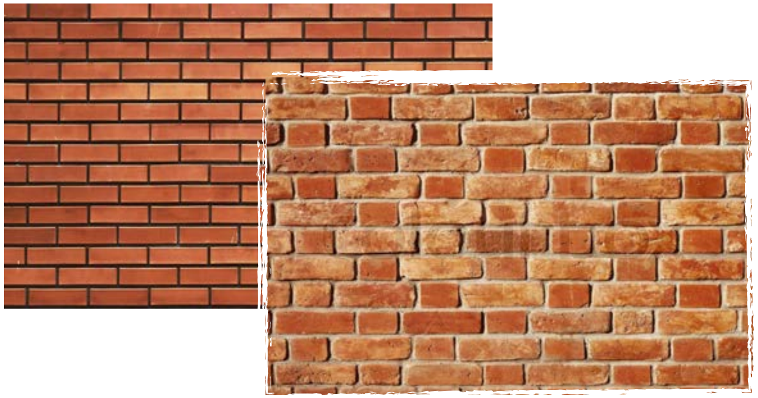
Part IV

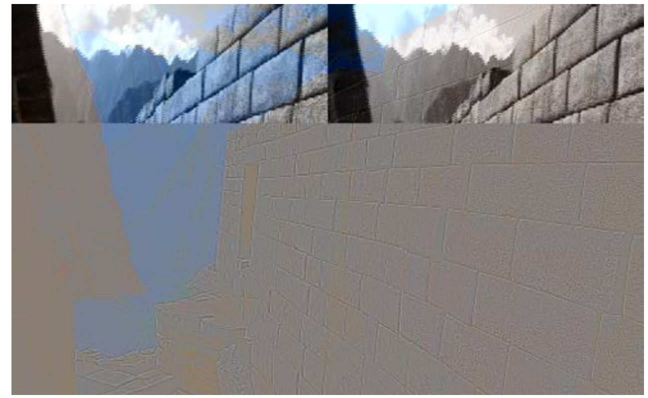
Waves in periodic discontinua

G/Bonnet

Motivation

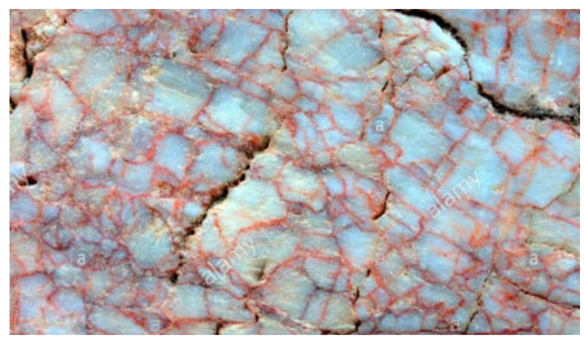
Masonry





Fractured rock

www.alamy.com/stock-photo/fracture-in-rock.html



Sulem & Mullhaus (1997)

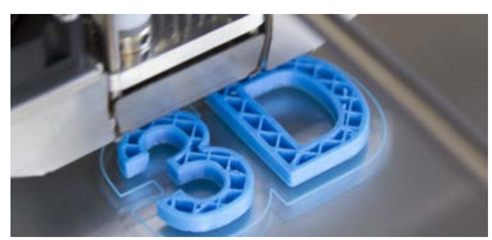
Stefanou, Sulem & Vardoulakis (2008,2010)

Bacigalupo & Gambarotta (2012)

Brajanovski, Gurevich, Schoenberg (2005)

Galvin & Gurevich (2015)

Manufacturing of phononic crystals



λ > 4-5 block sizes

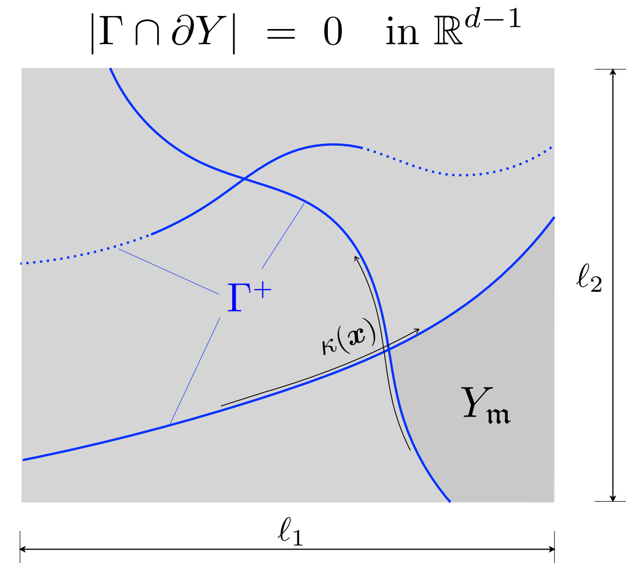
Setting

Governing system

$$-\nabla_{\boldsymbol{k}}\cdot\left(G\nabla_{\boldsymbol{k}}\tilde{u}\right) - \omega^2\rho\tilde{u} \;=\; \tilde{f} \quad \text{in } Y,$$

