\rangle_\Gamma \end{cases}$$

$Cr^{-1/3}$

$Cr^{-4/3}$




→ $(\mathcal{P}_1)$ has no regular solution (in $H^1(\Omega_T) \cap H^1(\Omega_B)$)

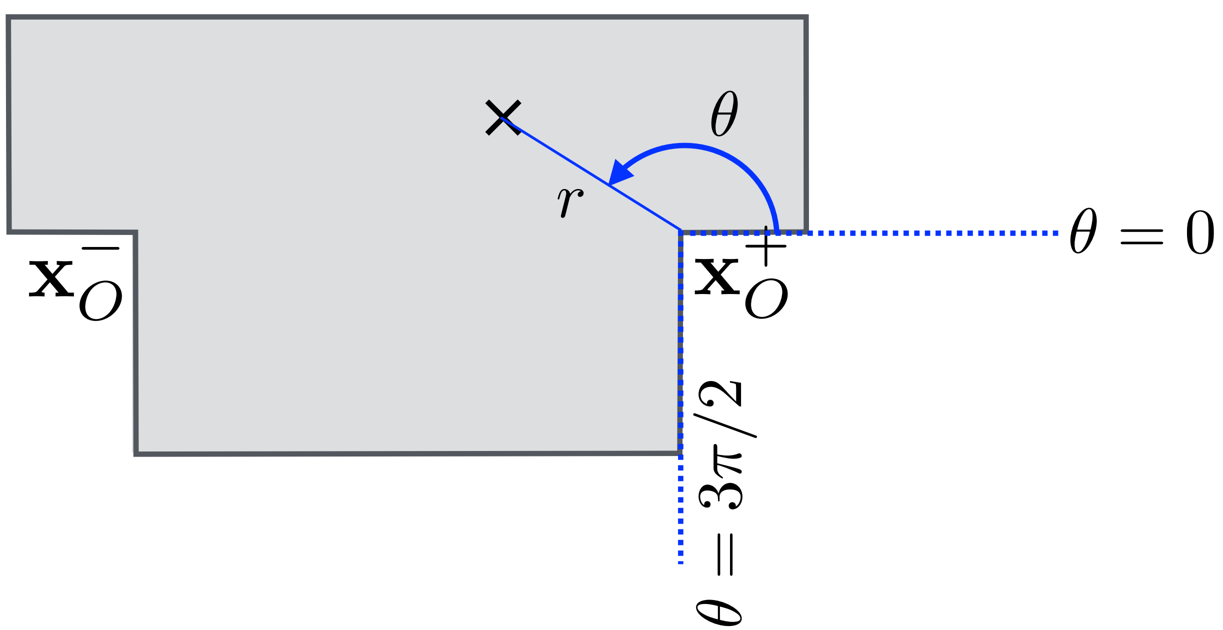
→ It is possible to construct a singular solution that behaves like $Cr^{-1/3}$ in the vicinity of the corners $\mathbf{x}_O^\pm$

*presence of a corner boundary layer effect*

# 4- Homogenization in presence of corners

## Construction of the full asymptotic expansion : two preliminary remarks

(1)- Asymptotic expansion of $u_{0,0}$ in the neighborhood of the corner $\mathbf{x}_O^+$




*Separation of variables:* $u_{0,0} = \sum_{n \in \mathbb{N}^*} c_n \, r^{\frac{2}{3}n} \, \sin(\frac{2}{3} n \, \theta)$ $\quad \mathbb{N}^* = \mathbb{N} \setminus \{0\}$

*Formal change of scale:* $R = r/\delta$ $\quad u_{0,0} = \sum_{n \in \mathbb{N}^*} c_n \, \boxed{\delta^{\frac{2}{3}n}} \, R^{\frac{2}{3}n} \, \sin(\frac{2}{3} n \, \theta)$

(2)- Purely periodic case: $u^\delta = \sum_{q \in \mathbb{N}} \boxed{\delta^q} \Big( \chi(\frac{x_2}{\delta}) u_q(\mathbf{x}) + \Pi_q(x_1, \frac{\mathbf{x}}{\delta}) \Big)$

