The role of phonons in dynamic homogenization: Dirac, Dirac-like, and almost-Dirac points

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Outline

(I) Dynamic homogenization at finite frequencies and wavenumbers

(II) Dirac, Dirac-like, and almost-Dirac points

(III) Effective Green's function

(IV) Waves in periodic discontinua

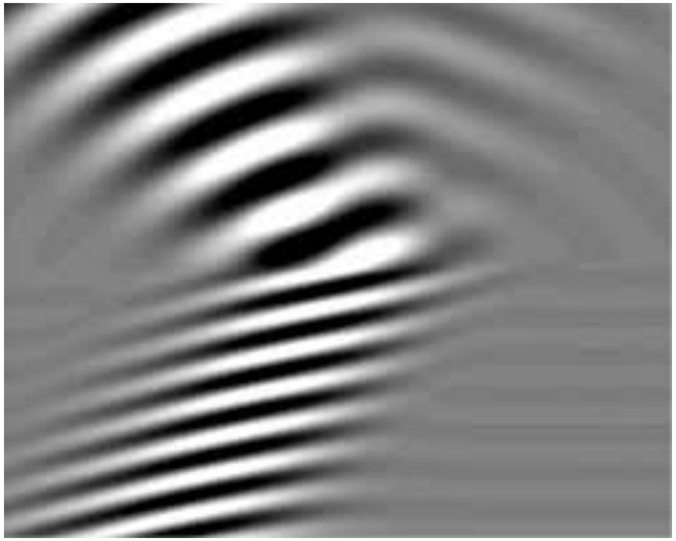
(V) Bravais lattices

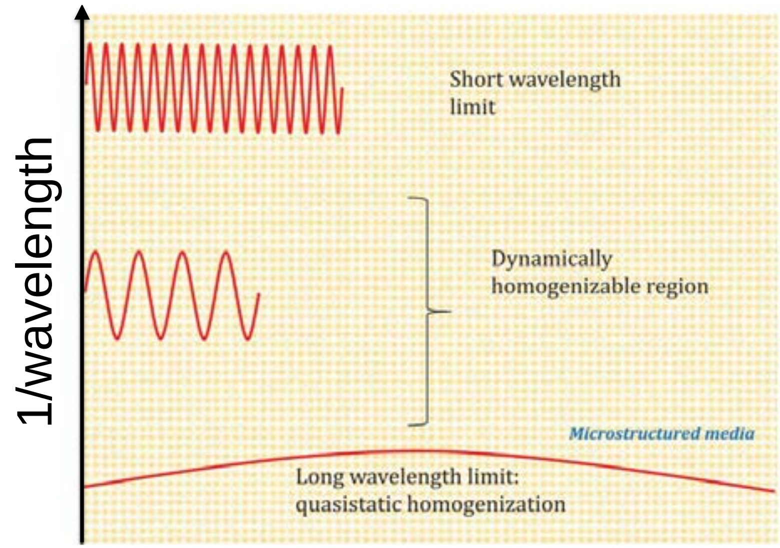
Part I

Dynamic homogenization at finite frequencies and wavenumbers

G/Meng/Oudghiri-Idrissi

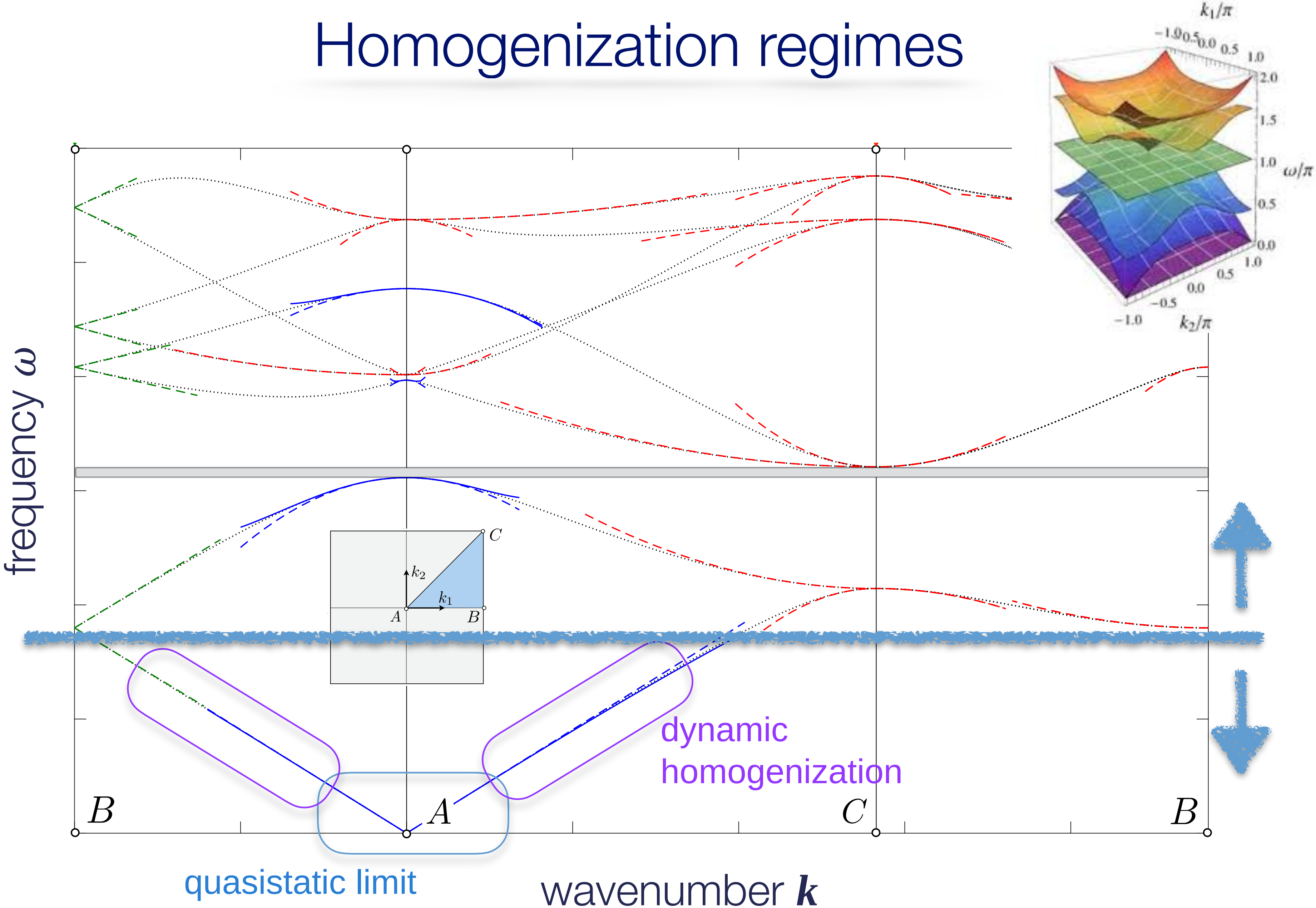
Background



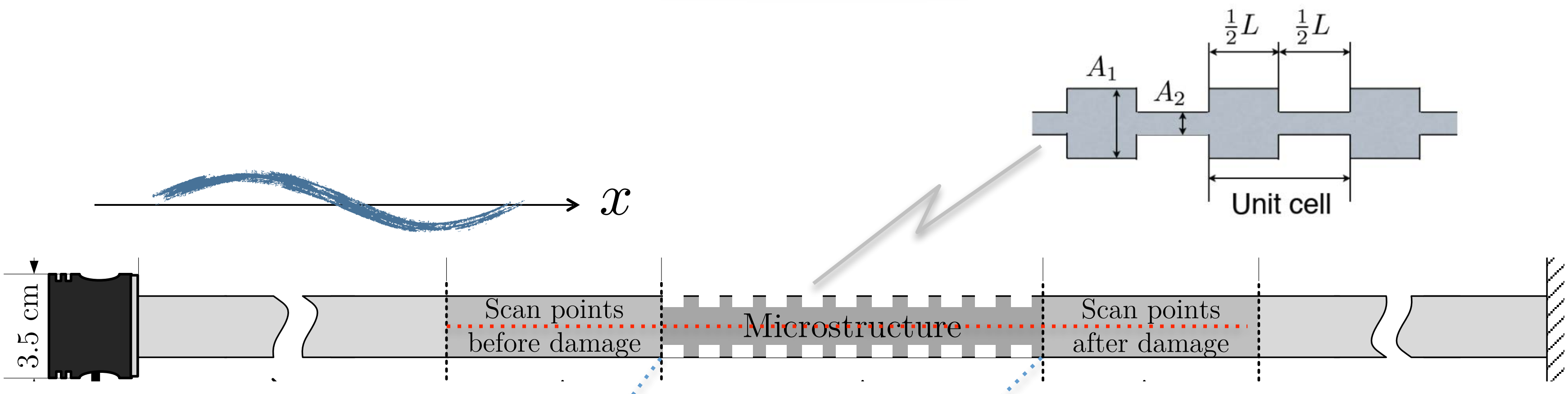


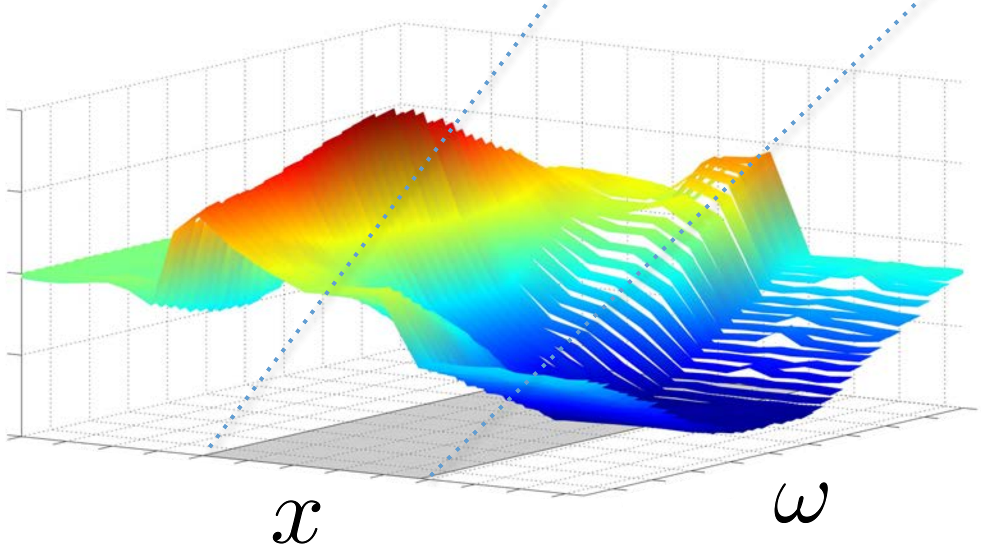
- Dispersion, band gaps, anisotropy, negative index of refraction, super-focusing, cloaking, topologically protected states….
- Eclipsing the quasistatic limit: effective field equation (non-locality, dispersion) and dynamic material properties
- Willis' approach (Willis (1981,2009), Norris et al. (2012)) vs. multiple-scales homogenization (Bensoussan et al. (1978), Bakhvalov et al. (1989))

