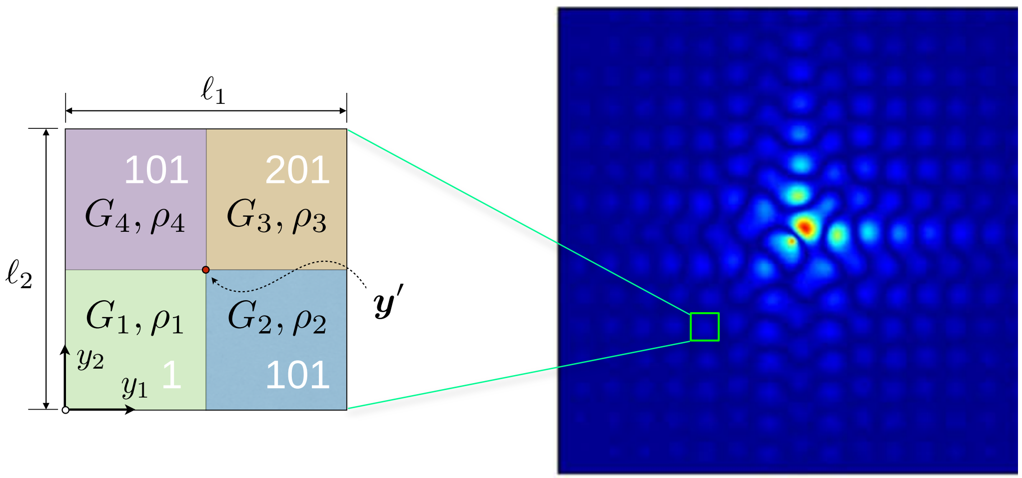
Part III

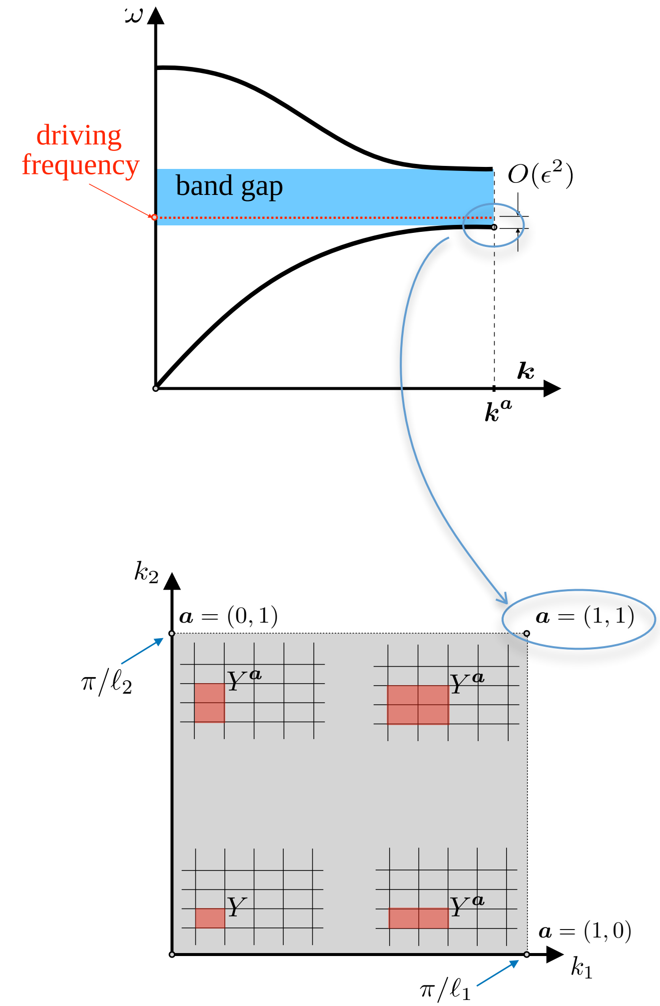
Effective Green's function

G/Meng/Oudghiri-Idrissi
Meng/Oudghiri-Idrissi/G

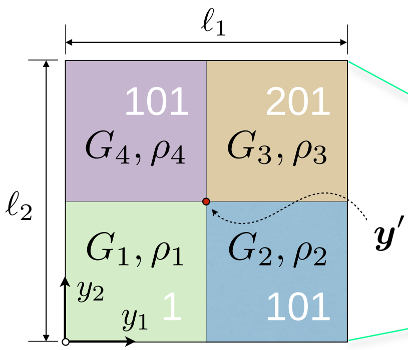
Green’s function near the edge of a band gap

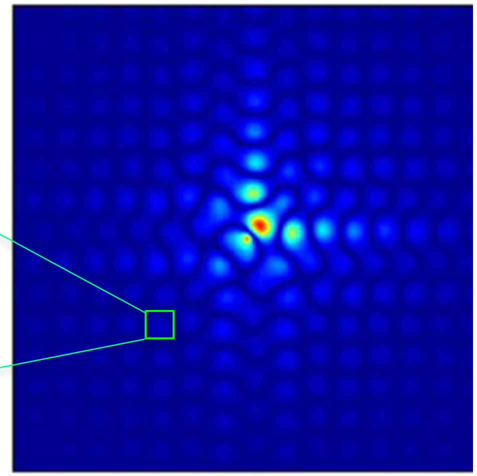


Craster et al. (2018)
J. Opt. Soc. Am., **28**



Green's function near the edge of a band gap





Craster et al. (2018)
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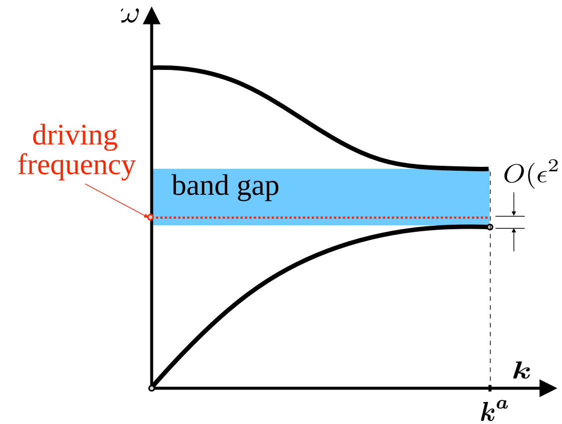
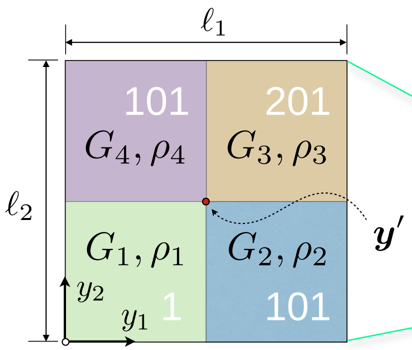


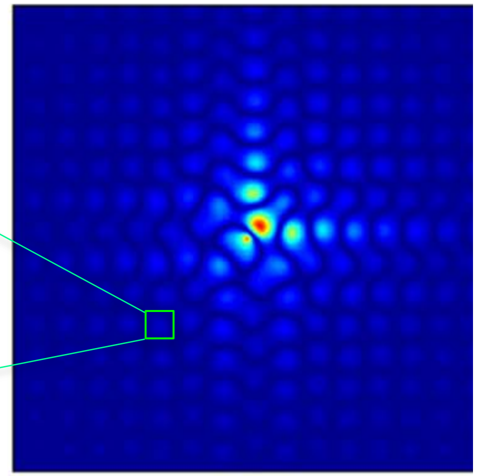
Effective equation

$$-\left(\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\hat{\omega}^2\right)\langle\tilde{u}\rangle - \epsilon^2\left(\boldsymbol{\mu}^{(2)} : (i\hat{\boldsymbol{k}})^4 + \sigma\boldsymbol{\rho}^{(2)} : (i\hat{\boldsymbol{k}})^2\hat{\omega}^2\right)\langle\tilde{u}\rangle \overset{\epsilon}{=} \epsilon^{-2}\tilde{f}\,M(\hat{\boldsymbol{k}},\hat{\omega}),$$

$M(\hat{\boldsymbol{k}},\hat{\omega}) = 1 + O(\epsilon)$

Green's function near the edge of a band gap





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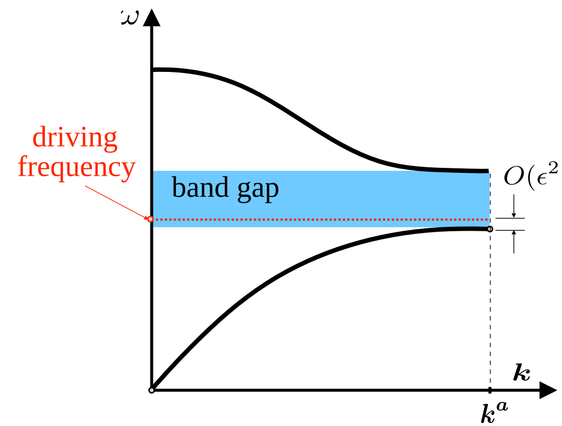
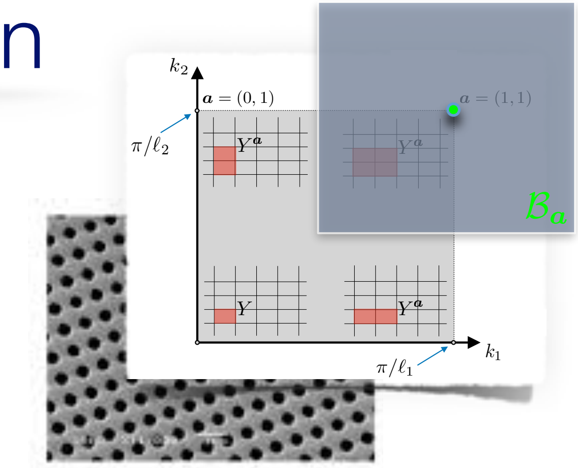


Effective PDE

$$-\left(\boldsymbol{\mu}^{(0)}\!:\!\nabla_{\boldsymbol{y}}^2 + \rho^{(0)}(\omega^2 - \tilde{\lambda}_{\hat{n}}^{\boldsymbol{a}})\right)\langle \tilde{u} \rangle \;=\; \left[\tilde{\varphi}_{\hat{n}}^{\boldsymbol{a}}(\boldsymbol{y}') \;-\; \boldsymbol{\chi}^{(1)}(\boldsymbol{y}')\cdot\nabla_{\boldsymbol{y}}\right]\delta(\boldsymbol{y}-\boldsymbol{y}')$$