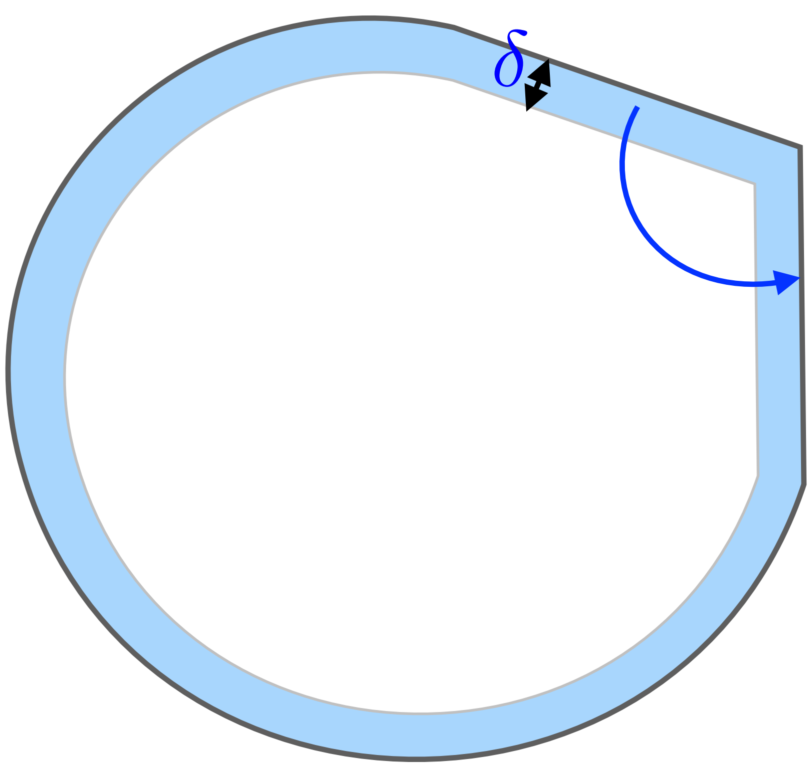
# 4- Homogenization in presence of corners

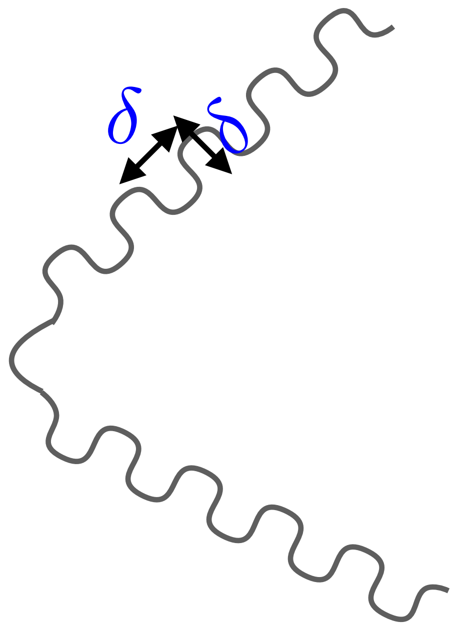
corner singularities
+
periodic layer correctors

$\delta^{\frac{2}{3}n+q}$

Methodology : we use the method of matched asymptotic expansions

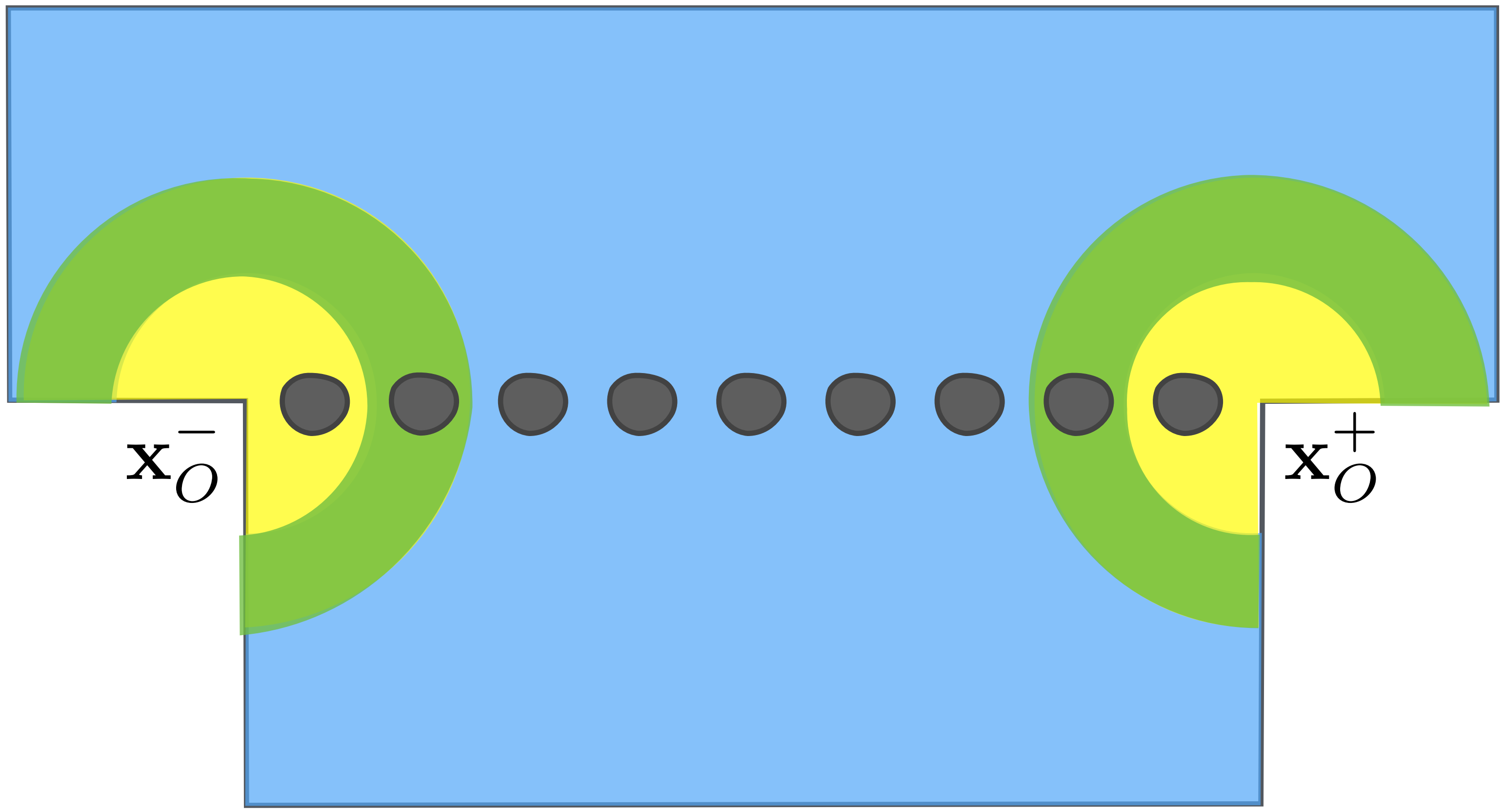
*(Caloz-Costabel-Dauge-Vial 06, Nazarov 08)*