Homogenization regimes

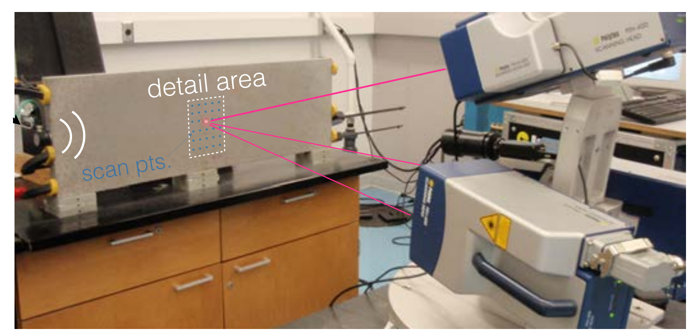


Band gaps



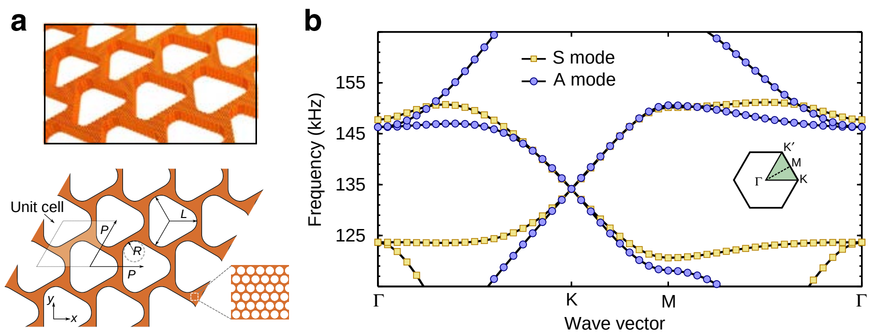


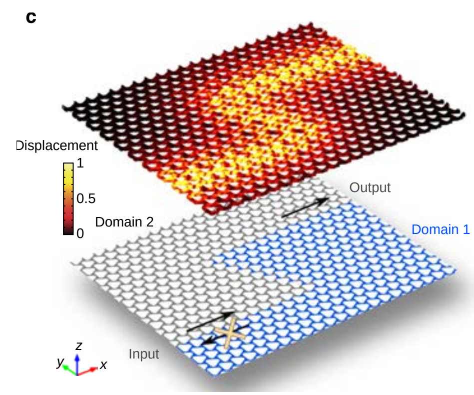
Scanning Laser
Doppler Vibrometer
(SLDV)



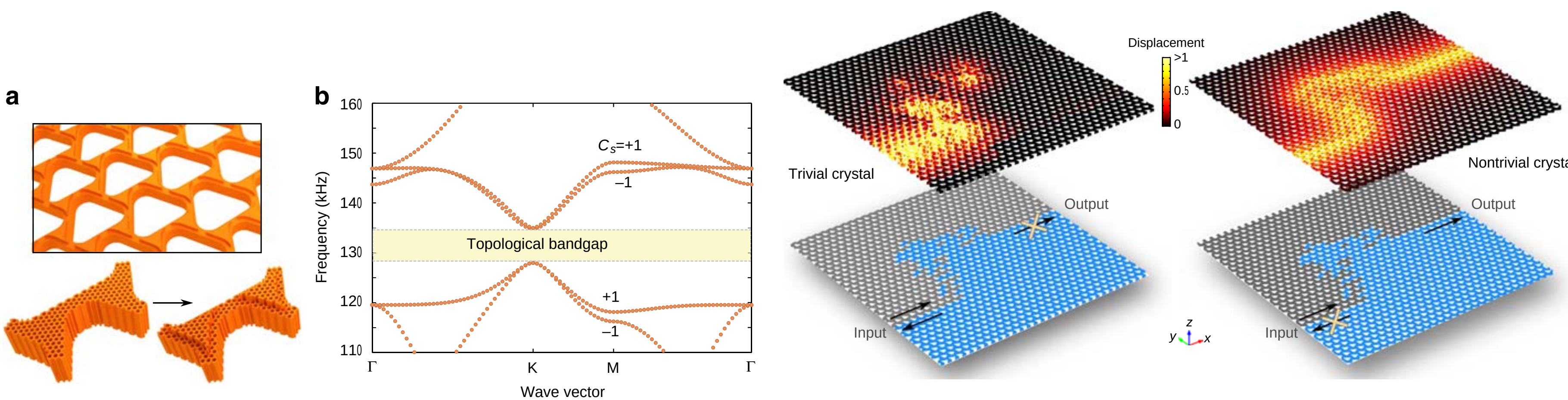
Dontsov, Tokmashev, G (2013) *IJSS*

Topologically-protected states





Mousavi, Khanikaev & Wang (2015)
Nature Communications, ***6***



LW-LF primer

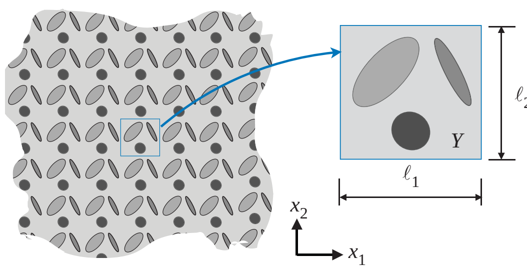
Willis (1981,2009), Norris et al. (2012)

Bensoussan, Lions & Papanicolaou (1978)

Floquet-Bloch approach + asymptotic expansion

Scalar wave equation (G, ρ are Y-periodic)

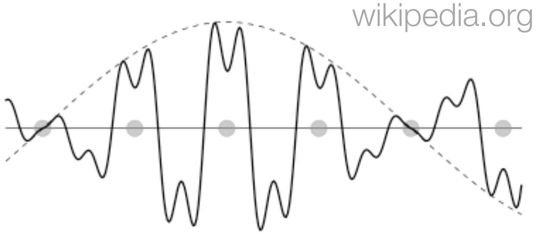
$$-\omega^2\rho(\boldsymbol{x})u - \nabla\cdot\big(G(\boldsymbol{x})\nabla u\big) \;=\; f(\boldsymbol{x}) \qquad \text{in} \;\; \mathbb{R}^d$$



Bloch-wave assumption (PWE)

wikipedia.org

$$u(\boldsymbol{x}) \;=\; \tilde{u}(\boldsymbol{x})\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \tilde{u} : Y\text{-periodic}$$