$$[\![t^{\boldsymbol{k}}_{\boldsymbol{\nu}}[\tilde{u},\tilde{u}]]\!] = 0, \quad [\![\tilde{u}]\!] + \kappa^{-1}t^{\boldsymbol{k}}_{\boldsymbol{\nu}}[\tilde{u},\tilde{u}] = 0 \quad \text{on } \Gamma,$$

$$\rceil\tilde{u}\lfloor_j^- \;=\; 0, \quad \rceil t^{\boldsymbol{k}}_{\boldsymbol{n}}[\tilde{u},\tilde{u}]\lfloor_j^+ \;=\; 0 \quad j=\overline{1,d} \quad \text{on } \partial Y,$$



Contact/surface traction

$$t^{\boldsymbol{k}}_{\boldsymbol{\nu}}[g,h](\boldsymbol{x}) \;=\; \boldsymbol{\nu}(\boldsymbol{x})\cdot G\left(\nabla g(\boldsymbol{x}) + i\boldsymbol{k}\,h(\boldsymbol{x})\right)$$

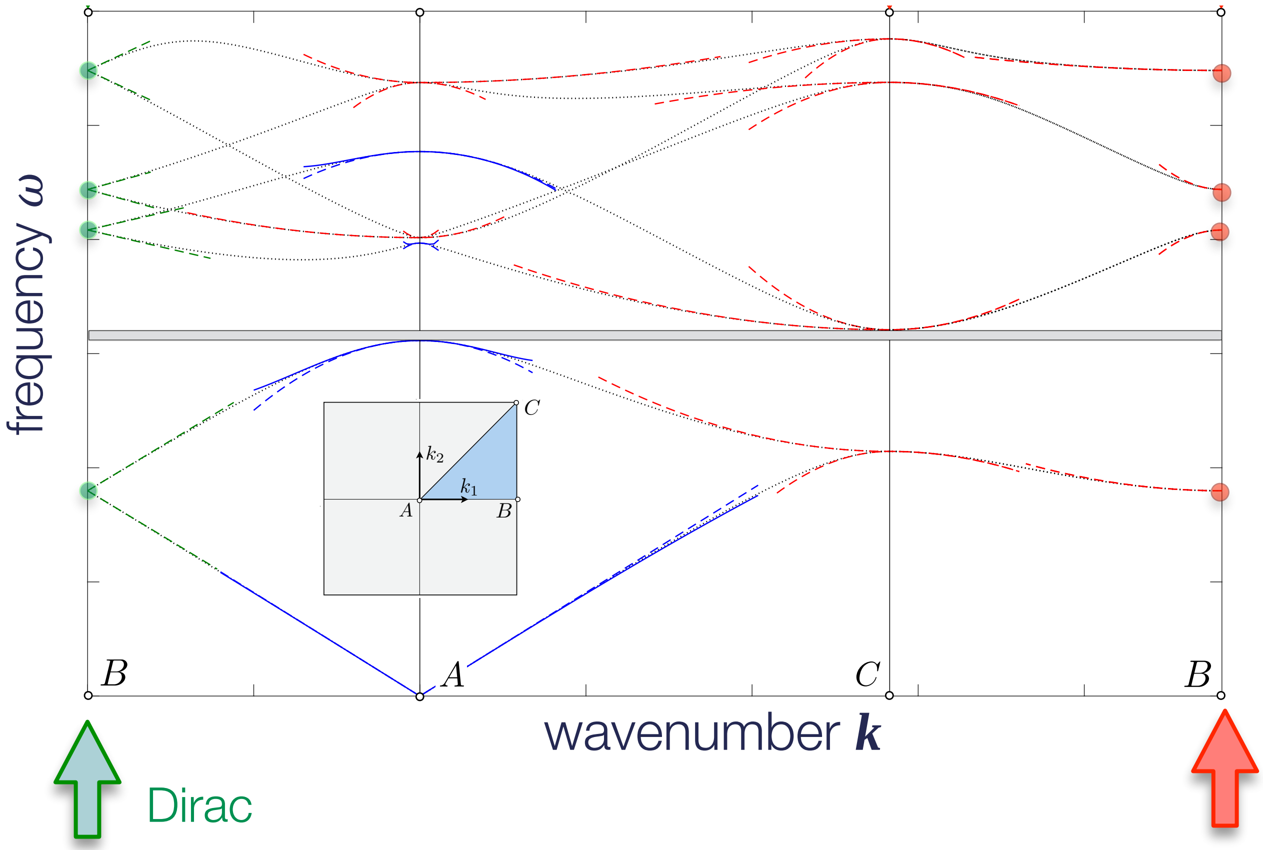
Jump

$$[\![g]\!](\boldsymbol{x}) = \lim_{\delta\to 0} g(\boldsymbol{x}+\delta\boldsymbol{\nu}(\boldsymbol{x})) - g(\boldsymbol{x}-\delta\boldsymbol{\nu}(\boldsymbol{x})), \quad \boldsymbol{x}\in\Gamma$$

Broken (periodic) Sobolev spaces

$$H^1_p(Y) \;=\; \{g\in L^2(Y):\, g\in L^2(Y_{\mathfrak{m}}), \nabla g\in (L^2(Y_{\mathfrak{m}}))^d,\; \rceil g\lfloor_j^- = 0 \text{ on } \partial Y,\; \mathfrak{m}=\overline{1,\mathfrak{M}}\}$$

Repeated eigenvalues



Uniform asymptotics?