# 4- Homogenization in presence of corners

## Method of matched asymptotic expansions 1: main ideas

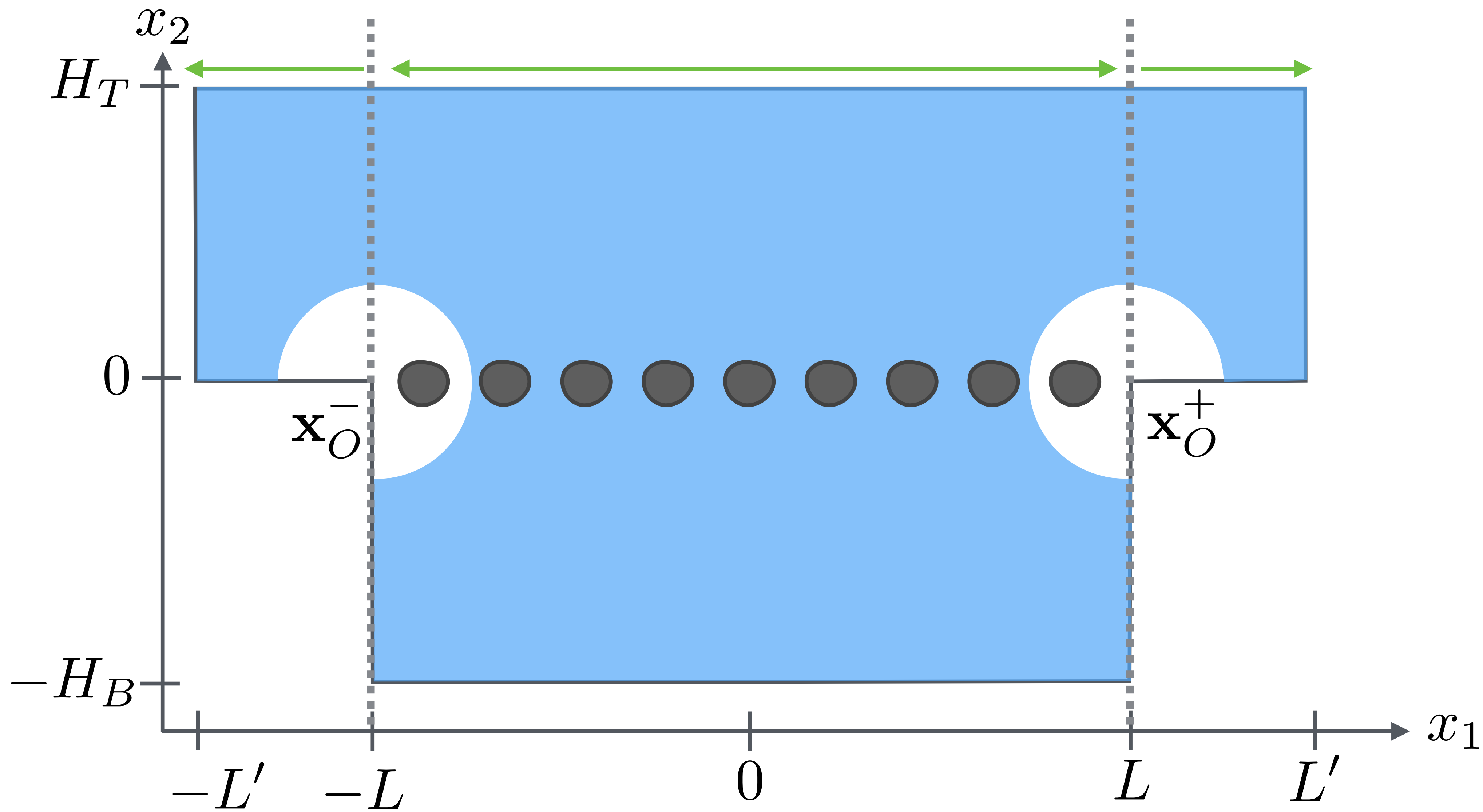



Far field zone : it is located far from the corners it includes the thin periodic layer

Near field zones : they are located close to the corners

Matching zones

**Method of matched asymptotic expansions I: far field expansion**




$$\sum_{(n,q)\in\mathbb{N}^2} \delta^{\frac{2}{3}n+q}\, u^\delta_{n,q}(\mathbf{x}) \qquad |x_1| > L$$

macroscopic terms

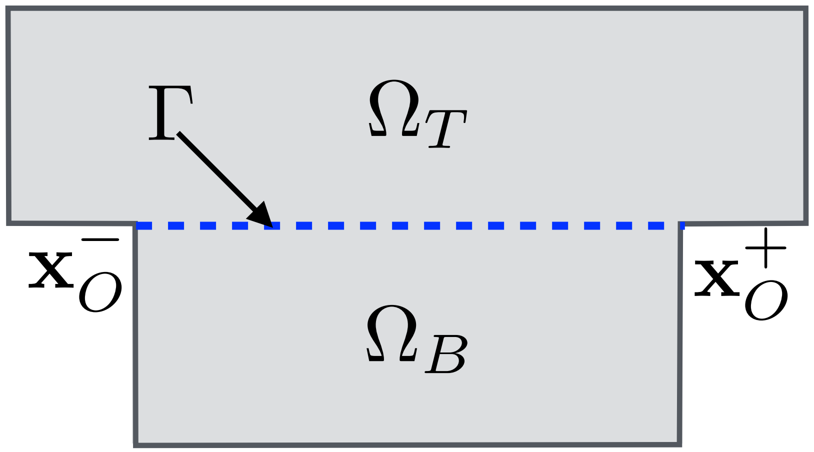
periodic correctors

$$\sum_{(n,q)\in\mathbb{N}^2} \delta^{\frac{2}{3}n+q} \left(\chi\left(\frac{x_2}{\delta}\right) u^\delta_{n,q}(\mathbf{x}) + \Pi^\delta_{n,q}\left(x_1, \frac{\mathbf{x}}{\delta}\right)\right) \qquad |x_1| < L$$

*as in the purely periodic case*