Reduced problem

$$-\omega^2\rho(\boldsymbol{x})\tilde{u} - \nabla_{\boldsymbol{k}}\cdot\big(G(\boldsymbol{x})\nabla_{\boldsymbol{k}}\tilde{u}\big) \;=\; \tilde{f} \;\; \text{in} \;\; Y,$$

$$\boldsymbol{\nu}\cdot G\nabla_{\boldsymbol{k}}\tilde{u}|_{x_j=0} \;=\; -\,\boldsymbol{\nu}\cdot G\nabla_{\boldsymbol{k}}\tilde{u}|_{x_j=\ell_j}$$

$$\nabla_{\boldsymbol{k}} = \nabla + i\boldsymbol{k}$$

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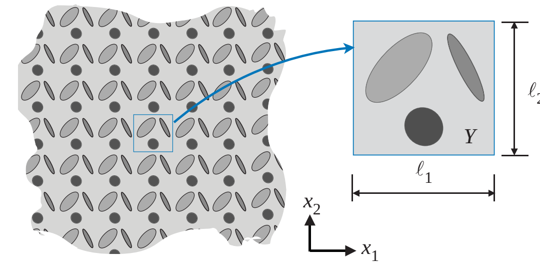
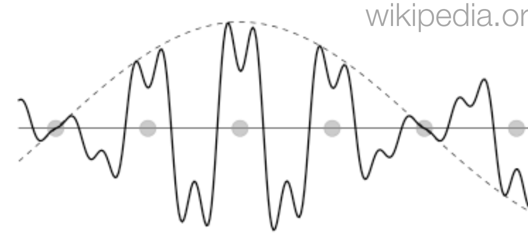


Bloch-wave assumption (PWE)

wikipedia.org



$$u(\boldsymbol{x}) \;=\; \tilde{u}(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \tilde{u}: Y\text{-periodic}$$

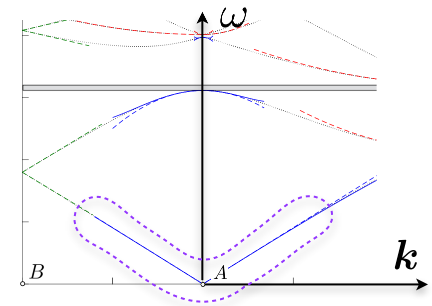
Reduced problem

$$-\omega^2\rho(\boldsymbol{x})\tilde{w} - \nabla_{\boldsymbol{k}}\cdot(G(\boldsymbol{x})\nabla_{\boldsymbol{k}}\tilde{w}) \;=1 \;\; \text{in} \;\; Y,$$

$$\boldsymbol{\nu}\cdot G\nabla_{\boldsymbol{k}}\tilde{w}|_{x_j=0} \;=\; -\boldsymbol{\nu}\cdot G\nabla_{\boldsymbol{k}}\tilde{w}|_{x_j=\ell_j}.$$

$$\nabla_{\boldsymbol{k}} = \nabla + i\boldsymbol{k}$$

Asymptotic analysis



LW-LF regime: $\boldsymbol{k} = \epsilon \hat{\boldsymbol{k}}, \qquad \omega = \epsilon \hat{\omega}$

Expansion

$$\tilde{w}(\boldsymbol{x}) \;=\; \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) \;+\; \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) \;+\; \tilde{w}_2(\boldsymbol{x}) \;+\; \cdots$$

⇨ Series in ϵ

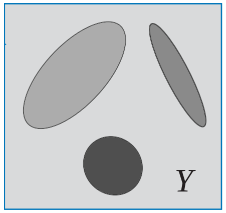
$$-\omega^2\rho(\boldsymbol{x})\tilde{w}(\boldsymbol{x}) - \nabla_{\boldsymbol{k}}\cdot(G(\boldsymbol{x})\nabla_{\boldsymbol{k}}\tilde{w}) \;=\; 1 \quad \text{in} \quad Y,$$

Mean motion: $\langle \tilde{w} \rangle = (\tilde{w}, 1) = \int_Y \tilde{w}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \qquad |Y| = 1$

⇨ $\langle \tilde{w} \rangle \;=\; \epsilon^{-2} w_0 \;+\; \epsilon^{-1} w_1 \;+\; w_2 \;+\; \epsilon\, w_3 \;+\; \cdots$

Meng & G (2018) *PRSA*, **474**

Cascade



O(ϵ^{-2}): $-\nabla\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_0\big) \;=\; 0 \quad \text{in} \quad Y$ ⇨ $\tilde{w}_0(\boldsymbol{x}) = w_0$

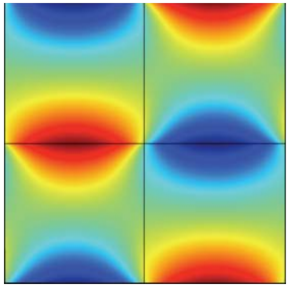
O(ϵ^{-1}): $-\nabla\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_1\big) - \nabla\cdot\big(G(\boldsymbol{x})i\hat{\boldsymbol{k}}\tilde{w}_0\big) - i\hat{\boldsymbol{k}}\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_0\big) \;=\; 0 \quad \text{in} \quad Y$

⇨ $\tilde{w}_1(\boldsymbol{x}) = \boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}}\,w_0 + w_1$

First (*zero-mean*) cell function $\boldsymbol{\chi}^{(1)} \in (H^1_{p0}(Y))^d$

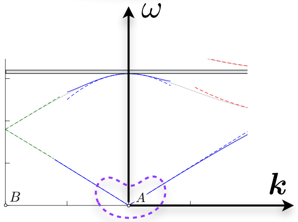
$$\nabla\cdot\big(G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})\big) \;=\; 0 \quad \text{in} \quad Y,$$

$$\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})|_{x_j=0} \;=\; -\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})|_{x_j=\ell_j}$$



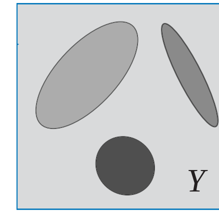
Effective LW-LF equation (quasistatic) $\langle\tilde{u}\rangle \;=\; \tilde{f}\langle\tilde{w}\rangle$

$$-\big(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\hat{\omega}^2\big)\langle\tilde{u}\rangle \;\overset{\epsilon^{-1}}{=}\; \epsilon^{-2}\tilde{f}$$



effective parameters ⇨ $\rho^{(0)} = \langle\rho\rangle, \qquad \boldsymbol{\mu}^{(0)} = \big\langle G\{\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\}\big\rangle$

Cascade



O(ϵ^{-2}): $-\nabla\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_0\big) = 0 \quad \text{in} \quad Y$

$\Rightarrow \quad \tilde{w}_0(\boldsymbol{x}) = w_0$

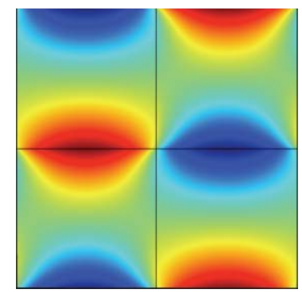
O(ϵ^{-1}): $-\nabla\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_1\big) - \nabla\cdot\big(G(\boldsymbol{x})i\hat{\boldsymbol{k}}\tilde{w}_0\big) - i\hat{\boldsymbol{k}}\cdot\big(G(\boldsymbol{x})\nabla\tilde{w}_0\big) = 0 \quad \text{in} \quad Y$

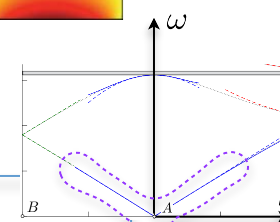
$\Rightarrow \quad \tilde{w}_1(\boldsymbol{x}) = \boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}}w_0 + w_1$

First (*zero-mean*) cell function $\boldsymbol{\chi}^{(1)} \in (H^1_{p0}(Y))^d$

$$\nabla\cdot\big(G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})\big) = 0 \quad \text{in} \quad Y,$$

$$\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})|_{x_j=0} = -\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I})|_{x_j=\ell_j}$$





Effective LW-LF equation (dynamic)

$$-\big(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\hat{\omega}^2\big)\langle\tilde{u}\rangle$$
$$- \epsilon^2\big(\boldsymbol{\mu}^{(2)} : (i\hat{\boldsymbol{k}})^4 + \sigma\boldsymbol{\rho}^{(2)} : (i\hat{\boldsymbol{k}})^2\hat{\omega}^2\big)\langle\tilde{u}\rangle \overset{\epsilon}{=} \epsilon^{-2}\tilde{f}M(\hat{\boldsymbol{k}},\hat{\omega}),$$