Bloch expansion thm…

Bloch expansion



$Y_{\boldsymbol{a}}$ -periodic medium

Rapidly decaying function $g: \mathbb{R}^d \to \mathbb{C}$

Bloch wave function @wavenumber $\boldsymbol{k}$

$$\tilde{g}(\boldsymbol{y};\boldsymbol{k}) \;=\; \sum_{\boldsymbol{\gamma_a}} g(\boldsymbol{y}+\boldsymbol{\gamma_a}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{y}+\boldsymbol{\gamma_a})}, \qquad g = u, f$$

lattice vectors

Bloch expansion thm Titchmarsh (1939), Odeh & Keller (1964)

$$u(\boldsymbol{y}) \;=\; |\mathcal{B}_{\boldsymbol{a}}|^{-1} \int_{\boldsymbol{k}_s+\mathcal{B}_{\boldsymbol{a}}} \tilde{u}(\boldsymbol{y};\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\, \mathrm{d}\boldsymbol{k}, \qquad \boldsymbol{y}\in\mathbb{R}^d \qquad \boldsymbol{k}_s = \boldsymbol{k}^{\boldsymbol{a}}$$

Local approximation (leading-order)

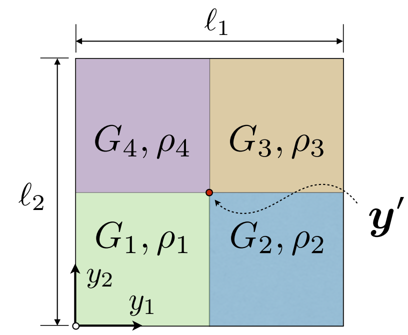
$$\tilde{u}(\boldsymbol{y};\boldsymbol{k}^{\boldsymbol{a}}+\epsilon\hat{\boldsymbol{k}}) \;\simeq\; \langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\, \tilde{\phi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y}), \qquad \boldsymbol{y}\in\mathbb{R}^d$$

$$\Rightarrow \quad u(\boldsymbol{y}) \overset{(i)}{\simeq} |\mathcal{B}_{\boldsymbol{a}}|^{-1}\, \tilde{\phi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y})\, e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{y}} \int_{\mathcal{B}_{\boldsymbol{a}}} \langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\, e^{i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\, \mathrm{d}(\epsilon\hat{\boldsymbol{k}}) \;=\; \tilde{\varphi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y})\, U_{\boldsymbol{a}}(\boldsymbol{y})$$

Field equation

Bloch wave function @ wavenumber $\boldsymbol{k}$

$$\tilde{g}(\boldsymbol{y};\boldsymbol{k}) = \sum_{\boldsymbol{\gamma_a}} g(\boldsymbol{y}+\boldsymbol{\gamma_a})e^{-i\boldsymbol{k}\cdot(\boldsymbol{y}+\boldsymbol{\gamma_a})}$$



Source term

$$\tilde{f}(\boldsymbol{y};\boldsymbol{k}) = \delta(\boldsymbol{y}-\boldsymbol{y}')e^{-i(\boldsymbol{k^a}+\epsilon\hat{\boldsymbol{k}})\cdot\boldsymbol{y}}$$

Field equation

$$M(\hat{\boldsymbol{k}},\hat{\omega}) = \langle e^{i\boldsymbol{k^a}\cdot\boldsymbol{x}}\rangle_{\boldsymbol{a}}^{\varphi} - \epsilon\,(e^{i\boldsymbol{k^a}\cdot\boldsymbol{x}}\boldsymbol{\chi}^{(1)},1)_{\overline{Y_a}}\cdot i\hat{\boldsymbol{k}}$$

$$-\left(\boldsymbol{\mu}^{(0)}\!:(i\epsilon\hat{\boldsymbol{k}})^2 + \rho^{(0)}\epsilon^2\sigma\hat{\omega}^2\right)\langle\tilde{u}\rangle_{\boldsymbol{a}} = \langle e^{-i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\delta(\boldsymbol{y}-\boldsymbol{y}')\rangle_{\boldsymbol{a}}^{\varphi} - (e^{-i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\boldsymbol{\chi}^{(1)}(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{y}'),1)_{\overline{Y_a}}\cdot i\epsilon\hat{\boldsymbol{k}}$$

$$= |Y_{\boldsymbol{a}}|^{-1}e^{-i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}'}\left(\tilde{\varphi}_{\hat{n}}^{\boldsymbol{a}}(\boldsymbol{y}') - \boldsymbol{\chi}^{(1)}(\boldsymbol{y}')\cdot i\epsilon\hat{\boldsymbol{k}}\right),$$

Effective term dependent on $\boldsymbol{y}'$

Field equation

$$-\left(\boldsymbol{\mu}^{(0)}\colon (i\epsilon\hat{\boldsymbol{k}})^2 + \rho^{(0)}\epsilon^2\sigma\hat{\omega}^2\right)\langle\tilde{u}\rangle_{\boldsymbol{a}} \;=\; |Y_{\boldsymbol{a}}|^{-1} e^{-i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}'}\left(\tilde{\varphi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y}') \;-\; \boldsymbol{\chi}^{(1)}(\boldsymbol{y}')\cdot i\epsilon\hat{\boldsymbol{k}}\right)$$

Local approximation (i)

$$\tilde{u}(\boldsymbol{y};\boldsymbol{k}^{\boldsymbol{a}}+\epsilon\hat{\boldsymbol{k}}) \;\simeq\; \langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\,\tilde{\phi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y}), \qquad \epsilon\hat{\boldsymbol{k}}\in\mathcal{B}_{\boldsymbol{a}}$$

Spatial domain

$$u(\boldsymbol{y}) \;\overset{(i)}{\simeq}\; |\mathcal{B}_{\boldsymbol{a}}|^{-1}\,\tilde{\phi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y})\,e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{y}}\int_{\mathcal{B}_{\boldsymbol{a}}}\langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\,e^{i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\,\mathrm{d}(\epsilon\hat{\boldsymbol{k}}) \;=\; \tilde{\varphi}^{\boldsymbol{a}}_{\hat{n}}(\boldsymbol{y})\,U_{\boldsymbol{a}}(\boldsymbol{y})$$

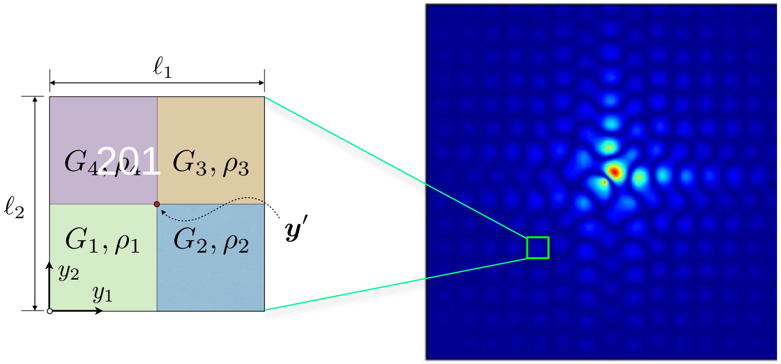
Approximation (ii)

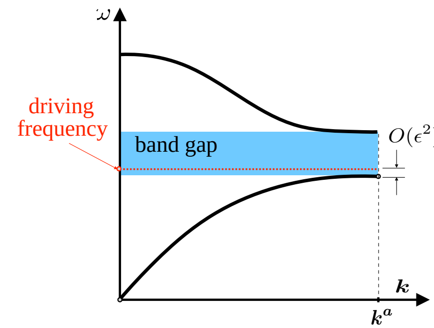
$$U_{\boldsymbol{a}}(\boldsymbol{y}) \;=\; |\mathcal{B}_{\boldsymbol{a}}|^{-1}\int_{\mathcal{B}_{\boldsymbol{a}}}\langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\,e^{i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\,\mathrm{d}(\epsilon\hat{\boldsymbol{k}}) \;\overset{(ii)}{\simeq}\; |\mathcal{B}_{\boldsymbol{a}}|^{-1}\int_{\mathbb{R}^d}\langle\tilde{u}\rangle_{\boldsymbol{a}}[\epsilon\hat{\boldsymbol{k}}]\,e^{i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\,\mathrm{d}(\epsilon\hat{\boldsymbol{k}})$$

$$O(\epsilon^2 + \|\epsilon\hat{\boldsymbol{k}}\|^2)^{-1} \quad \text{for} \quad \epsilon\hat{\boldsymbol{k}}\in\mathbb{R}^d$$

Dossou et al. (2008)
Phys. Rev. A, **77**

Green's function near the edge of a band gap





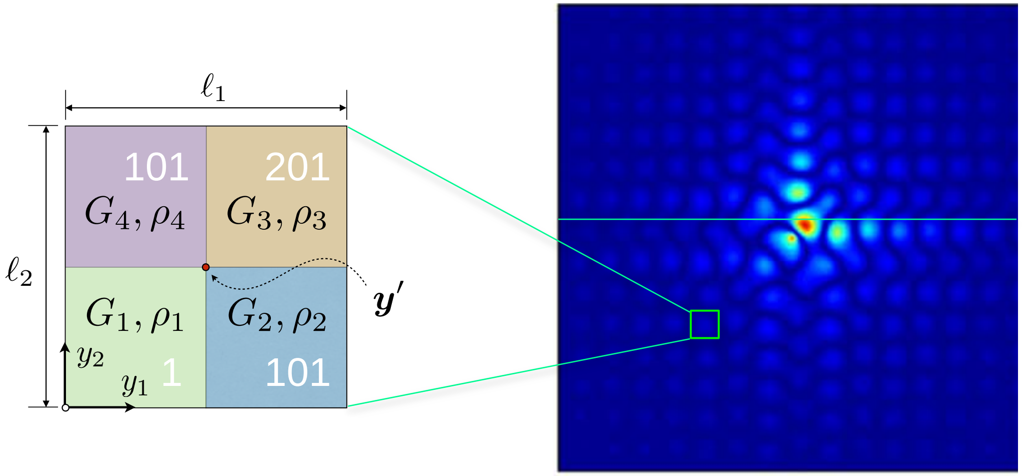
$$-\left(\boldsymbol{\mu}^{(0)}\!:\!\nabla_{\boldsymbol{y}}^2 + \rho^{(0)}(\omega^2 - \tilde{\lambda}_{\hat{n}}^{\boldsymbol{a}})\right) U_{\boldsymbol{a}} \;=\; \left[\tilde{\varphi}_{\hat{n}}^{\boldsymbol{a}}(\boldsymbol{y}') \;-\; \boldsymbol{\chi}^{(1)}(\boldsymbol{y}')\cdot\nabla_{\boldsymbol{y}}\right] F(\boldsymbol{y}-\boldsymbol{y}')$$