# 4- Homogenization in presence of corners

$u^{\delta}_{n,q}$ are the macroscopic terms




- ✓ They are defined in $\Omega_T \cup \Omega_B$
- ✓ They are not necessarily continuous across $\Gamma$

*as in the purely periodic case*

- ✓ They (might) only have a polynomial dependence w.r.t. $\ln \delta$ :

$$u^{\delta}_{n,q} = \sum_{k=0}^{K} (\ln \delta)^k \, u_{n,q,k} \quad u_{n,q,k} \text{ independant of } \delta$$

- ✓ They might blow up in the vicinity of the corners $\mathbf{x}_O^{\pm}$

*due to the corners*

# 4- Homogenization in presence of corners

## Method of matched asymptotic expansions 1: far field expansion

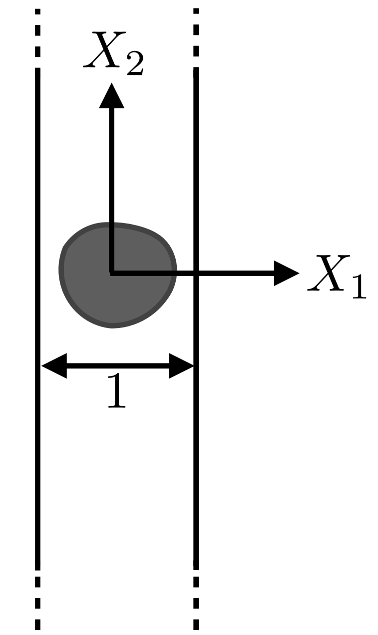
$\Pi^{\delta}_{n,q}$ are the periodic correctors

$$\Pi^{\delta}_{n,q}(x_1, X_1, X_2)$$

1-periodic w.r.t $X_1$

exponentially decaying w.r.t $X_2$

(boundary layer effect)