$1 + O(\epsilon^2)$

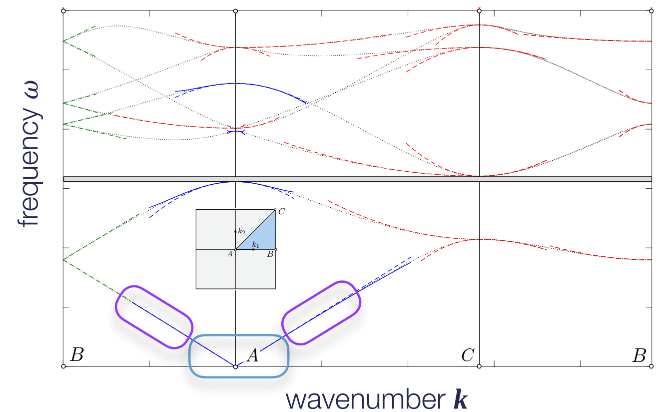
PDE interpretation

$$\boldsymbol{k} = \epsilon\hat{\boldsymbol{k}}, \qquad \omega = \epsilon\hat{\omega}$$

$$-\big(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\hat{\omega}^2\big)\langle\tilde{u}\rangle - \epsilon^2\big(\boldsymbol{\mu}^{(2)} : (i\hat{\boldsymbol{k}})^4 + \boldsymbol{\rho}^{(2)} : (i\hat{\boldsymbol{k}})^2\hat{\omega}^2\big)\langle\tilde{u}\rangle \overset{\epsilon}{=} \epsilon^{-2}\tilde{f}\, M(\hat{\boldsymbol{k}}, \hat{\omega}),$$

$M(\hat{\boldsymbol{k}}, \hat{\omega})$: $1 + O(\epsilon^2)$

Meng & G (2018) *PRSA*, **474**

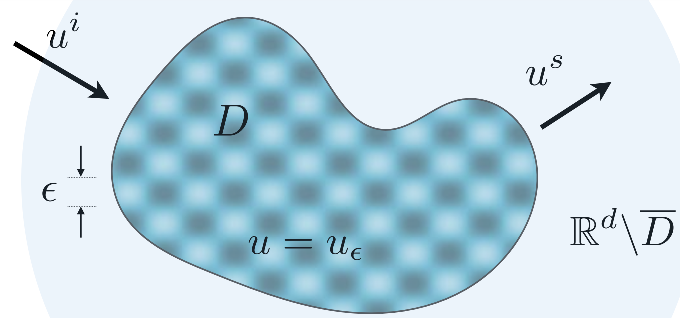


Wautier & G (2015) *JMPS*, **78**

$$-\big(\boldsymbol{\mu}^{(0)} : \nabla^2 + \omega^2\rho^{(0)}\big)\langle u\rangle - \epsilon^2\big(\boldsymbol{\mu}^{(2)} : \nabla^4 + \omega^2\boldsymbol{\rho}^{(2)} : \nabla^2\big)\langle u\rangle \overset{\epsilon^2}{=} f + \epsilon^2\big(\langle G(\nabla\boldsymbol{\eta}^{(1)} + \boldsymbol{I}\eta^{(0)})\rangle : \nabla^2 + \omega^2\langle\rho\eta^{(0)}\rangle\big) f$$

Boundary vs bulk correction

C



THEOREM 2. *Let u_ϵ be the solution to (3), u_0 the solution to (4), and let the bulk correction $u^{(1)}$ be given by (13) in the interior of D and zero on the exterior of D. Then for any ball B_{R} of radius $R > 0$ which contains D,*

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)})\|_{H^1(D)} + \|u_\epsilon - u_0\|_{H^1(B_R \backslash D)} \leq C_{\mathrm{R}}\, \epsilon^{1/2}$$

and

$$\|u_\epsilon - u_0\|_{L^2(B_{\mathrm{R}})} \leq C_{\mathrm{R}}\, \epsilon,$$

Cakoni, G, Moskow (2016) *SIMA*, **48**

THEOREM 5. *Let u_ϵ be the solution to (3), u_0 the solution to (4), and let the bulk and boundary corrections $u^{(1)}$ and θ_ϵ be given by (13) and (18), respectively. We also note that in the definition of θ_ϵ, $v^{(1)}$ must be given by (63). Then for any ball B_{R} of radius $R > 0$ which contains D, we have*

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon\theta_\epsilon)\|_{H^1(B_{\mathrm{R}} \backslash D)} \leq C_{\mathrm{R}}\, \epsilon^{3/2} \|u_0\|_{H^4(D)}$$

and

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{L^2(B_{\mathrm{R}})} \leq C_{\mathrm{R}}\, \epsilon^2 \|u_0\|_{H^4(D)},$$

Finite frequencies?

$$\boldsymbol{k} = \epsilon \hat{\boldsymbol{k}}, \qquad \omega = \epsilon \hat{\omega}$$

Primal problem

$$-\omega^2 \rho(\boldsymbol{x}) \tilde{w} - \nabla_{\boldsymbol{k}} \cdot (G(\boldsymbol{x}) \nabla_{\boldsymbol{k}} \tilde{w}) \;=\; 1 \quad \text{in } \; Y,$$

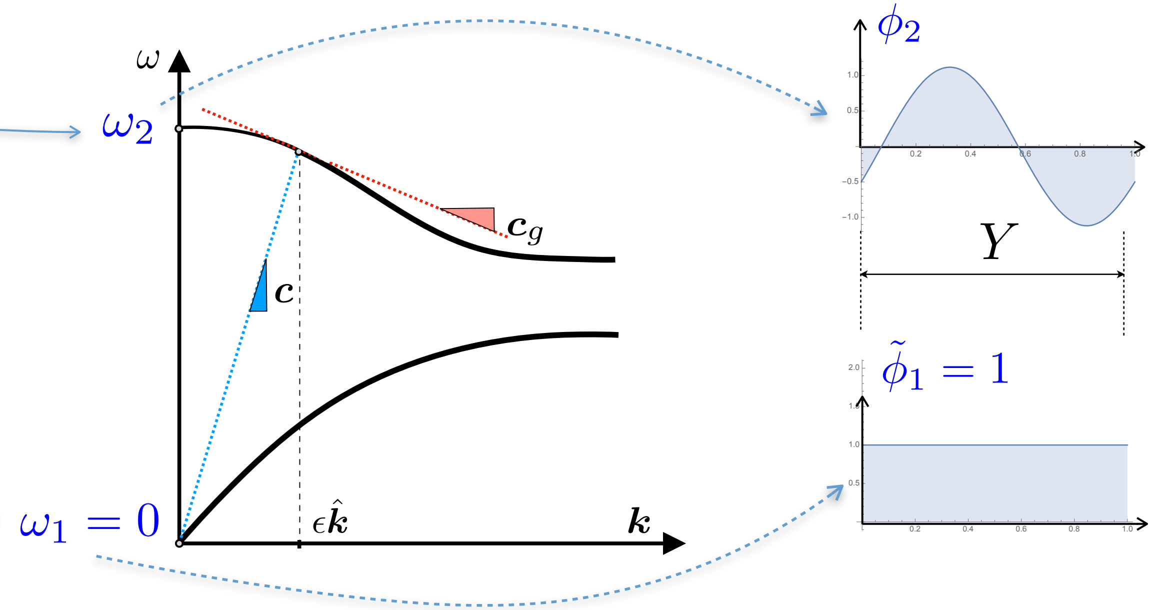
$$\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{w}|_{x_j=0} \;=\; -\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{w}|_{x_j=\ell_j}.$$

Eigensystem

$$\{\tilde{\phi}_m(\cdot;\boldsymbol{k}) \in H^1_p(Y),\ \tilde{\lambda}_m(\boldsymbol{k}) \in \mathbb{R}\} \qquad (\rho\tilde{\phi}_m, \tilde{\phi}_n) \;=\; \delta_{mn}(\rho\tilde{\phi}_n, \tilde{\phi}_n)$$

$$-\tilde{\lambda}_m \rho(\boldsymbol{x})\, \tilde{\phi}_m - \nabla_{\boldsymbol{k}} \cdot \big(G(\boldsymbol{x}) \nabla_{\boldsymbol{k}} \tilde{\phi}_m\big) \;=\; 0 \quad \text{in } \; Y,$$

$$\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{\phi}_m|_{x_j=0} \;=\; -\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{\phi}_m|_{x_j=\ell_j}$$



$$\tilde{\lambda}_m = \omega_m^2$$

Effective motion

$\boldsymbol{k} = \epsilon\hat{\boldsymbol{k}}, \qquad \omega = \epsilon\hat{\omega}$

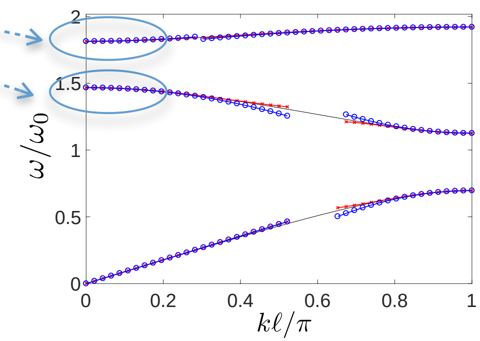
$$\langle\tilde{u}\rangle \;=\; (\tilde{u}, \tilde{\phi}_n)$$

LW-FF approximation

$$\tilde{\phi}_n^\circ(\boldsymbol{x}) = \tilde{\phi}_n(\boldsymbol{x};\boldsymbol{0})$$

Scaling (isolated eigenvalues)

$$\boldsymbol{k} \;=\; \epsilon\hat{\boldsymbol{k}}, \qquad \omega^2 \;=\; \tilde{\lambda}_n^\circ + \epsilon^2\sigma\hat{\omega}^2$$