Cone or no cone?

Ansatz

O(ϵ^{-2}): $-\tilde{\lambda}_n^{\boldsymbol{a}} \rho \tilde{w}_0 - \nabla \cdot (G \nabla \tilde{w}_0) = 0 \quad \text{in} \quad Y_{\boldsymbol{a}}$ ⇨ $\tilde{w}_0(\boldsymbol{x}) = \sum_q w_{0q} \tilde{\varphi}_{nq}^{\boldsymbol{a}}(\boldsymbol{x})$

O(ϵ^{-1}): ➡ Identity

O(ϵ^{0}): ➡ Effective equation

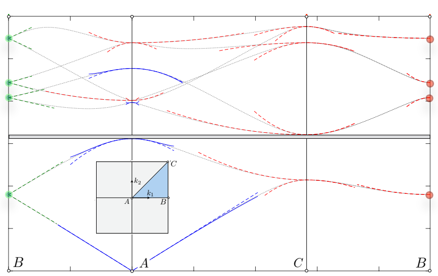
Generalized EVP

$$\tau = -\sigma \breve{\omega}^2$$

$$\sum_q A_{pq} w_{0q} - \tau \sum_q D_{pq} w_{0q} = 0, \qquad p = \overline{1, Q}$$

$$A_{pq} = \boldsymbol{\theta}_{pq}^{(0)} \cdot i \hat{\boldsymbol{k}}, \qquad \boldsymbol{\theta}_{pq}^{(0)} = \langle G \nabla \tilde{\varphi}_{nq}^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{p\varphi} - \langle G \nabla \tilde{\varphi}_{np}^{\boldsymbol{a}} \rangle_{\boldsymbol{a}}^{q\varphi}$$

A_{pq}: fixed rank



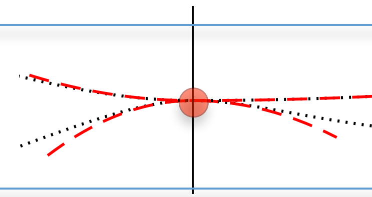
Near-trivial A_{pq}

Generalized EVP

$O(\epsilon)$

O(ϵ^{-1}): $$\sum_q A_{pq} w_{0q} - \tau \sum_q D_{pq} w_{0q} = 0, \qquad p = \overline{1,Q}$$

$$\rightarrow \quad \tau = -\sigma \breve{\omega}^2 = 0$$



Residual: $$-\epsilon \sum_q \bar{A}_{pq} w_{0q}, \qquad \bar{A}_{pq} = \epsilon^{-1} A_{pq}, \qquad p = \overline{1,Q}$$

Effective system

O(1): $$-\sum_q (B_{pq} + \bar{A}_{pq}) w_{0q} - \sigma \hat{\omega}^2 \sum_q D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}} \cdot \boldsymbol{x}} \rangle_{\boldsymbol{a}}^{p\varphi}, \qquad p = \overline{1,Q}$$

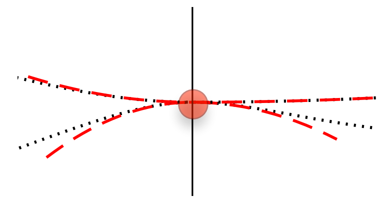
Near-trivial A_{pq}

O(1): $$-\sum_q (B_{pq} + \bar{A}_{pq}) w_{0q} - \sigma\hat{\omega}^2 \sum_q D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle_{\boldsymbol{a}}^{p\varphi}, \qquad p = \overline{1, Q}$$

$\boldsymbol{\mu}_{pq}^{(0)} : (i\hat{\boldsymbol{k}})^2$ $\qquad \epsilon^{-1}\,\boldsymbol{\theta}_{pq}^{(0)} \cdot i\hat{\boldsymbol{k}}$

$\hat{\boldsymbol{k}}$ ~ orthogonal to $\boldsymbol{\theta}_{pq}^{(0)}$

2nd-order polynomial in $i\hat{\boldsymbol{k}}$



Scaling: $$\omega^2 = \tilde{\lambda}_n^{\boldsymbol{a}} + \epsilon\sigma\breve{\omega}^2 + \epsilon^2\sigma\hat{\omega}^2$$

$$\sigma\breve{\omega}^2 = \frac{\omega^2 - \tilde{\lambda}_n^{\boldsymbol{a}}}{\epsilon}, \qquad \sigma\hat{\omega}^2 = \frac{\omega^2 - \tilde{\lambda}_n^{\boldsymbol{a}}}{\epsilon^2}, \qquad \Longrightarrow \qquad \sigma\hat{\omega}^2 = \epsilon^{-1}\sigma\breve{\omega}^2$$