Periodicity cell $\mathcal{B}$

- ✓ They are defined in $\Gamma \times \mathcal{B}$

*as in the purely periodic case*

- ✓ They (might) only have a polynomial dependence w.r.t. $\ln \delta$ :

$$\Pi^{\delta}_{n,q} = \sum_{k=0}^{K} (\ln \delta)^k \, \Pi_{n,q,k}$$

$\Pi_{n,q,k}$ independant of $\delta$

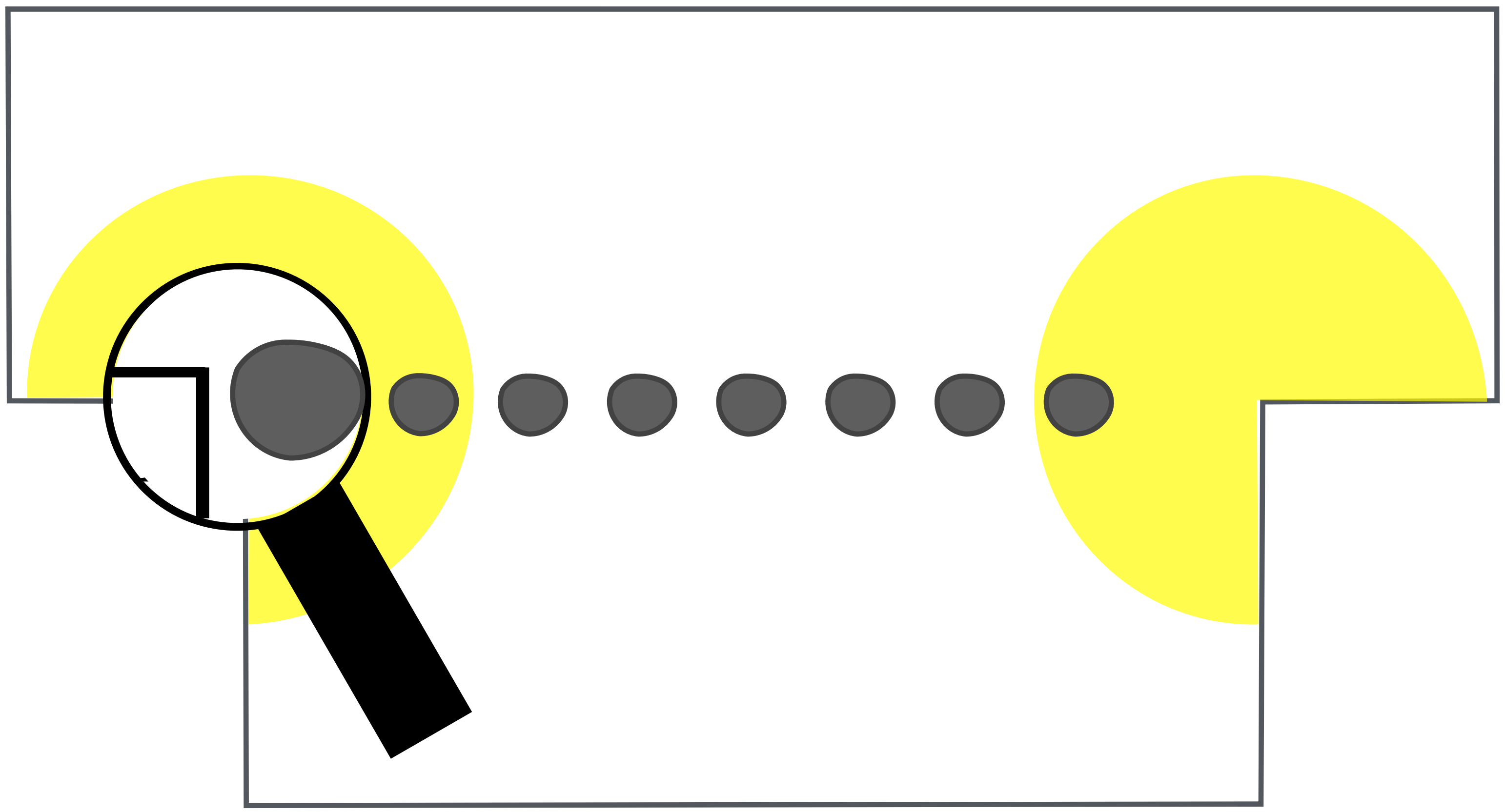
- ✓ They might blow up in the vicinity of the corners $\mathbf{x}_O^{\pm}$ $(x_1 \to \pm L)$