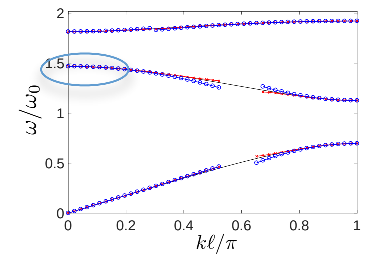
Effective LW-FF equation

$$-\big(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\hat{\omega}^2\big)\langle\tilde{u}\rangle - \epsilon^2\big(\boldsymbol{\mu}^{(2)} : (i\hat{\boldsymbol{k}})^4 + \sigma\boldsymbol{\rho}^{(2)} : (i\hat{\boldsymbol{k}})^2\hat{\omega}^2\big)\langle\tilde{u}\rangle \;\stackrel{\epsilon}{=}\; \epsilon^{-2}\tilde{f}\, M(\hat{\boldsymbol{k}},\hat{\omega})$$

$1 + O(\epsilon)$

$$\rho^{(0)} \;=\; \langle\rho\tilde{\phi}_n^\circ\rangle, \qquad \boldsymbol{\mu}^{(0)} \;=\; \big\langle G\{\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\tilde{\phi}_n^\circ\}\big\rangle - \big(G\{\boldsymbol{\chi}^{(1)}\otimes\nabla\tilde{\phi}_n^\circ\}, 1\big)$$

Local fields



$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \cdots$$

$$\langle\tilde{w}\rangle = \epsilon^{-2}w_0 + \epsilon^{-1}w_1 + w_2 + \epsilon w_3 + \cdots \quad \Leftarrow \text{ Effective eq.}$$

$$\tilde{w}_0(\boldsymbol{x}) = w_0\,\tilde{\phi}_n^{\circ}(\boldsymbol{x})$$

$$\tilde{w}_1(\boldsymbol{x}) = w_0\,\boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}} + w_1\,\tilde{\phi}_n^{\circ}(\boldsymbol{x}) \quad \Rightarrow \quad \tilde{w}_1 \in \operatorname{span}\{\tilde{\phi}_n^{\circ}, \chi_1^{(1)}, \chi_2^{(1)}\}$$

$$\tilde{w}_2(\boldsymbol{x}) = w_0\,\boldsymbol{\chi}^{(2)}(\boldsymbol{x}) : (i\hat{\boldsymbol{k}})^2 + w_1\,\boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}} + w_2\,\tilde{\phi}_n^{\circ}(\boldsymbol{x}) + \eta^{(0)}(\boldsymbol{x})$$

$$\tilde{\lambda}_n^{\circ}\rho\boldsymbol{\chi}^{(1)} + \nabla\cdot\left(G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\tilde{\phi}_n^{\circ})\right) + G\nabla\tilde{\phi}_n^{\circ} = 0 \quad \text{in } Y,$$

$$\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\tilde{\phi}_n^{\circ})|_{x_j=0} = -\boldsymbol{\nu}\cdot G(\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\tilde{\phi}_n^{\circ})|_{x_j=\ell_j}$$

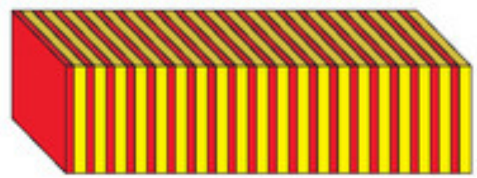
$$\tilde{\lambda}_n^{\circ}\rho\boldsymbol{\chi}^{(2)} + \nabla\cdot\left(G\left(\nabla\boldsymbol{\chi}^{(2)} + \{\boldsymbol{I}\otimes\boldsymbol{\chi}^{(1)}\}'\right)\right) + G\{\nabla\boldsymbol{\chi}^{(1)} + \boldsymbol{I}\tilde{\phi}_n^{\circ}\} - \frac{\rho}{\rho^{(0)}}\boldsymbol{\mu}^{(0)}\tilde{\phi}_n^{\circ} = 0 \quad \text{in } Y,$$

$$\boldsymbol{\nu}\cdot G\left(\nabla\boldsymbol{\chi}^{(2)} + \{\boldsymbol{I}\otimes\boldsymbol{\chi}^{(1)}\}'\right)|_{x_j=0} = -\boldsymbol{\nu}\cdot G\left(\nabla\boldsymbol{\chi}^{(2)} + \{\boldsymbol{I}\otimes\boldsymbol{\chi}^{(1)}\}'\right)|_{x_j=\ell_j},$$

1D BVP

C

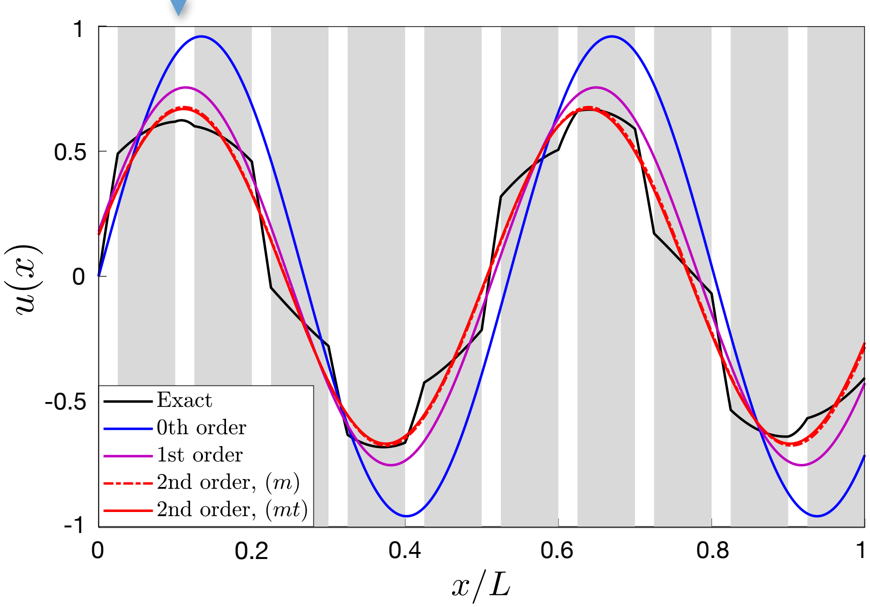
Dirichlet

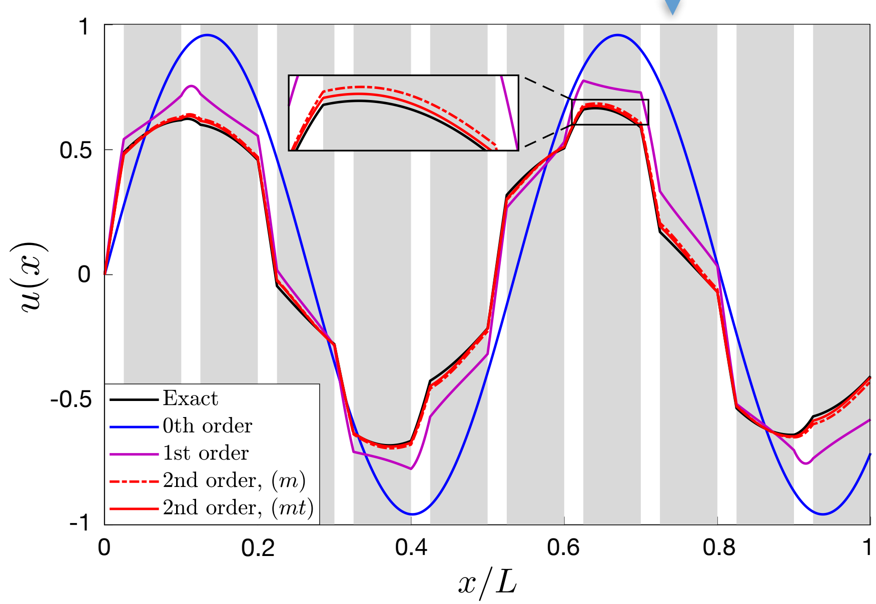


Neumann

$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \cdots$$

$$\langle \tilde{w} \rangle = \epsilon^{-2} w_0 + \epsilon^{-1} w_1 + w_2 + \epsilon w_3 + \cdots$$





Cornaggia, G (2019) in review

Role of eigenfunctions

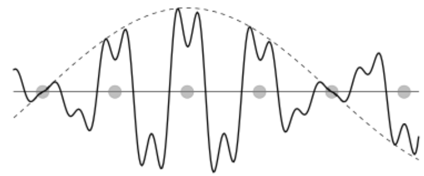
Brillouin (1953) *Wave Propagation in Periodic Structures*

Bensoussan, Lions & Papanicolaou (1978) *Asymptotic Analysis for Periodic Structures*