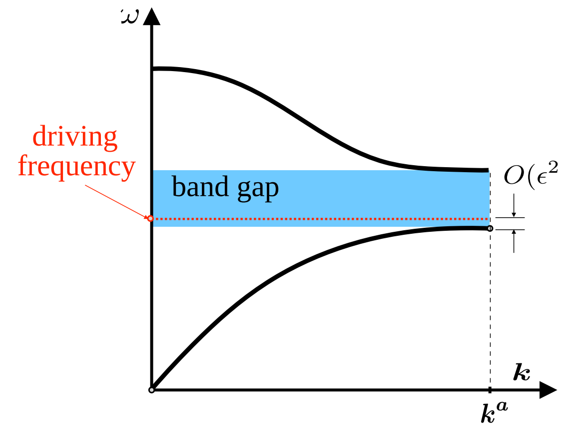
$$F(\boldsymbol{y}-\boldsymbol{y}') \;=\; \begin{cases} |Y_{\boldsymbol{a}}|^{-1} \prod_{j=1}^{d} \operatorname{sinc}\left[\frac{\pi(y_j - y_j')}{(1+a_j)\ell_j}\right], & \text{approx. (i)} \\ \delta(\boldsymbol{y}-\boldsymbol{y}'), & \text{approx. (ii)} \end{cases}$$

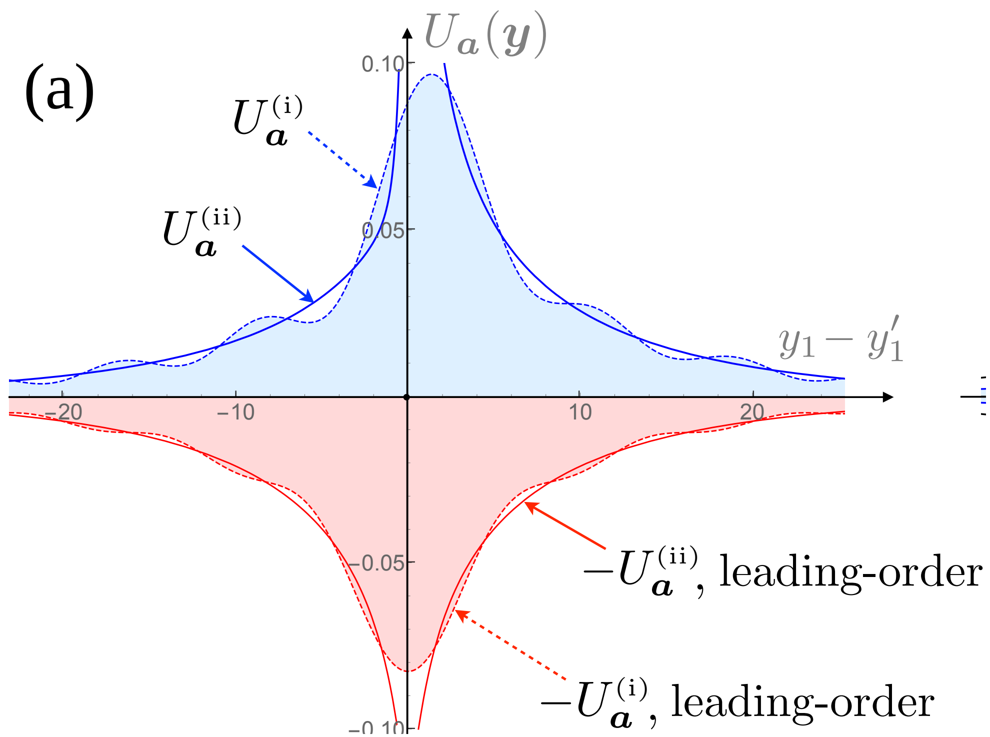
Effective PDE

$$-\left(\boldsymbol{\mu}^{(0)}\!:\!\nabla_{\boldsymbol{y}}^2 + \rho^{(0)}(\omega^2 - \tilde{\lambda}_{\hat{n}}^{\boldsymbol{a}})\right)\langle\tilde{u}\rangle \;=\; \left[\tilde{\varphi}_{\hat{n}}^{\boldsymbol{a}}(\boldsymbol{y}') \;-\; \boldsymbol{\chi}^{(1)}(\boldsymbol{y}')\cdot\nabla_{\boldsymbol{y}}\right] \delta(\boldsymbol{y}-\boldsymbol{y}')$$

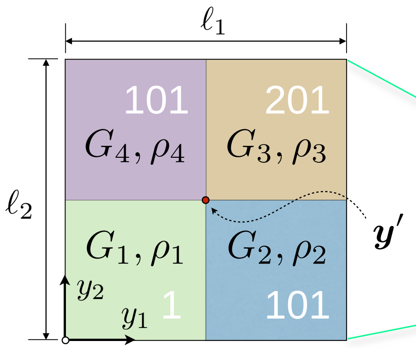
Effective Green's function

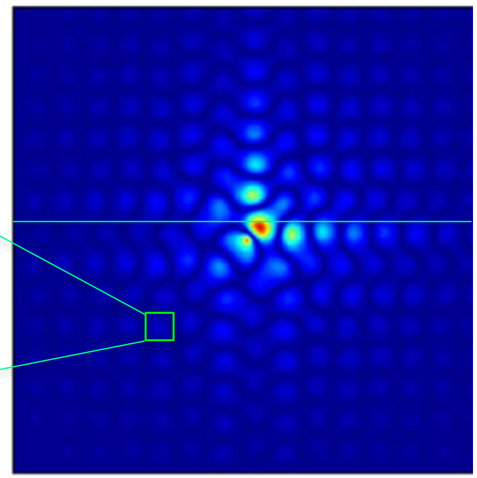


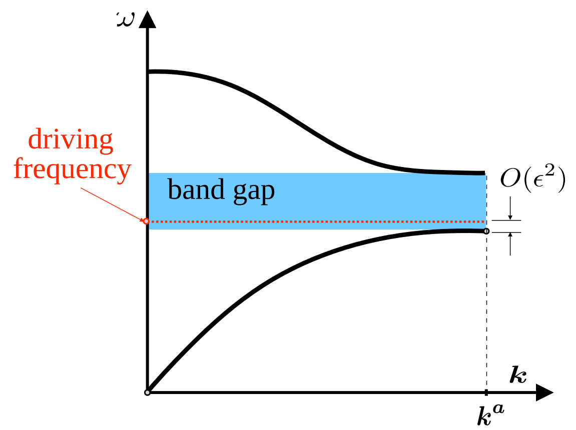


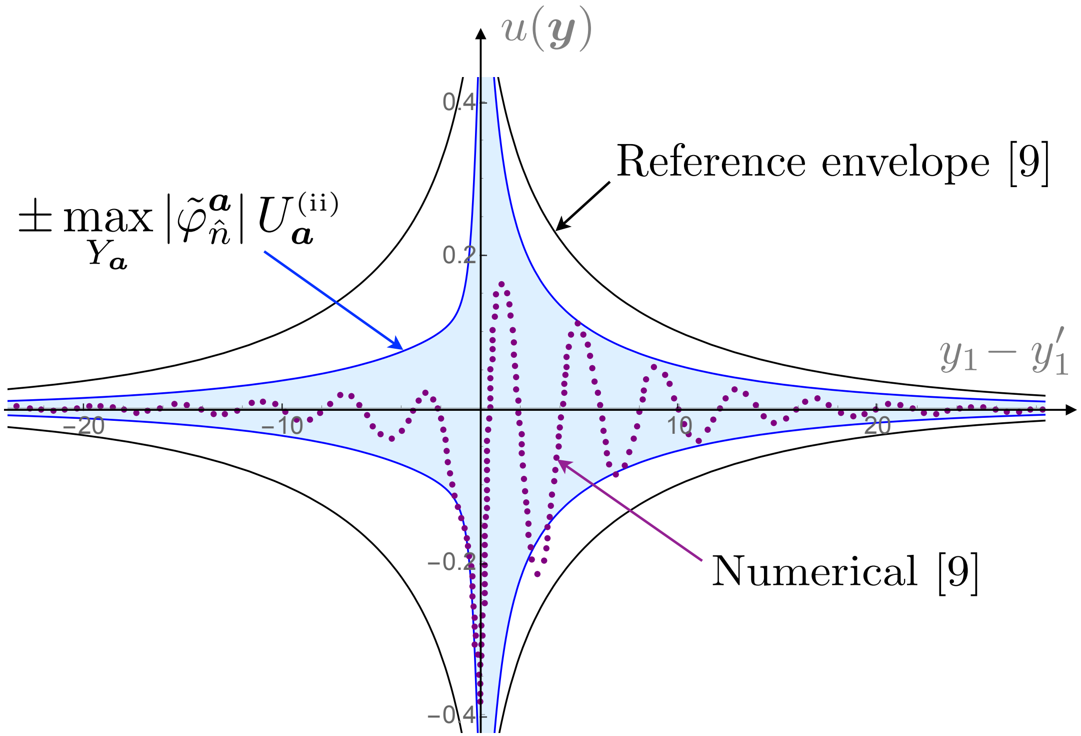


Effective Green's function







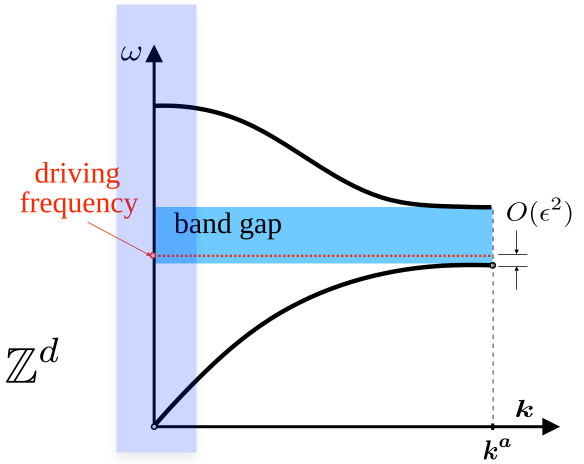


[9] Craster et al. (2018)
J. Opt. Soc. Am., **28**

A bit more rigor…

Meng/Oudghiri-Idrissi/G

Scalar wave equation, small $\boldsymbol{k}$, square grid



$$-\nabla\cdot\big(G(\boldsymbol{x}/\epsilon)\nabla U\big) - \Omega^2\rho(\boldsymbol{x}/\epsilon)U \;=\; f(\boldsymbol{x}) \quad \text{in} \;\; \mathbb{R}^d$$

$$G(\boldsymbol{y}+\boldsymbol{z}) = G(\boldsymbol{y}), \quad \rho(\boldsymbol{y}+\boldsymbol{z}) = \rho(\boldsymbol{y}) \quad \forall \boldsymbol{y}\in\mathbb{R}^d, \;\; \boldsymbol{z}\in\mathbb{Z}^d$$