*due to the corners*

# 4- Homogenization in presence of corners

**Method of matched asymptotic expansions 2: near field expansion**

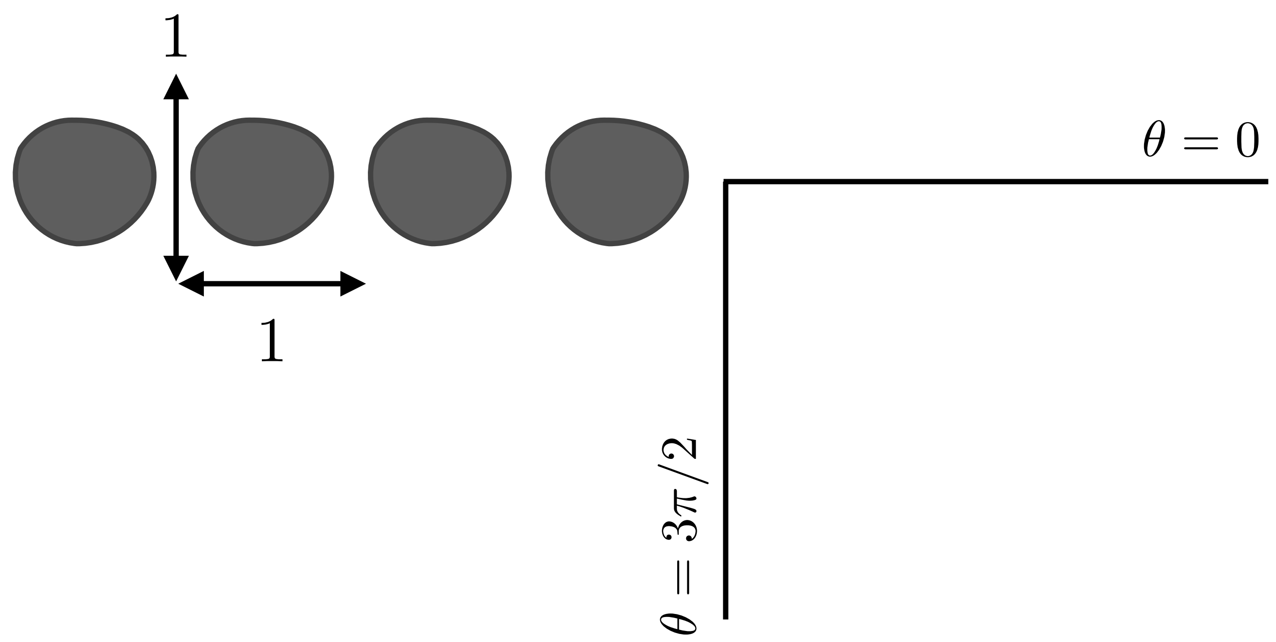
Near field areas (close to the corners)



$$u^{\delta} = \sum_{(n,q) \in \mathbb{N}^2} \delta^{\frac{2}{3}n+q} \, U_{n,q}^{\delta,\pm}\left(\frac{\mathbf{x} - \mathbf{x}_{\mathbf{O}}^{\pm}}{\delta}\right)$$