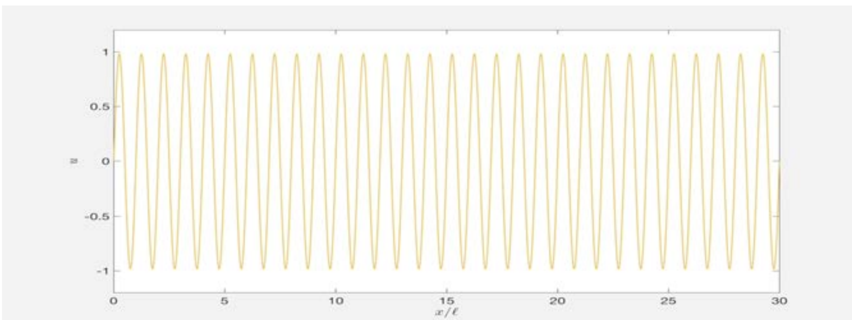
Birman & Suslina (2006), J. Math. Sci., **136**

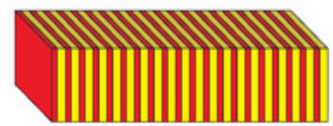
Daya et al. (2002). *Compt. Rend. Mecanique*, **330**

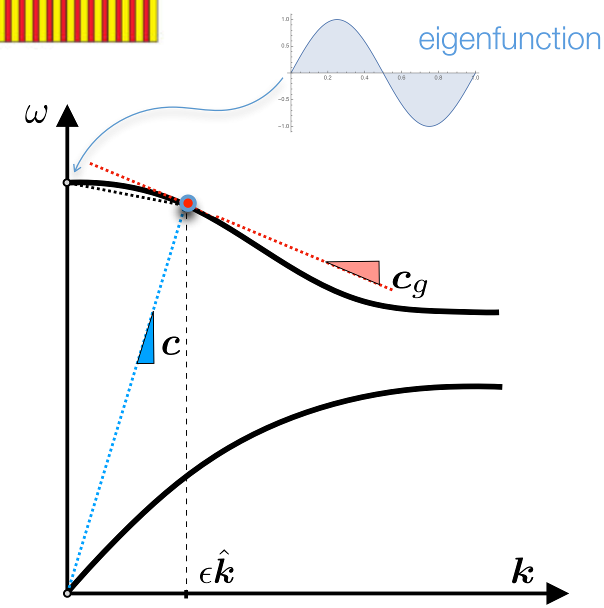
Dossou et al. (2006), *PRA,* **77**; Craster et al. (2010), *PRSA,* **466**



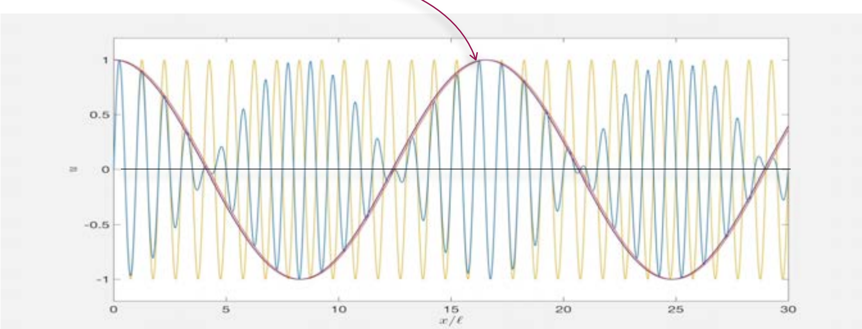
https://en.wikipedia.org/wiki/Bloch_wave



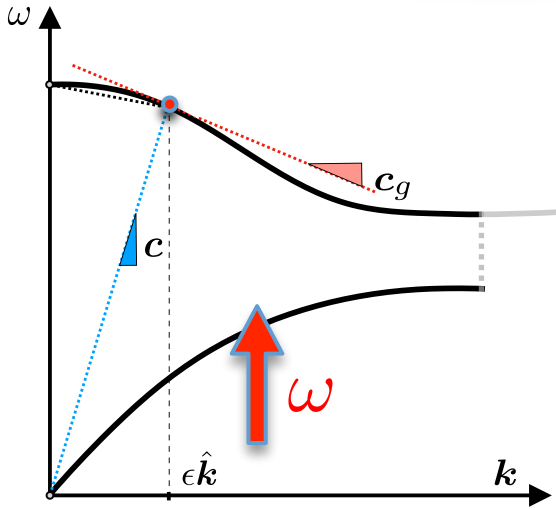


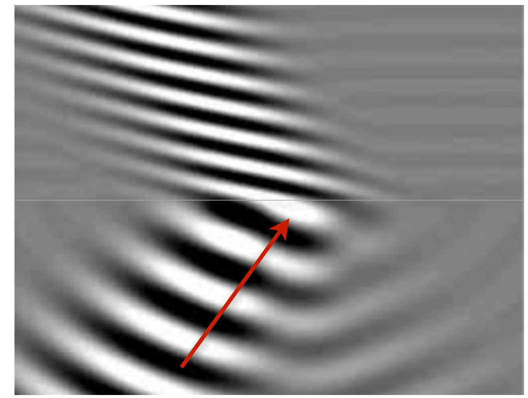


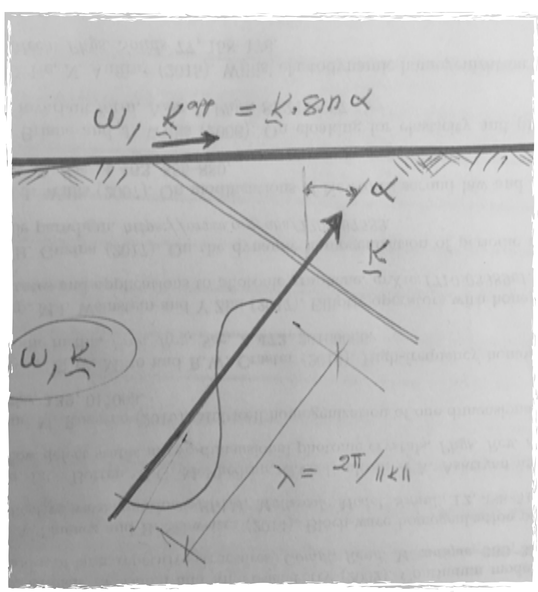
$\Leftarrow$ $\langle \tilde{u} \rangle = (\tilde{u}, \tilde{\phi}_n^\circ)$