Let $\boldsymbol{y} = \epsilon^{-1}\boldsymbol{x}, \;\; \omega = \epsilon\Omega, \;\; u(\boldsymbol{y}) = U(\boldsymbol{x})$

$$-\nabla\cdot\big(G(\boldsymbol{y})\nabla u\big) - \omega^2\rho(\boldsymbol{y})u \;=\; \epsilon^2 f(\epsilon\boldsymbol{y}) \quad \text{in} \;\; \mathbb{R}^d$$

Long-wavelength motion: $\boldsymbol{k} = \epsilon\hat{\boldsymbol{k}} \;\; \rightarrow \;\; \lambda = \dfrac{2\pi}{||\boldsymbol{k}||} = O(\epsilon^{-1})$

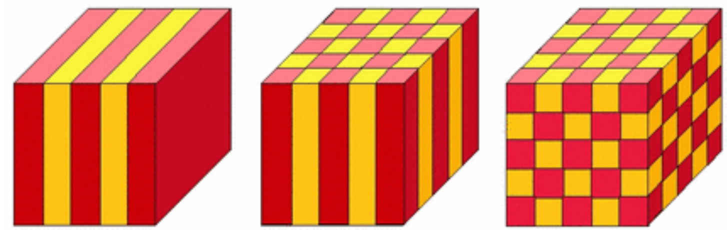
Preliminaries

Scalar wave equation, small $\boldsymbol{k}$, finite frequency

$$-\nabla \cdot \big(G(\boldsymbol{y})\nabla u\big) - \omega^2 \rho(\boldsymbol{y}) u \;=\; \epsilon^2 f(\epsilon \boldsymbol{y}) \quad \text{in} \quad \mathbb{R}^d$$

Elliptic operator: $H^1_{loc}(\mathbb{R}^d) \mapsto L^2_{loc}(\mathbb{R}^d)$

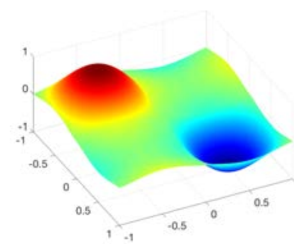
$$A \;=\; -\frac{1}{\rho(\boldsymbol{y})}\nabla \cdot \big(G(\boldsymbol{y})\nabla u\big)$$

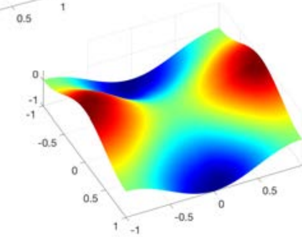


d-dimensional taurus $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$

Bloch waves: $H^1_{loc}(\mathbb{T}^d) \mapsto L^2_{loc}(\mathbb{T}^d)$

$$A(\boldsymbol{k}) = -\frac{1}{\rho(\boldsymbol{y})}(\nabla + i\boldsymbol{k}) \cdot \big[G(\boldsymbol{y})(\nabla + i\boldsymbol{k})\big]$$





Eigenvalues $\{\omega_m^2(\boldsymbol{k})\}_{m=0}^{\infty}$, eigenfunctions $\{\phi_m(\boldsymbol{y};\boldsymbol{k})\}_{m=0}^{\infty}$

$$A(\boldsymbol{k})\phi_m(\boldsymbol{y};\boldsymbol{k}) = \omega_m^2(\boldsymbol{k})\phi_m(\boldsymbol{y};\boldsymbol{k})$$

$$\langle \phi_m(\boldsymbol{y};\boldsymbol{k}), \phi_n(\boldsymbol{y};\boldsymbol{k}) \rangle := \int_{\mathbb{T}^d} \rho(\boldsymbol{y})\phi_m(\boldsymbol{y};\boldsymbol{k})\overline{\phi}_n(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{y} = \delta_{mn}$$

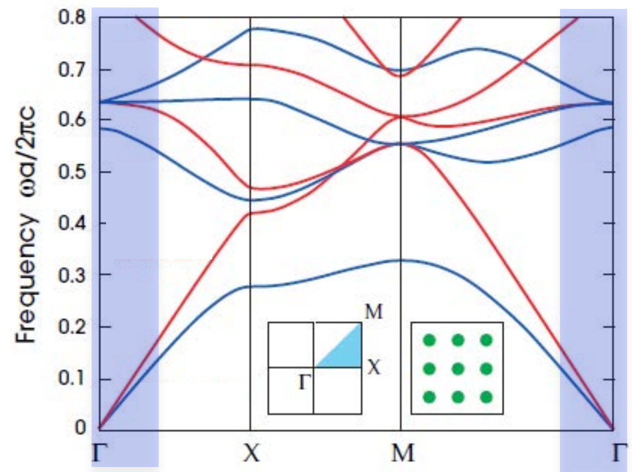
Bloch waves

Bloch variety

$$\mathcal{B} := [-\tfrac{1}{2}, \tfrac{1}{2}]^d$$

$$B_A = \cup_{m=0}^{\infty} \{(\boldsymbol{k}, \omega_m(\boldsymbol{k})); \boldsymbol{k} \in \mathbb{R}^d\}$$

p-th branch: $\{(\boldsymbol{k}, \omega_p(\boldsymbol{k})); \boldsymbol{k} \in \mathcal{B}\}$



Bloch expansion thm.

$$u(\boldsymbol{y}) = \int_{\mathcal{B}} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \tilde{u}(\boldsymbol{y};\boldsymbol{k}) d\boldsymbol{k} \quad \leftrightarrow \quad \tilde{u}(\boldsymbol{y};\boldsymbol{k}) = \sum_{\boldsymbol{\gamma}\in 2\pi\mathbb{Z}^d} u(\boldsymbol{y}+\boldsymbol{\gamma}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{y}+\boldsymbol{\gamma})}$$

Eigenfunction expansion

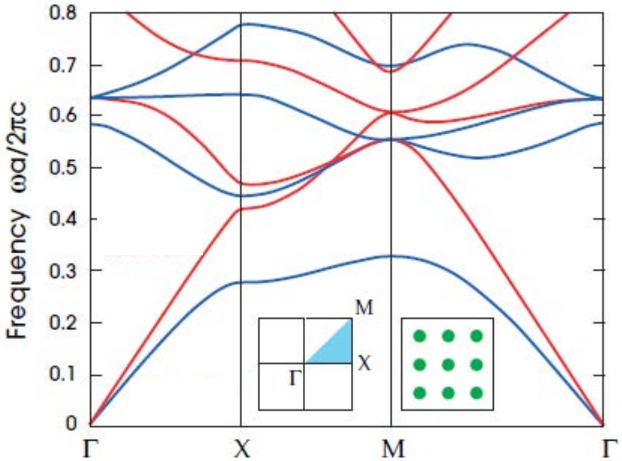
$$\tilde{u}(\boldsymbol{y};\boldsymbol{k}) = \sum_{m=0}^{\infty} \hat{u}_m(\boldsymbol{k}) \phi_m(\boldsymbol{y};\boldsymbol{k}), \qquad \hat{u}_m(\boldsymbol{k}) = \int_{\mathbb{T}^d} \rho(\boldsymbol{y}) \tilde{u}(\boldsymbol{y};\boldsymbol{k}) \overline{\phi}_m(\boldsymbol{y};\boldsymbol{k}) d\boldsymbol{y}$$

Formal solution

Representation

$$u(\boldsymbol{y}) = \int_{\mathcal{B}} \sum_{m=0}^{\infty} \hat{u}_m(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_m(\boldsymbol{y};\boldsymbol{k}) d\boldsymbol{k},$$

$$\hat{u}_m(\boldsymbol{k}) = \int_{\mathbb{R}^d} \rho(\boldsymbol{y}) u(\boldsymbol{y}) \overline{e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_m(\boldsymbol{y};\boldsymbol{k})} d\boldsymbol{y},$$



Field equation

$$Au - \omega^2 u = \frac{1}{\rho(\boldsymbol{y})} \epsilon^2 f(\epsilon \boldsymbol{y}) := g(\boldsymbol{y}) \quad \text{in} \quad \mathbb{R}^d$$

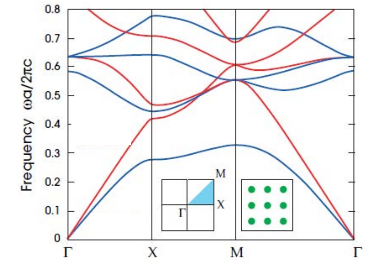
Relationship

$$A e^{i\boldsymbol{k}\cdot\boldsymbol{y}} = e^{i\boldsymbol{k}\cdot\boldsymbol{y}} A(\boldsymbol{k})$$

$$A = -\frac{1}{\rho(\boldsymbol{y})} \nabla \cdot (G(\boldsymbol{y}) \nabla u)$$

$$A(\boldsymbol{k}) = -\frac{1}{\rho(\boldsymbol{y})} (\nabla + i\boldsymbol{k}) \cdot [G(\boldsymbol{y})(\nabla + i\boldsymbol{k})]$$

Formal solution



$$Au - \omega^2 u \;=\; g(\boldsymbol{y}) \quad \text{in} \quad \mathbb{R}^d$$

Operator

$$Au(\boldsymbol{y}) = \int_{\mathcal{B}} A\left[e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\tilde{u}(\boldsymbol{y};\boldsymbol{k})\right]d\boldsymbol{k} = \int_{\mathcal{B}} e^{i\boldsymbol{k}\cdot y} A(\boldsymbol{k})\tilde{u}(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k}$$

$$= \int_{\mathcal{B}} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} A(\boldsymbol{k}) \sum_{m=0}^{\infty} \hat{u}_m(\boldsymbol{k})\phi_m(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k}$$

$$= \int_{\mathcal{B}} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \sum_{m=0}^{\infty} \hat{u}_m(\boldsymbol{k})\omega_m^2(\boldsymbol{k})\phi_m(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k},$$