# 4- Homogenization in presence of corners

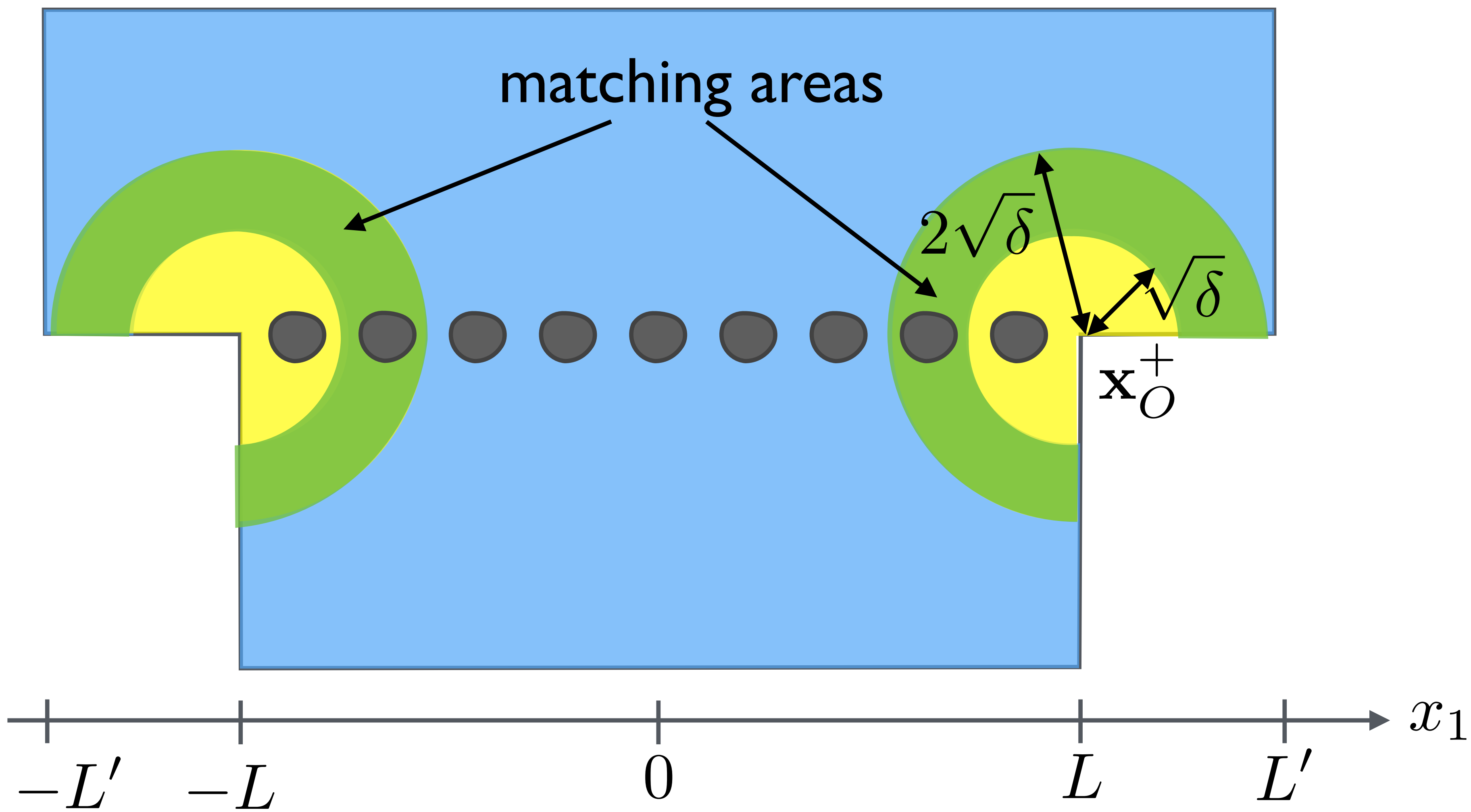
## Method of matched asymptotic expansions 2: near field expansion




The infinite angular domain $\hat{\Omega}^+$

- ✓ They are defined in the infinite angular domain $\hat{\Omega}^{\pm}$
- ✓ They might have a polynomial dependance w.r.t $\ln \delta$
- ✓ They might blow up at infinity

## Method of matched asymptotic expansions 3: matching principle




Far and near field expansions coincide in the matching zones

Neighborhood of the corners for the far field ($r$ small)

$R$ going to $+\infty$ for the near field

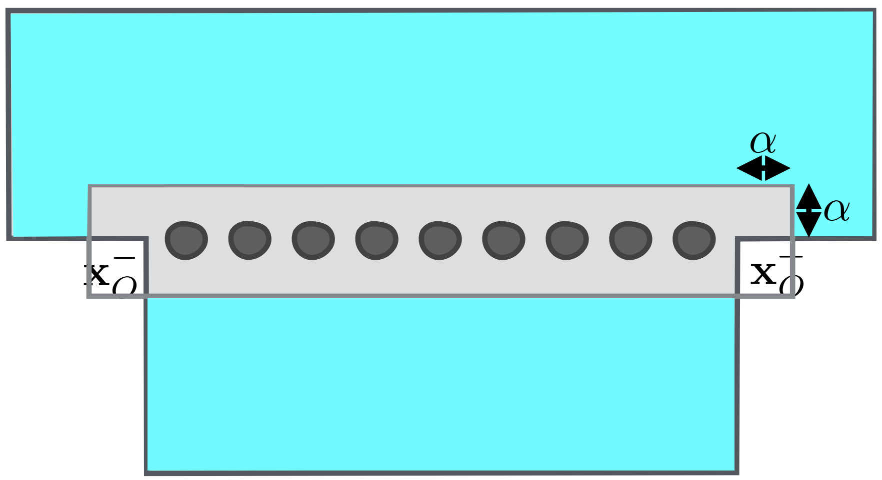
# 4- Homogenization in presence of corners

## Justification of the asymptotic expansion: convergence

- ✓ Far field equations
- ✓ Near field equations
- ✓ Matching procedure

→ Existence of all the terms of the asymptotic expansion

recurrence procedure to define the different terms




**Proposition:** Let $\alpha > 0$, and

$$\Omega_\alpha = \Omega^\delta \setminus (-L-\alpha, L+\alpha) \times (-\alpha, \alpha).$$