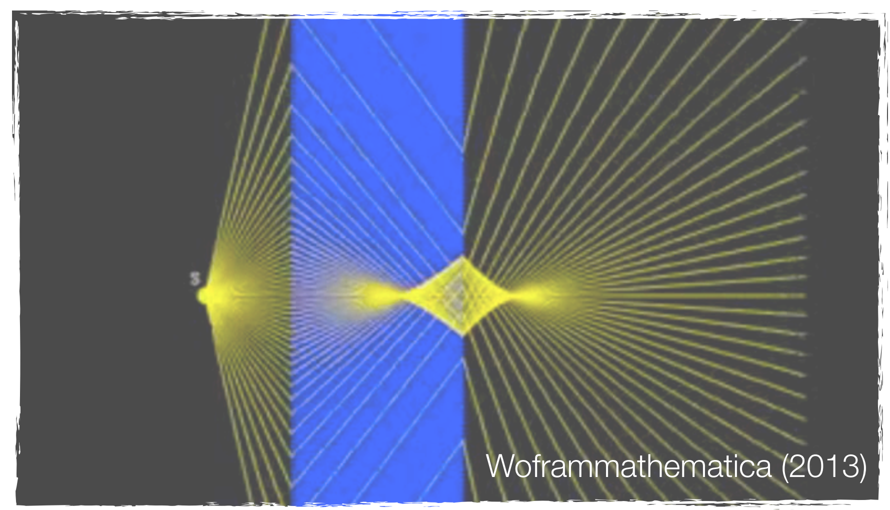
Negative-index metamaterials



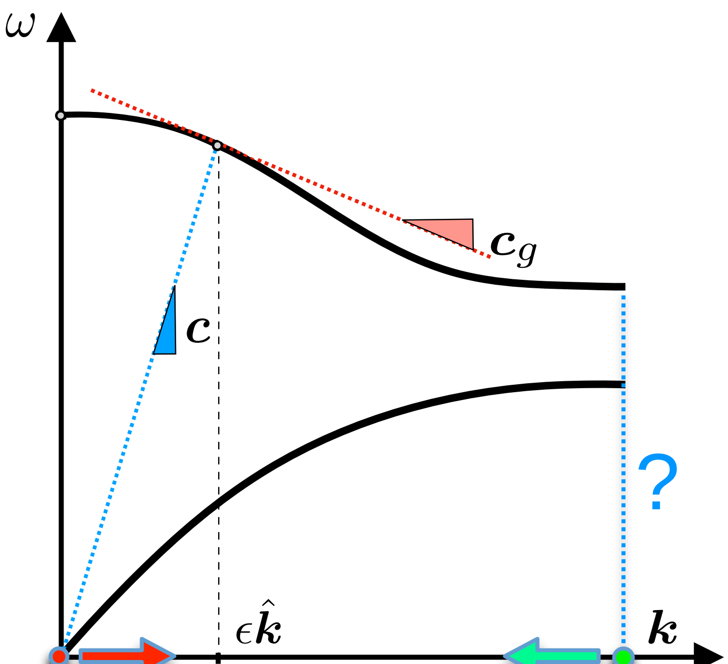


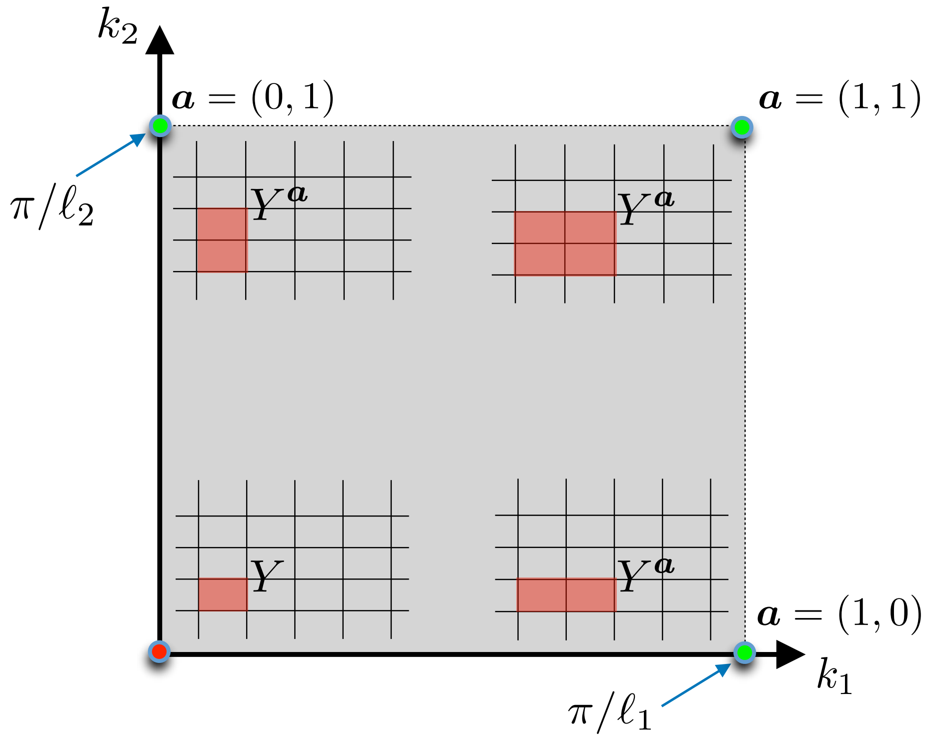


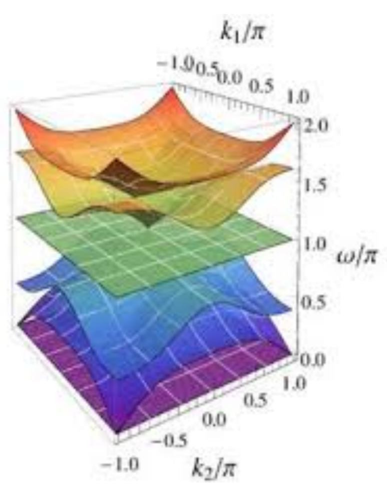
$$sign(\boldsymbol{c} \cdot \boldsymbol{c}_g) \;=\; sign(\boldsymbol{\mu}^{(0)} : \hat{\boldsymbol{k}}^2)$$



Apexes of the 1st Brillouin zone



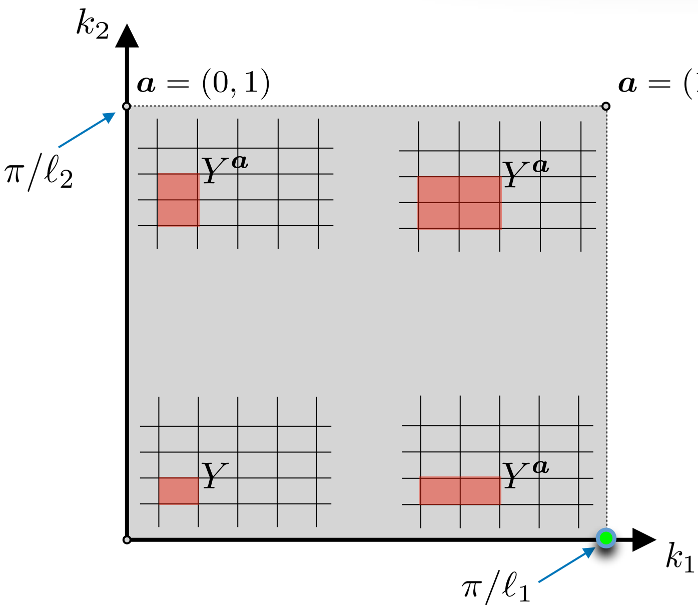




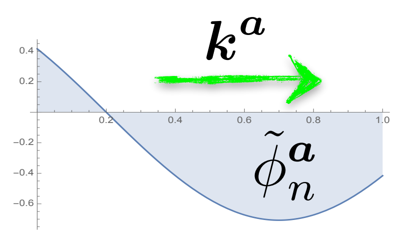
Multi-cell analysis

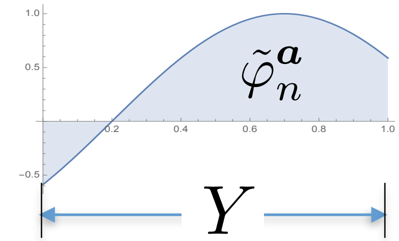
Vasiliev et al (2005) *IJSS*, **42**

Propagating eigenfunction

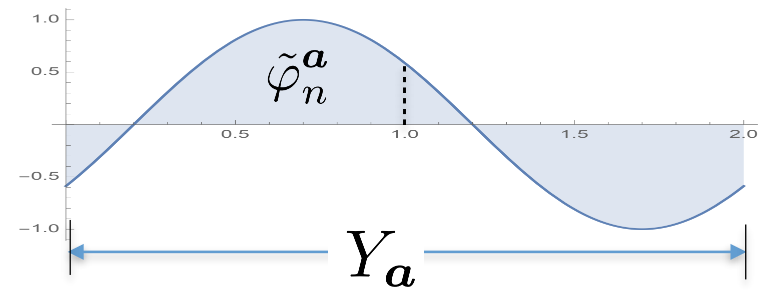


$$\tilde{\phi}_n^{\boldsymbol{a}}(\boldsymbol{x}) = \tilde{\varphi}_n^{\boldsymbol{a}}(\boldsymbol{x})\, e^{-i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}$$





Extending ⇨

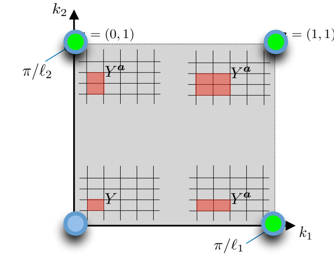


Eigenfunction

$$-(\nabla + i\boldsymbol{k}^{\boldsymbol{a}})\cdot\big(G(\nabla + i\boldsymbol{k}^{\boldsymbol{a}})\tilde{\phi}_n^{\boldsymbol{a}}\big) = \tilde{\lambda}_n^{\boldsymbol{a}}\,\rho\,\tilde{\phi}_n^{\boldsymbol{a}} \quad \text{in} \quad Y,$$

$$-\nabla\cdot\big(G\nabla\tilde{\varphi}_n^{\boldsymbol{a}}\big) = \tilde{\lambda}_n^{\boldsymbol{a}}\,\rho\,\tilde{\varphi}_n^{\boldsymbol{a}} \quad \text{in} \quad Y_{\boldsymbol{a}},$$

Factorized ansatz



Origin, $\boldsymbol{k}^{\boldsymbol{a}} = \boldsymbol{0}$

$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \epsilon\tilde{w}_3(\boldsymbol{x}) + \cdots$$

$$w = \langle \tilde{w} \rangle = (\tilde{w}, \tilde{\phi}_n^{\circ})_{\overline{Y}}$$

$$w_m = \langle \tilde{w} \rangle = (\tilde{w}_m, \tilde{\phi}_n^{\circ})_{\overline{Y}}$$

$$\langle \tilde{w} \rangle = \epsilon^{-2} w_0 + \epsilon^{-1} w_1 + w_2 + \epsilon w_3 + \cdots$$