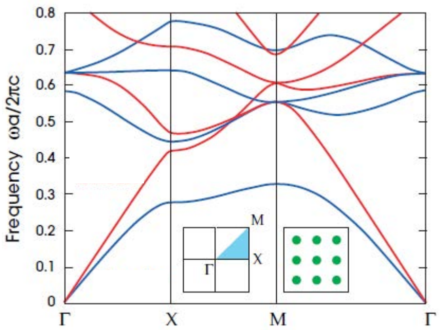
LHS: $$Au(\boldsymbol{y}) - \omega^2 u \;=\; \int_{\mathcal{B}} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \sum_{m=0}^{\infty} \hat{u}_m(\boldsymbol{k})\left(\omega_m^2(\boldsymbol{k}) - \omega^2\right)\phi_m(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k}$$

RHS: $$g(\boldsymbol{y}) = \int_{\mathcal{B}} \sum_{m=0}^{\infty} \hat{g}_m(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\phi_m(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k}$$

$$\hat{g}_m(\boldsymbol{k}) = \int_{\mathbb{R}^d} \epsilon^2 f(\epsilon\boldsymbol{y})\overline{e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\phi_m(\boldsymbol{y};\boldsymbol{k})}d\boldsymbol{y}$$

Formal solution

$$Au - \omega^2 u = g(\boldsymbol{y}) \quad \text{in} \quad \mathbb{R}^d$$



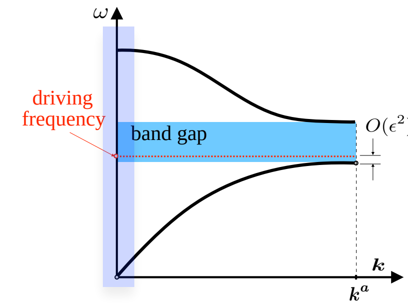
By Parseval's identity $(\omega^2 \neq \omega_m^2(\boldsymbol{k}))$

$$\hat{u}_m(\boldsymbol{k}) = \frac{\hat{g}_m(\boldsymbol{k})}{\omega_m^2(\boldsymbol{k}) - \omega^2}, \qquad m = 0, 1, \cdots$$

Solution

$$\begin{aligned} u(\boldsymbol{y}) &= \int_{\mathcal{B}} \sum_{m=0}^{\infty} \frac{\hat{g}_m(\boldsymbol{k})}{\omega_m^2(\boldsymbol{k}) - \omega^2} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_m(\boldsymbol{y}; \boldsymbol{k}) d\boldsymbol{k} \\ &= \int_{\mathcal{B}} \sum_{m=0}^{\infty} \frac{\int_{\mathbb{R}^d} \epsilon^2 f(\epsilon\boldsymbol{\xi}) \overline{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \phi_m(\boldsymbol{\xi}; \boldsymbol{k})} d\boldsymbol{\xi}}{\omega_m^2(\boldsymbol{k}) - \omega^2} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_m(\boldsymbol{y}; \boldsymbol{k}) d\boldsymbol{k} \end{aligned}$$

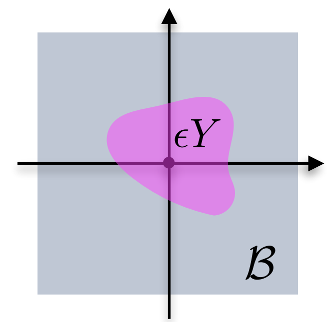
Source term



$$u(\boldsymbol{y}) = \int_{\mathcal{B}} \sum_{m=0}^{\infty} \frac{\int_{\mathbb{R}^d} \epsilon^2 f(\epsilon\boldsymbol{\xi})\overline{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\phi_m(\boldsymbol{\xi};\boldsymbol{k})}d\boldsymbol{\xi}}{\omega_m^2(\boldsymbol{k}) - \omega^2} e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\phi_m(\boldsymbol{y};\boldsymbol{k})d\boldsymbol{k}$$

Assumption: small $\boldsymbol{k}$ — pth branch

$$f(\epsilon\boldsymbol{\xi}) = \frac{1}{(2\pi)^d}\left[\int_{\mathbb{R}^d} F(\boldsymbol{k})e^{i\epsilon\boldsymbol{k}\cdot\boldsymbol{\xi}}d\boldsymbol{k}\right]\rho(\boldsymbol{\xi})\,\phi_p(\boldsymbol{\xi};\boldsymbol{0}), \qquad p \geqslant 1$$



$F \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$: compactly supported in $\boldsymbol{0} \ni Y \subset \mathbb{R}^d$

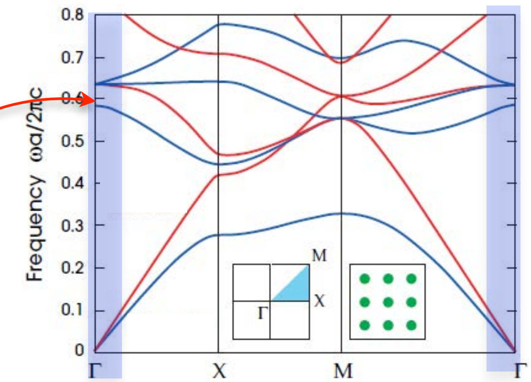
LEMMA 2.1. *Let ϕ be a bounded $2\pi\mathbb{Z}^d$-periodic function. Assume that the Fourier series of $\rho(\boldsymbol{\xi})\overline{\phi_p(\boldsymbol{\xi};\boldsymbol{0})}\phi(\boldsymbol{\xi})$ converges pointwise almost everywhere, i.e.*

$$\rho(\boldsymbol{\xi})\overline{\phi_p(\boldsymbol{\xi};\boldsymbol{0})}\phi(\boldsymbol{\xi}) \sim \sum_{\boldsymbol{n}\in\mathbb{Z}^d} a_{\boldsymbol{n}} e^{i\boldsymbol{n}\cdot\boldsymbol{\xi}}. \qquad .3)$$

$$\int_{\mathbb{R}^d} f(\epsilon\boldsymbol{\xi})\overline{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\phi(\boldsymbol{\xi})}d\boldsymbol{\xi} = \begin{cases} \epsilon^{-d}\overline{a}_{\boldsymbol{0}} F(\boldsymbol{k}/\epsilon), & \boldsymbol{k} \in \epsilon Y \\ 0, & \boldsymbol{k} \in \mathcal{B}\setminus\overline{\epsilon Y} \end{cases}$$

$\hat{\boldsymbol{k}}$

Asymptotic model



Driving frequency

$$\omega^2 = \omega_p^2(\mathbf{0}) + \epsilon^2 \sigma \hat{\omega}^2$$

Contribution of the pth branch

$$u_p(\boldsymbol{y}) = \int_{\epsilon Y} \frac{\int_{\mathbb{R}^d} \epsilon^2 f(\epsilon\boldsymbol{\xi}) \overline{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \phi_p(\boldsymbol{\xi};\boldsymbol{k})} d\boldsymbol{\xi}}{\omega_p^2(\boldsymbol{k}) - \omega^2} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_p(\boldsymbol{y};\boldsymbol{k}) d\boldsymbol{k}.$$

Zeroth-order approximation, $u_p = u_p^{(0)} + O(\epsilon)$

$$u_p^{(0)}(\boldsymbol{y}) = \phi_p(\boldsymbol{y};\mathbf{0})\, \mathrm{w}_0(\epsilon\boldsymbol{y})$$

$$\mathrm{w}_0(\epsilon\boldsymbol{y}) = \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{F(\hat{\boldsymbol{k}})\, e^{i\epsilon\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}}{\boldsymbol{\mu}^{(0)} : (i\hat{\boldsymbol{k}})^2 + \rho^{(0)}\sigma\hat{\omega}^2} d\hat{\boldsymbol{k}}$$

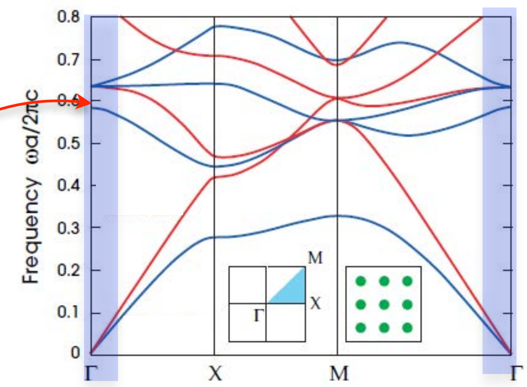
First-order approximation, $u_p = u_p^{(1)} + O(\epsilon^2)$

$$u_p^{(1)}(\boldsymbol{y}) = \phi_p(\boldsymbol{y};\mathbf{0})\, \mathrm{w}_0(\epsilon\boldsymbol{y}) + \boldsymbol{\chi}^{(1)}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \mathrm{w}_0(\epsilon\boldsymbol{y})$$