Let $\hat{\boldsymbol{k}}$ depart from the normal to $\boldsymbol{\theta}_{pq}^{(0)}$ $\longrightarrow$ $A_{pq} \to O(1)$ $\qquad \bar{A}_{pq} = \epsilon^{-1} A_{pq}$;

O(1): $$-\sum_q A_{pq} w_{1q} - \sigma\breve{\omega}^2 \sum_q D_{pq} w_{1q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle_{\boldsymbol{a}}^{p\varphi}$$

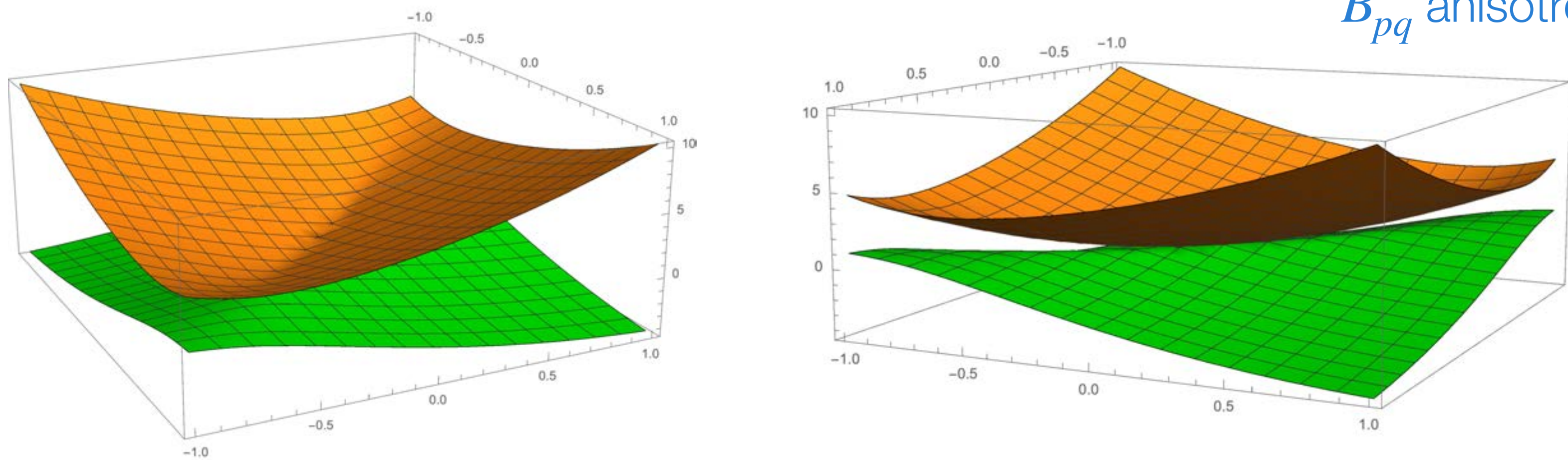
$w_{0q} \mapsto \epsilon w_{1q}$

Merging dispersion surfaces

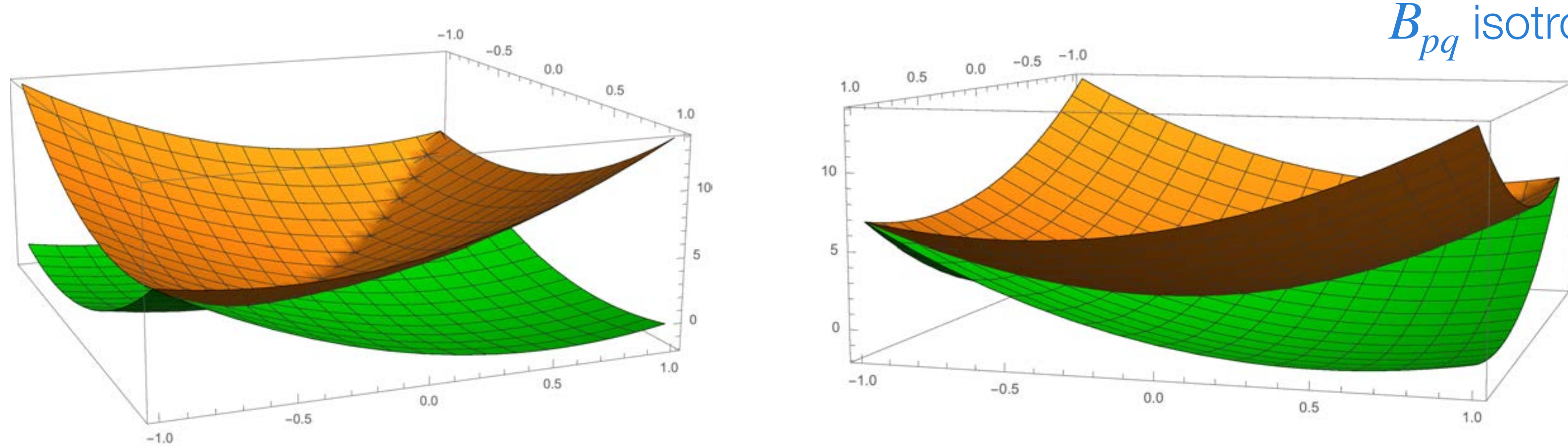
$$\boldsymbol{\mu}^{(0)}_{pq} : (i\hat{\boldsymbol{k}})^2 \qquad \epsilon^{-1}\, \boldsymbol{\theta}^{(0)}_{pq} \cdot i\hat{\boldsymbol{k}}$$