Apex, $\boldsymbol{k}^{\boldsymbol{a}} \neq \boldsymbol{0}$

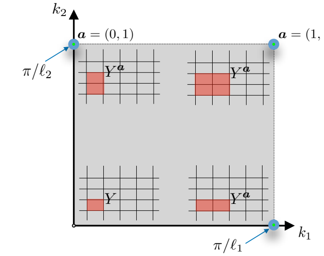
$$\tilde{w}(\boldsymbol{x}) = \underline{e^{-i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}}(\epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \epsilon\tilde{w}_3(\boldsymbol{x}) + \cdots)$$

$$w = \langle \tilde{w} \rangle_{\boldsymbol{a}} = (\tilde{w}, \tilde{\phi}_n^{\boldsymbol{a}})_{\overline{Y}_{\boldsymbol{a}}}$$

$$w_m = \langle \tilde{w}_m \rangle_{\boldsymbol{a}}^{\varphi} = (\tilde{w}_m, \tilde{\varphi}_m^{\boldsymbol{a}})_{\overline{Y}_{\boldsymbol{a}}}$$

$$\langle \tilde{w} \rangle_{\boldsymbol{a}} = \epsilon^{-2} w_0 + \epsilon^{-1} w_1 + w_2 + \epsilon w_3 + \cdots$$

Effective model



Averages

physics description

$$\langle \tilde{u} \rangle_{\boldsymbol{a}} = \frac{1}{|Y_{\boldsymbol{a}}|} \big(\tilde{u}, \tilde{\phi}_n^{\boldsymbol{a}}\big)_{Y_{\boldsymbol{a}}} \qquad \langle \tilde{g} \rangle_{\boldsymbol{a}}^{\varphi} = \frac{1}{|Y_{\boldsymbol{a}}|} \big(\tilde{g}, \tilde{\varphi}_n^{\boldsymbol{a}}\big)_{Y_{\boldsymbol{a}}}$$

Effective PDE

$$\langle \tilde{u} \rangle_{\boldsymbol{a}} = f \, \langle \tilde{w} \rangle_{\boldsymbol{a}}$$

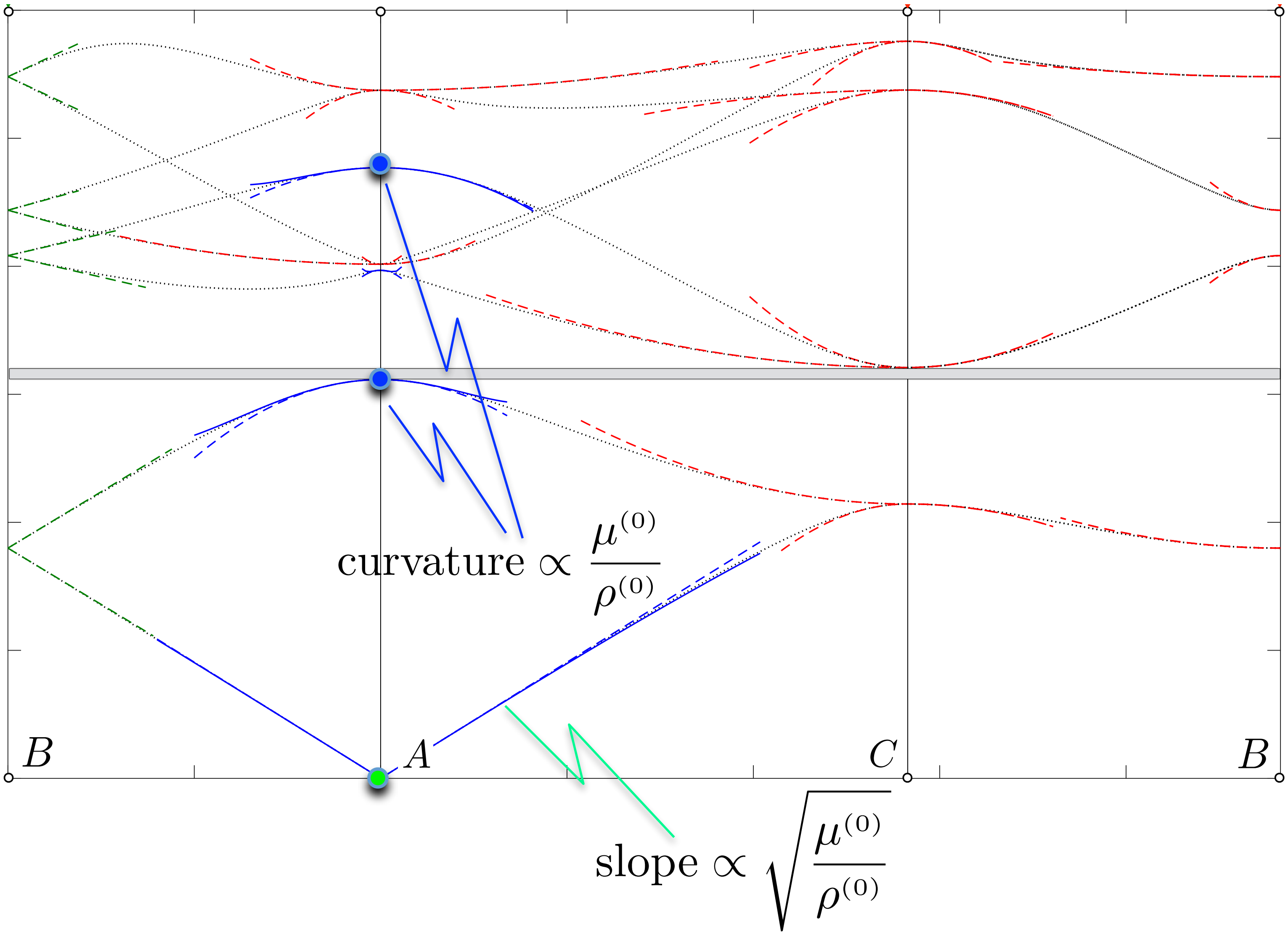
$$-\Big(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)} \hat{\omega}^2\Big) \langle \tilde{u} \rangle_{\boldsymbol{a}} - \epsilon^2 \Big(\boldsymbol{\mu}^{(2)} : (i\hat{\boldsymbol{k}})^4 + \sigma \boldsymbol{\rho}^{(2)} : (i\hat{\boldsymbol{k}})^2 \hat{\omega}^2\Big) \langle \tilde{u} \rangle_{\boldsymbol{a}} \overset{\epsilon}{=} \epsilon^{-2} \tilde{f} M(\hat{\boldsymbol{k}}, \hat{\omega})$$

$e^{i\boldsymbol{k}^{\boldsymbol{a}} \cdot \boldsymbol{x}}$

$$\rho^{(0)} = \langle \rho \tilde{\varphi}_n^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{\varphi}, \qquad \boldsymbol{\mu}^{(0)} = \big\langle G\{\nabla \boldsymbol{\chi}^{(1)} + \boldsymbol{I} \tilde{\varphi}_n^{\boldsymbol{a}}\} \big\rangle_{\boldsymbol{a}}^{\varphi} - \big(G\{\boldsymbol{\chi}^{(1)} \otimes \nabla \tilde{\varphi}_n^{\boldsymbol{a}}\}, 1\big)_{\overline{Y_{\boldsymbol{a}}}}$$

G, Meng & Oudghiri-Idrissi (2019) *PRSA*, **475**

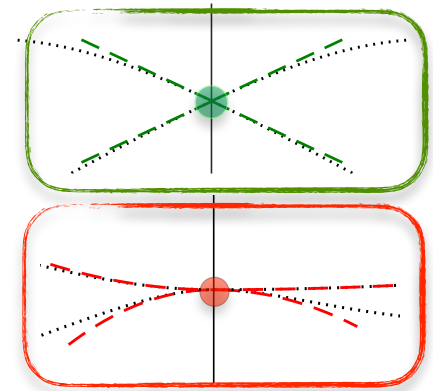
Effective parameters



Uncoupling

Average effective density (energy): invariant

$$\rho^{(0)} = \frac{1}{Q}\sum_q \rho_q^{(0)} \longrightarrow D_{pq}^{-1/2} = \delta_{pq}\,(\rho_q^{(0)})^{-1/2}$$



→ fixed direction $\hat{\boldsymbol{k}}/||\hat{\boldsymbol{k}}||$

Transformation ($\boldsymbol{R}_{ps}$ orthogonal, real-valued)

$$T_{pq} = \sqrt{\rho^{(0)}}\sum_s R_{ps}^T D_{sq}^{-1/2}$$

$$\widetilde{\tilde{\varphi}_{np}^{\boldsymbol{a}}} = \sum_q T_{pq}\,\tilde{\varphi}_{nq}^{\boldsymbol{a}}$$

$$\begin{bmatrix} 0 & A_{12} & 0 & 0 \\ -A_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} \\ 0 & 0 & -A_{34} & 0 \end{bmatrix}$$

$$\widetilde{A_{pq}} = \widetilde{\boldsymbol{\theta}_{pq}^{(0)}}\cdot(i\hat{\boldsymbol{k}})$$ 2x2 block-diagonal

$$\widetilde{D_{pq}} = \delta_{pq}\,\rho^{(0)}$$ ∝ identity

$$\widetilde{B_{pq}} = \widetilde{\boldsymbol{\mu}_{pq}^{(0)}} : (i\hat{\boldsymbol{k}})^2$$ diagonal