Asymptotic model

Driving frequency

$$\omega^2 = \omega_p^2(\mathbf{0}) + \epsilon^2 \sigma \hat{\omega}^2$$



Contribution of the remaining branches

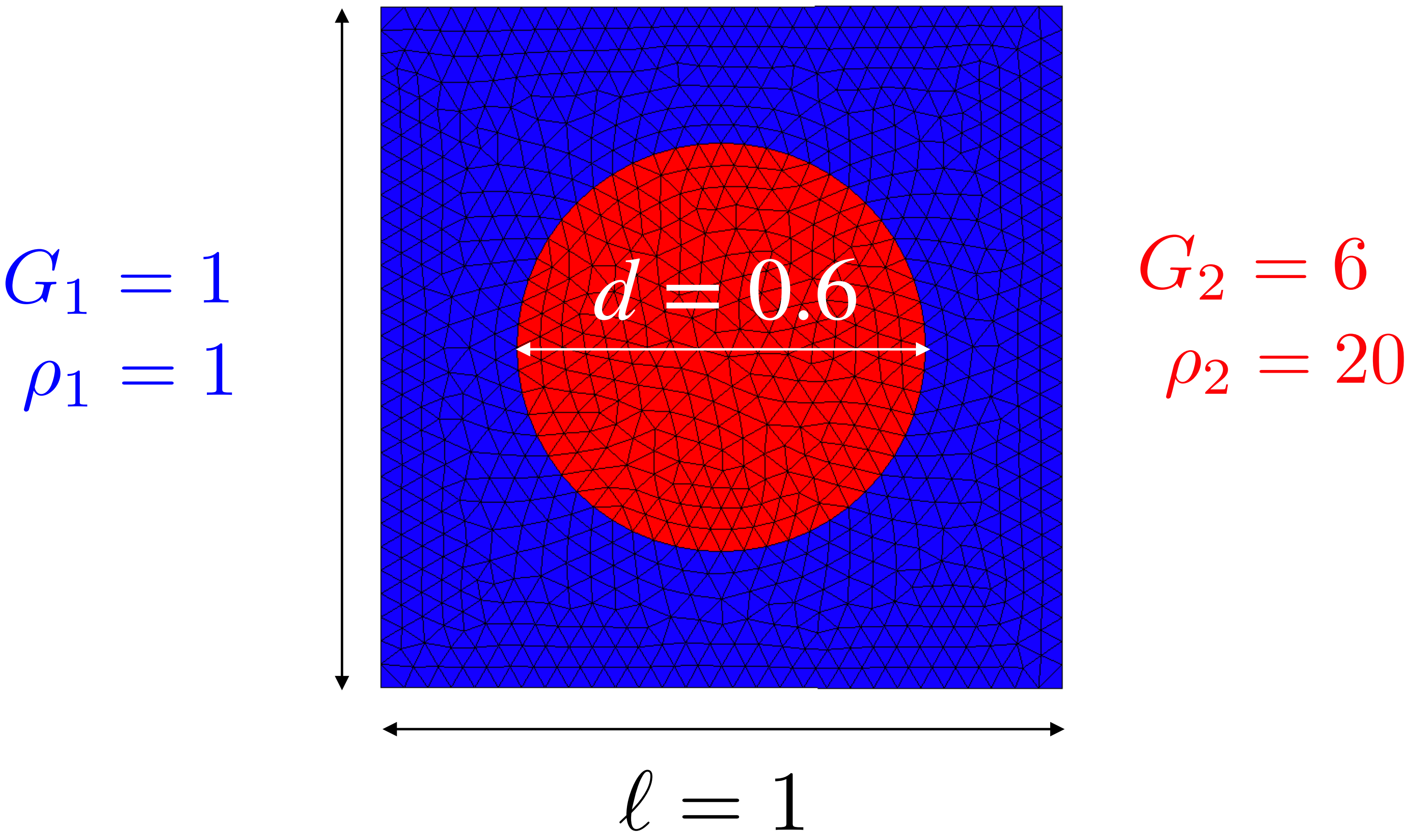
LEMMA 4.1. *Assume that* $f \in H^1(\mathbb{R}^d)$ *and let*

$$u_{res}(\boldsymbol{y}) = \int_{\epsilon Y} \sum_{m \neq p} \frac{\int_{\mathbb{R}^d} \epsilon^2 f(\epsilon \boldsymbol{\xi}) \overline{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \phi_m(\boldsymbol{\xi};\boldsymbol{k})} d\boldsymbol{\xi}}{\omega_m^2(\boldsymbol{k}) - \omega^2} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \phi_m(\boldsymbol{y};\boldsymbol{k}) d\boldsymbol{k}.$$

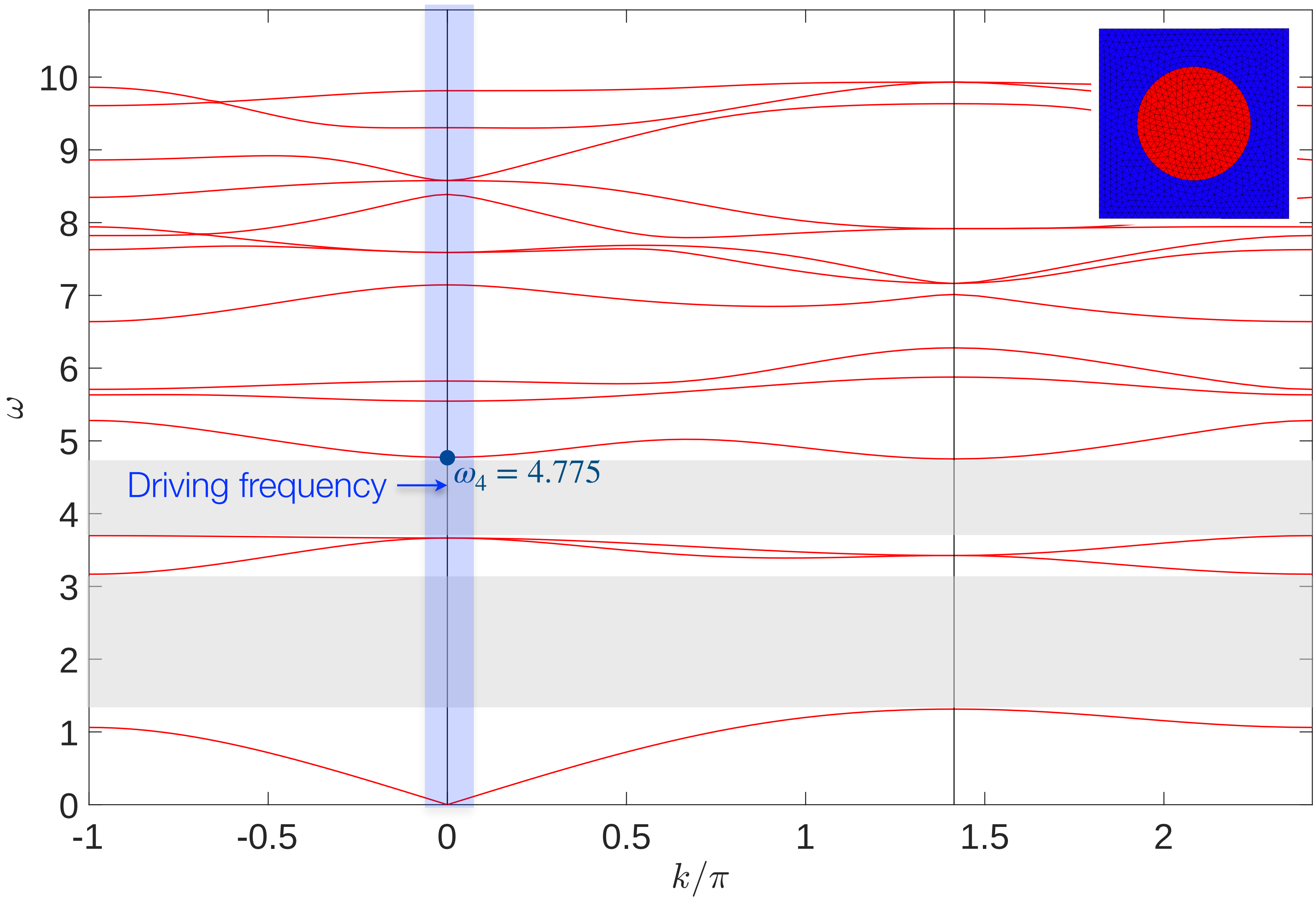
→ $$\|u_{res}\|_{L^2(\mathbb{R}^d)} \leqslant c\, \epsilon^{3-d/2} \|F\|_{L^2(\mathbb{R})}$$