There exists , such that for $\delta$ sufficiently small

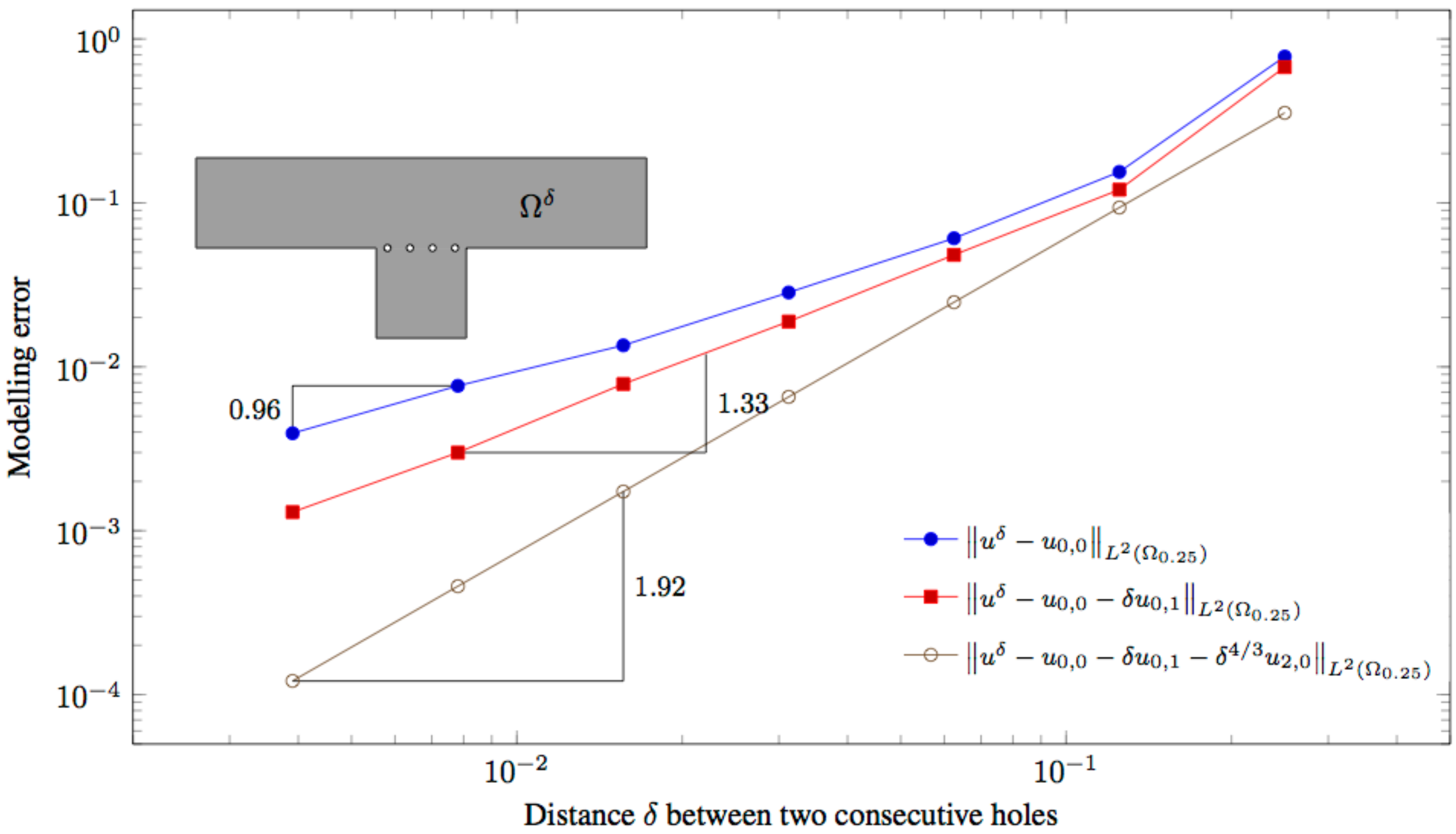
$$\| u^\delta - \sum_{(n,q)\in\mathbb{N}^2,\ \frac{2}{3}n+q<m} \delta^{\frac{2}{3}n+q}\, u^\delta_{n,q} \|_{H^1(\Omega_\alpha)} \leq C\delta^m \ln \delta^r$$

$$\| u^\delta - u_{0,0} - \delta u_{0,1} - \delta^{\frac{4}{3}} u_{2,0} \|_{H^1(\Omega_\alpha)} \leq C\delta^2 \ln \delta$$

# 4- Homogenization in presence of corners

**Numerical illustration of the convergence estimates.**

$$\|u^\delta - u_{0,0} - \delta u_{0,1} - \delta^{\frac{4}{3}} u_{2,0}\|_{H^1(\Omega_\alpha)} \leq C\delta^2 \ln \delta$$



# Thank your for your attention !

*In collaboration with Xavier Claeys, Tung Doan, Houssem Haddar, David P. Hewett, Patrick Joly, Adrien Semin, Kersten Schmidt.*