$$-\sum_q (B_{pq} + \bar{A}_{pq})\, w_{0q} - \sigma\hat{\omega}^2 \sum_q D_{pq}\, w_{0q} \;=\; \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}}, \qquad p = \overline{1, Q}$$

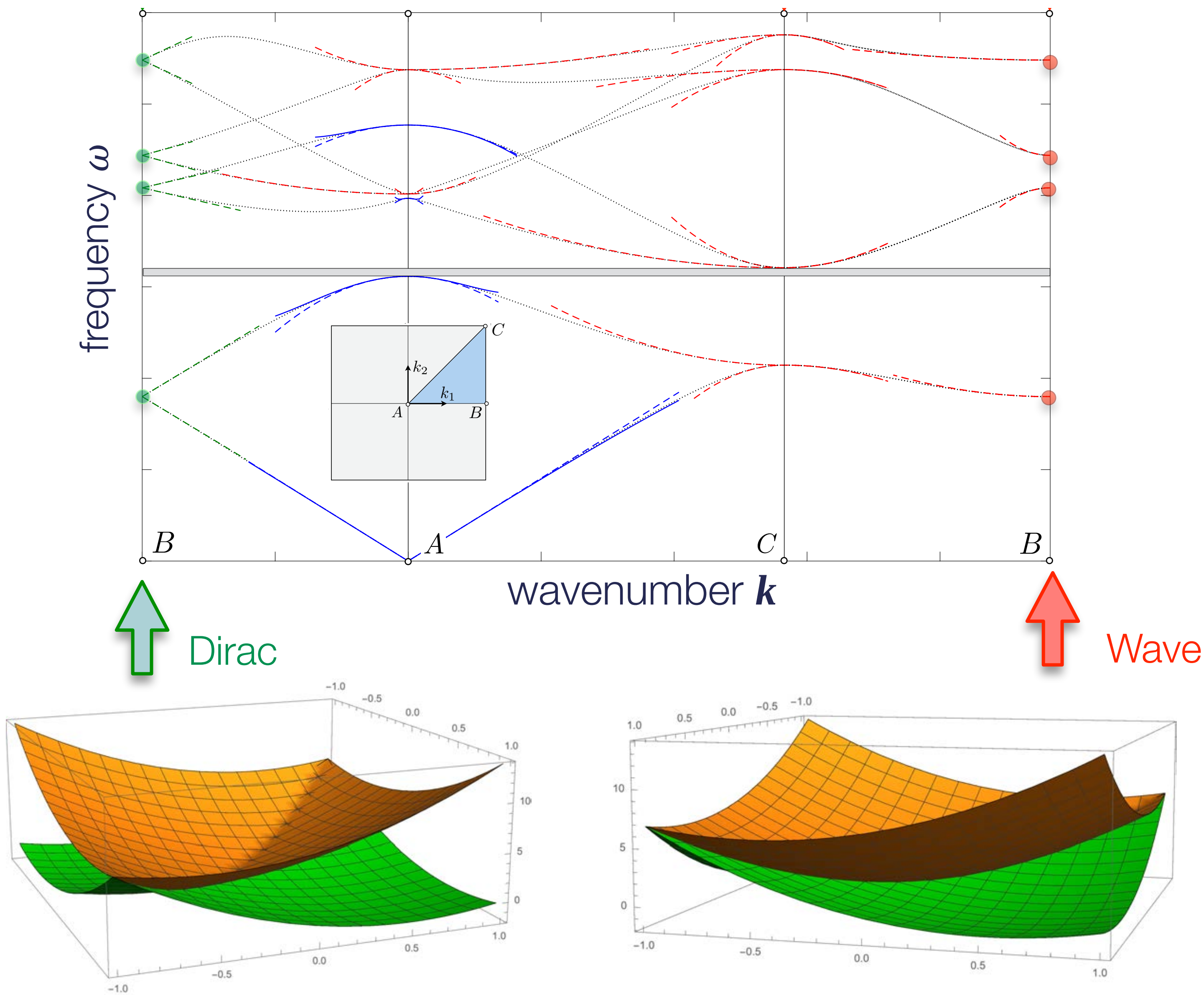
B_{pq} anisotropic



B_{pq} isotropic



Merging dispersion surfaces



Partial rank A_{pq}

Block-diagonal form

$$A_{pq} = \sum_{r,s} U_{pr} \Sigma_{rs} U^{\mathrm{T}}_{sq}$$

$$\left[\Sigma_{rs}\right] = \mathrm{diag}\left\{ \mathbb{O}_{N_0}, \underbrace{\begin{bmatrix} 0 & i\tau_{N_0+1} \\ -i\tau_{N_0+1} & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & i\tau_{N} \\ -i\tau_{N} & 0 \end{bmatrix}}_{O(\epsilon)}, \underbrace{\begin{bmatrix} 0 & i\tau_{N+1} \\ -i\tau_{N+1} & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & i\tau_{Q} \\ -i\tau_{Q} & 0 \end{bmatrix}}_{O(1)} \right\}$$

N "parabolas"

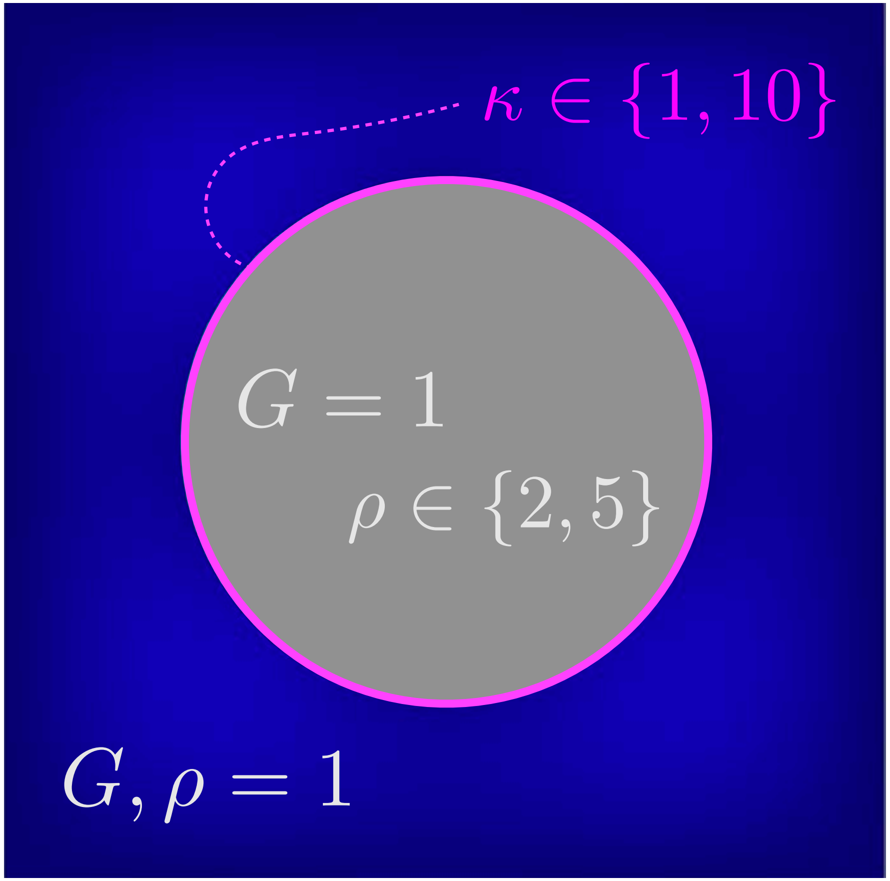
$$-\sum_{q\leqslant N} \left(B_{pq} + \bar{A}_{pq}\right) w_{0q} - \sigma\hat{\omega}^2 \sum_{q\leqslant N} D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}}, \qquad p = \overline{1, N}$$

Q-N "cones"