Contribution of the pth branch: $\|u_p\|_{L^2(\mathbb{R}^d)} = O(1)$

Example

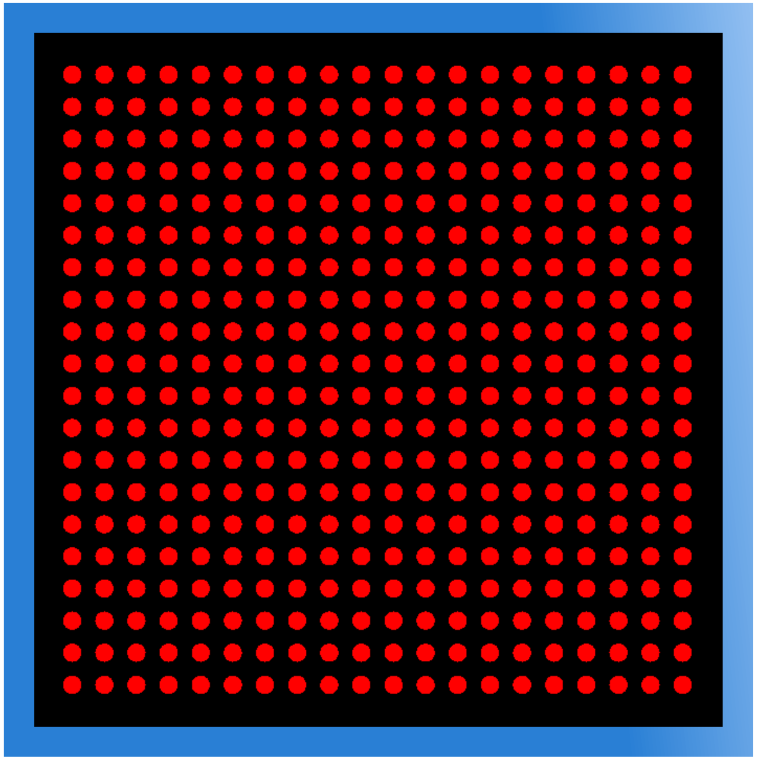


Dispersion relationship



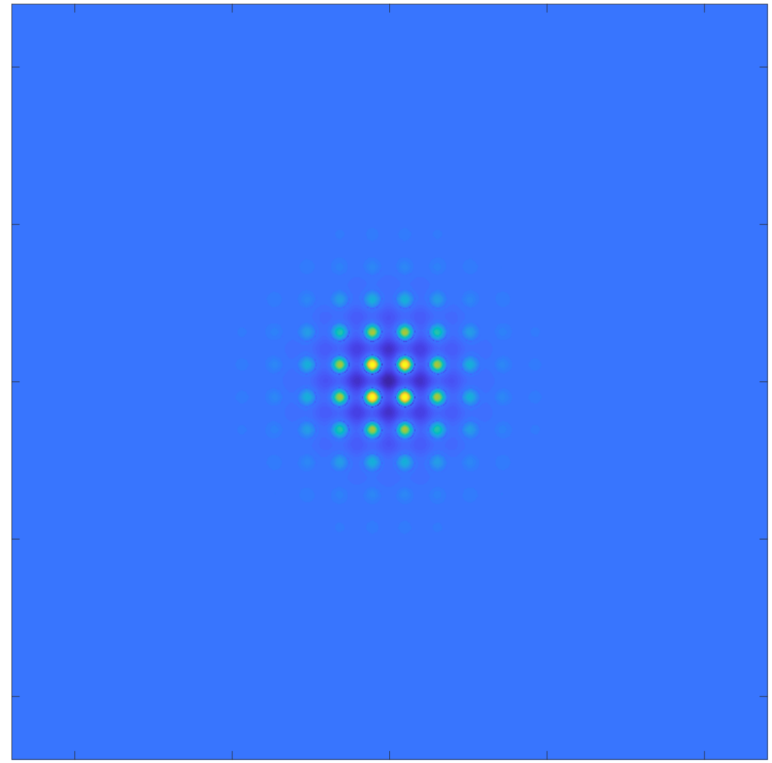
Excitation of a finite domain

20x20 cells



Absorbing BCs

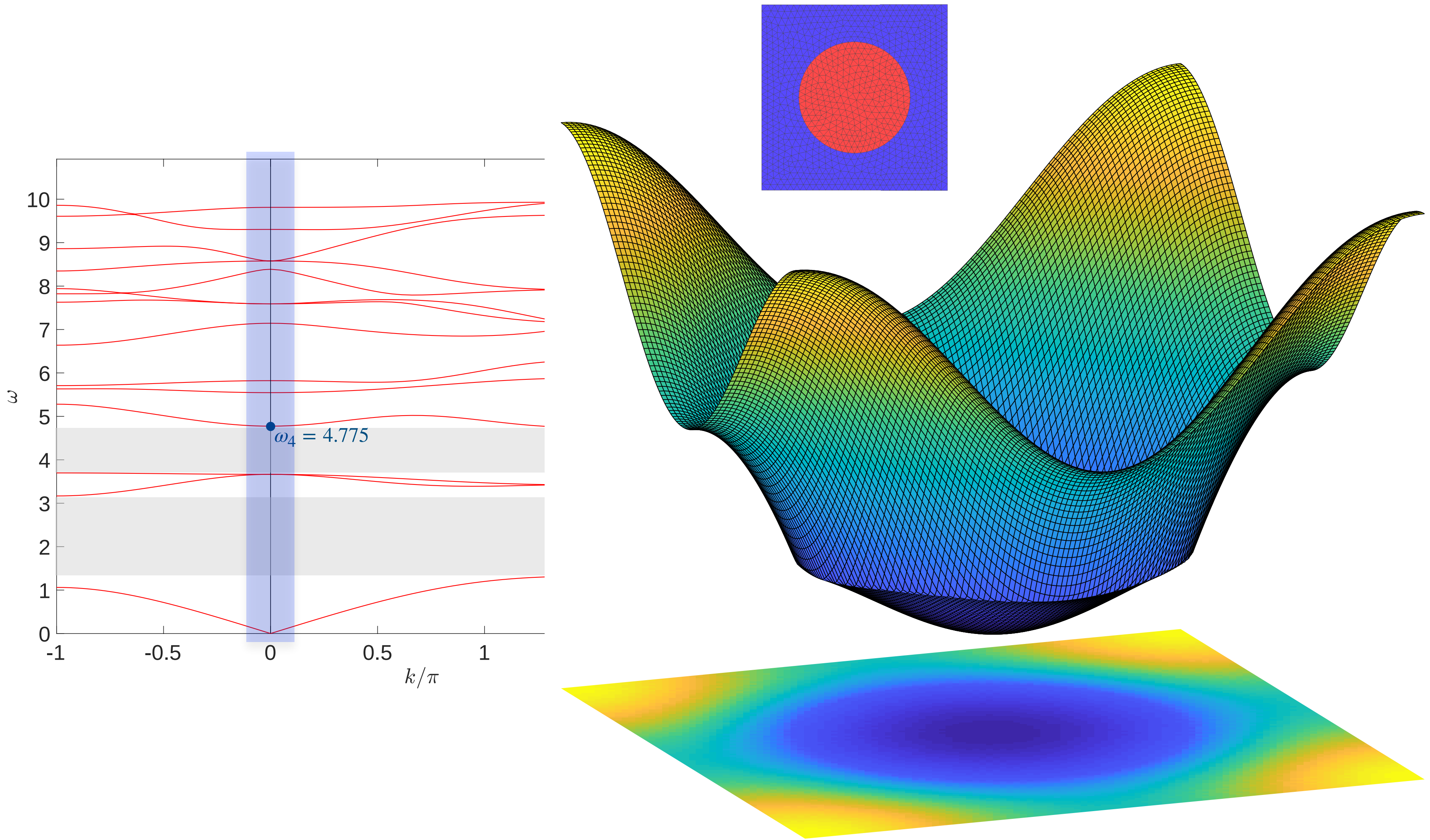
Source density



$$f(\epsilon \boldsymbol{y}) = \frac{1}{\sqrt{\pi}} \left[e^{-\epsilon^2 \|\boldsymbol{y}\|^2} \right] \rho(\boldsymbol{y}) \tilde{\phi}_4(\boldsymbol{y})$$

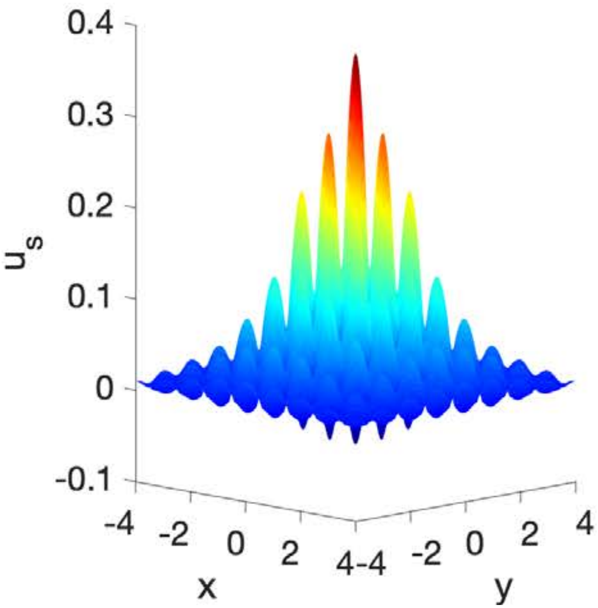
$$\epsilon^2 = |\omega^2 - \omega_4^2|$$

Eigenfunction $\tilde{\phi}_4$

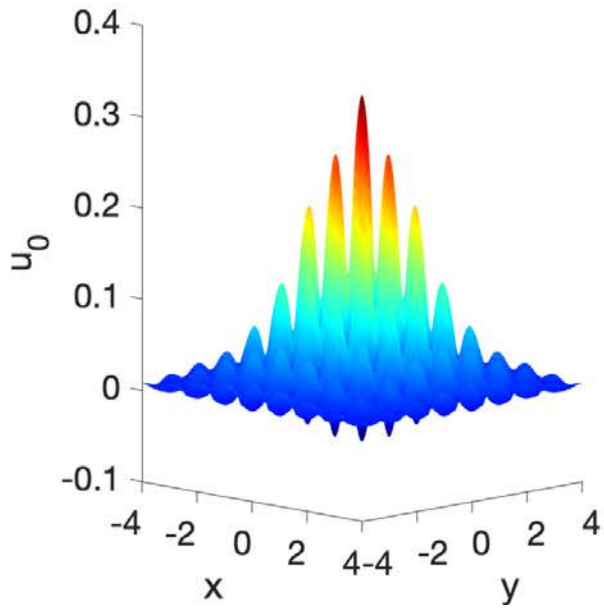


Asymptotic approximations

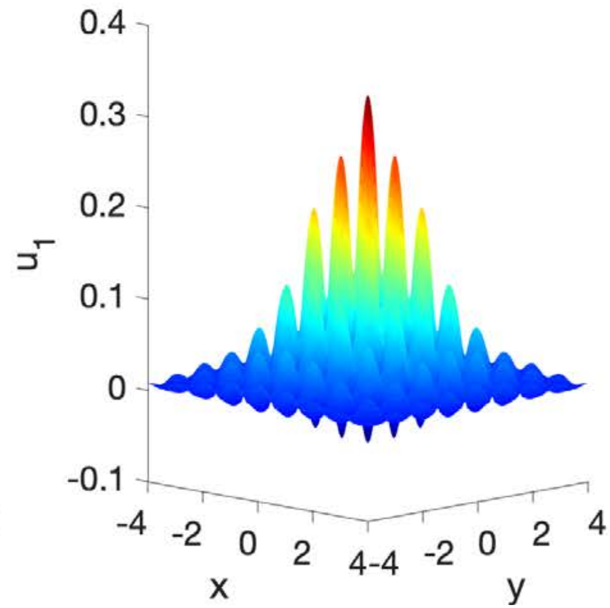
$\epsilon = 0.75$



Numerical simulation



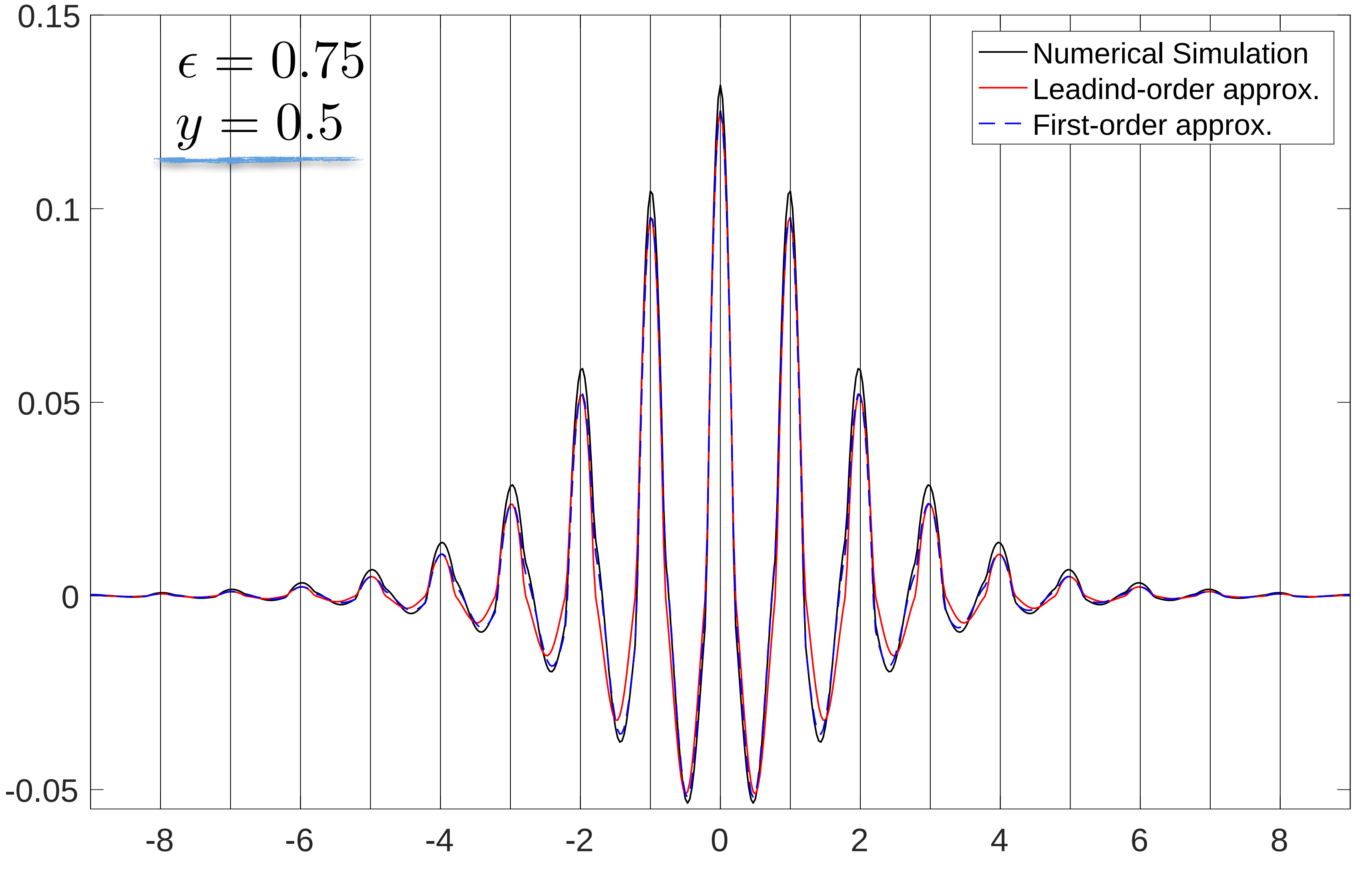
Leading-order approximation



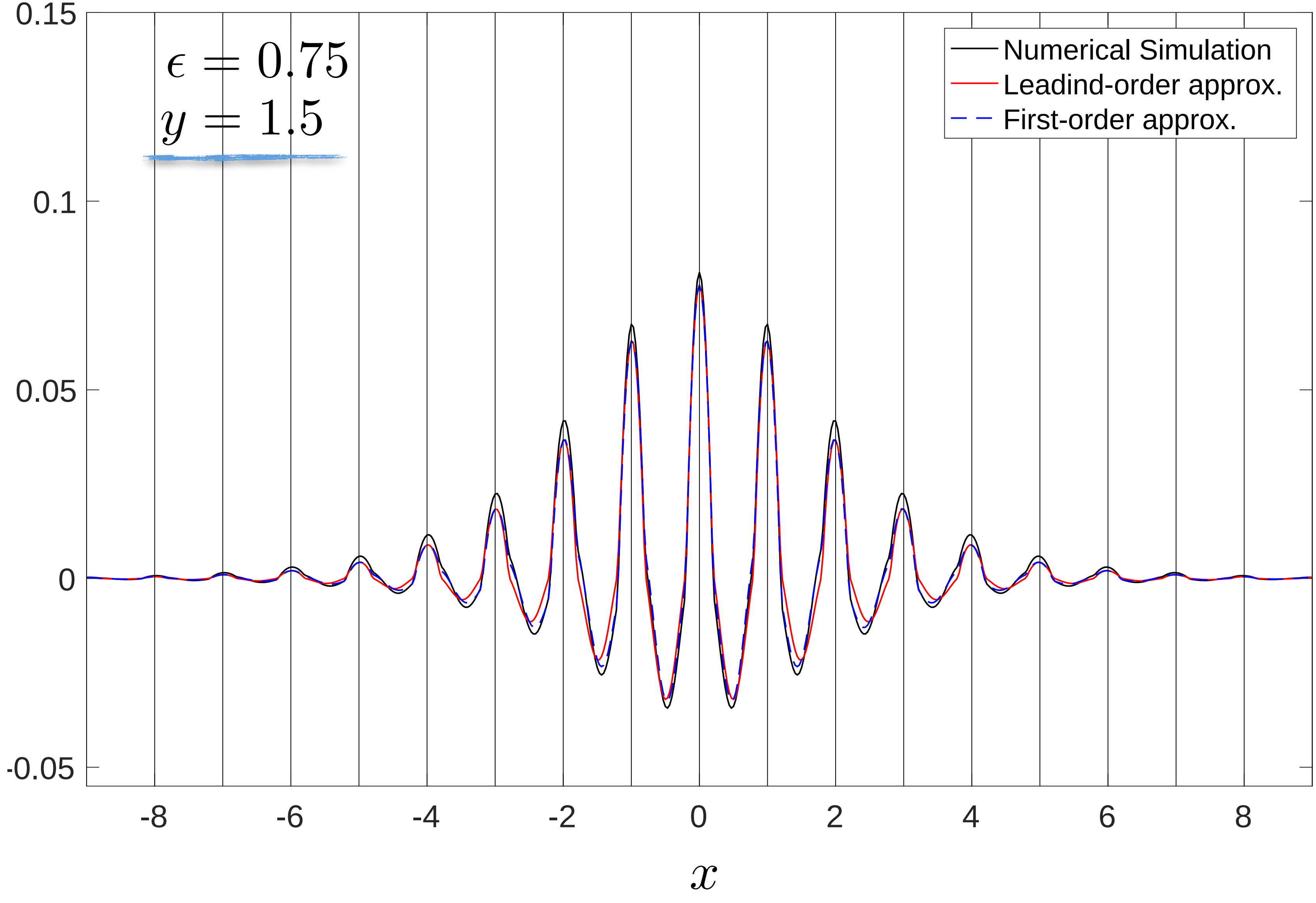
First-order approximation

$$\|u_s - u_0\|_2 = 1.3380 = 0.75 \times 1.784$$
$$\|u_s - u_1\|_2 = 1.2832 = 0.75^2 \times 2.281$$

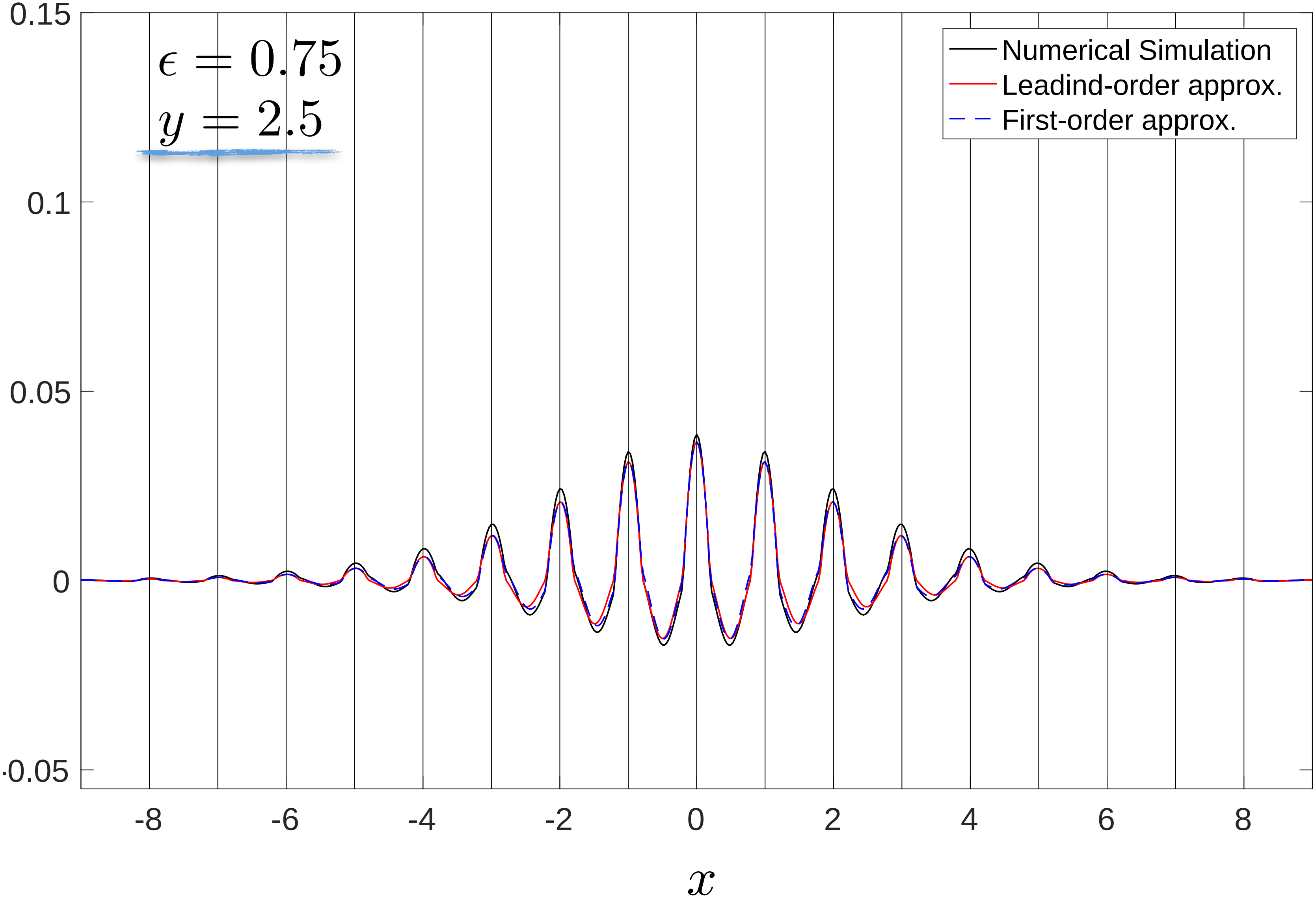
Asymptotic approximations



Asymptotic approximations



Asymptotic approximations



Asymptotic approximations