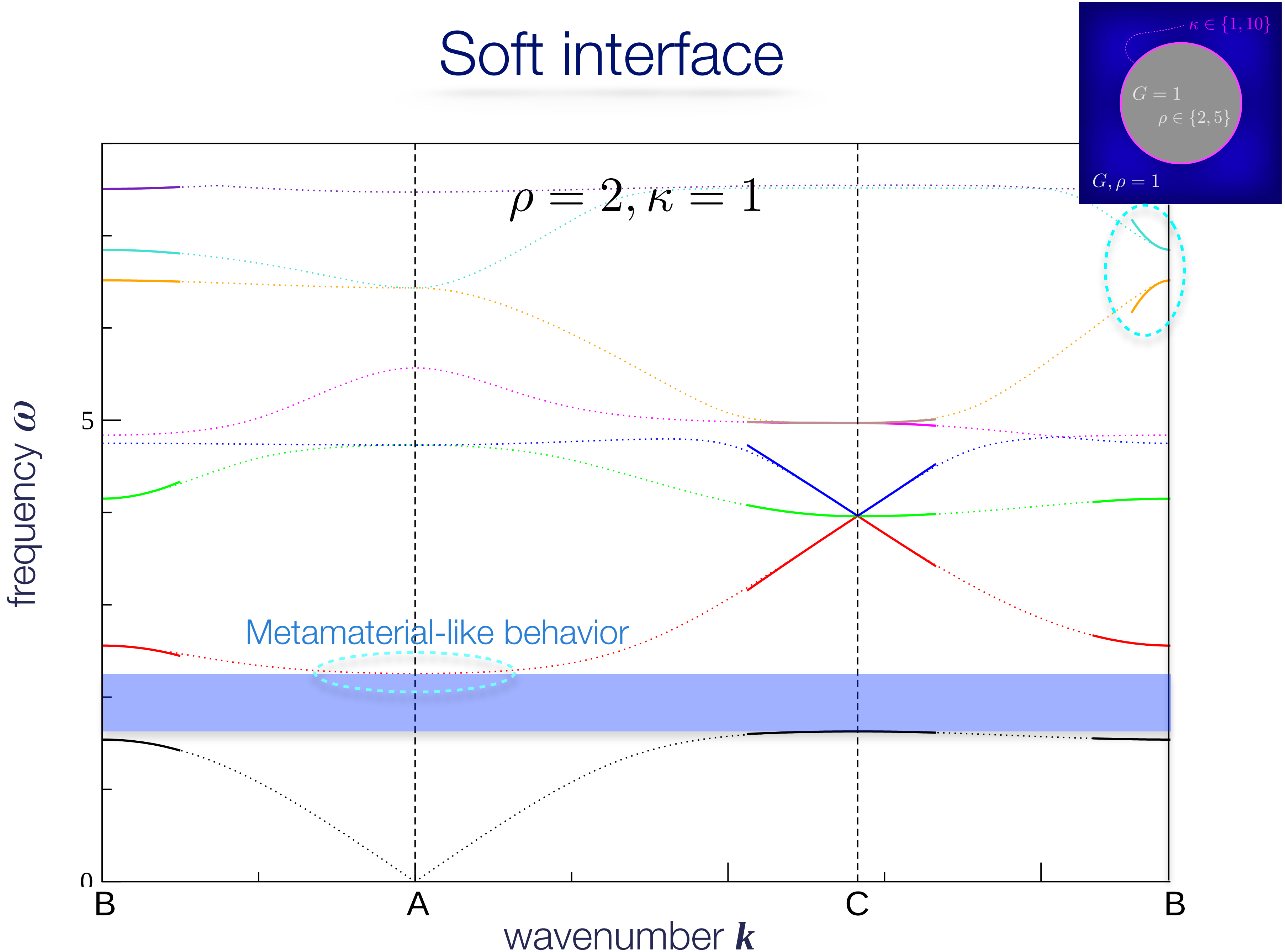
$$-\sum_{q>N} A_{pq} w_{1q} - \sigma\breve{\omega}^2 \sum_{q>N} D_{pq} w_{1q} =$$

$$= \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}} + \sum_{q\leqslant N} w_{0q} \left(B_{pq} + \sigma\breve{\omega}^2 \langle\rho\boldsymbol{\chi}^{(1)}_q\rangle^{p\varphi}_{\boldsymbol{a}} \cdot i\hat{\boldsymbol{k}}\right), \quad p = \overline{N+1, Q}$$

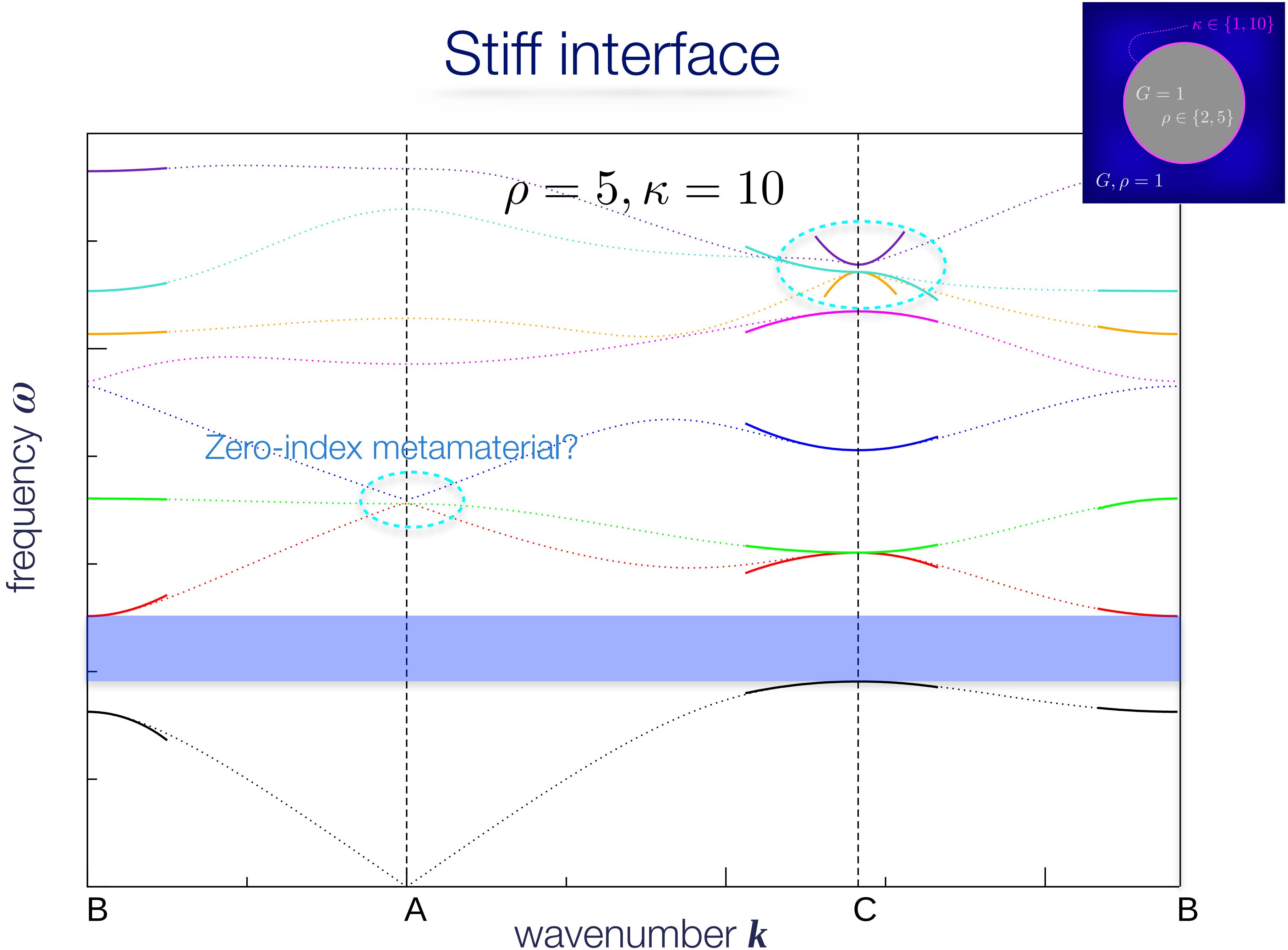
Oscillator



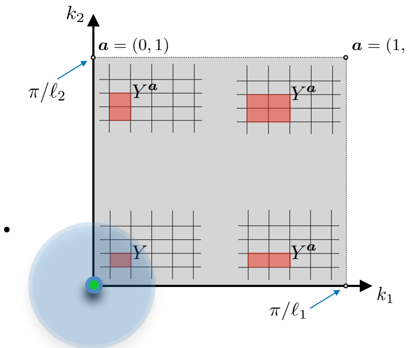
Soft interface



Stiff interface



"Microscopic" behavior



Ansatz

$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \epsilon\tilde{w}_3(\boldsymbol{x}) + \cdots$$

$$\langle\tilde{w}\rangle = \epsilon^{-2}w_0 + \epsilon^{-1}w_1 + w_2 + \epsilon w_3 + \cdots$$

$$-\big(\boldsymbol{\mu}^{(0)}\!:(i\hat{\boldsymbol{k}})^2 + \sigma\rho^{(0)}\hat{\omega}^2\big)w_0 = \langle 1\rangle \quad \times\epsilon^{-2}$$

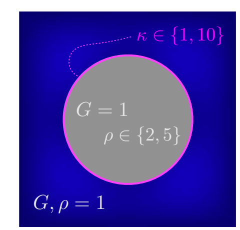
Macro

$$-\big(\boldsymbol{\mu}^{(0)}\!:(i\hat{\boldsymbol{k}})^2 + \sigma\rho^{(0)}\hat{\omega}^2\big)w_1 = -(\boldsymbol{\chi}^{(1)},1)\cdot i\hat{\boldsymbol{k}} \quad \times\epsilon^{-1}$$

$$-\big(\boldsymbol{\mu}^{(0)}\!:(i\hat{\boldsymbol{k}})^2 + \sigma\rho^{(0)}\hat{\omega}^2\big)w_2 - \big(\boldsymbol{\mu}^{(2)}\!:(i\hat{\boldsymbol{k}})^4 + \sigma\boldsymbol{\rho}^{(2)}\!:(i\hat{\boldsymbol{k}})^2\hat{\omega}^2\big)w_0 = \sigma\langle\rho\eta^{(0)}\rangle\hat{\omega}^2 + \ldots$$

$$\tilde{w}_0(\boldsymbol{x}) = w_0\,\tilde{\phi}_n^{\circ}(\boldsymbol{x})$$

Micro



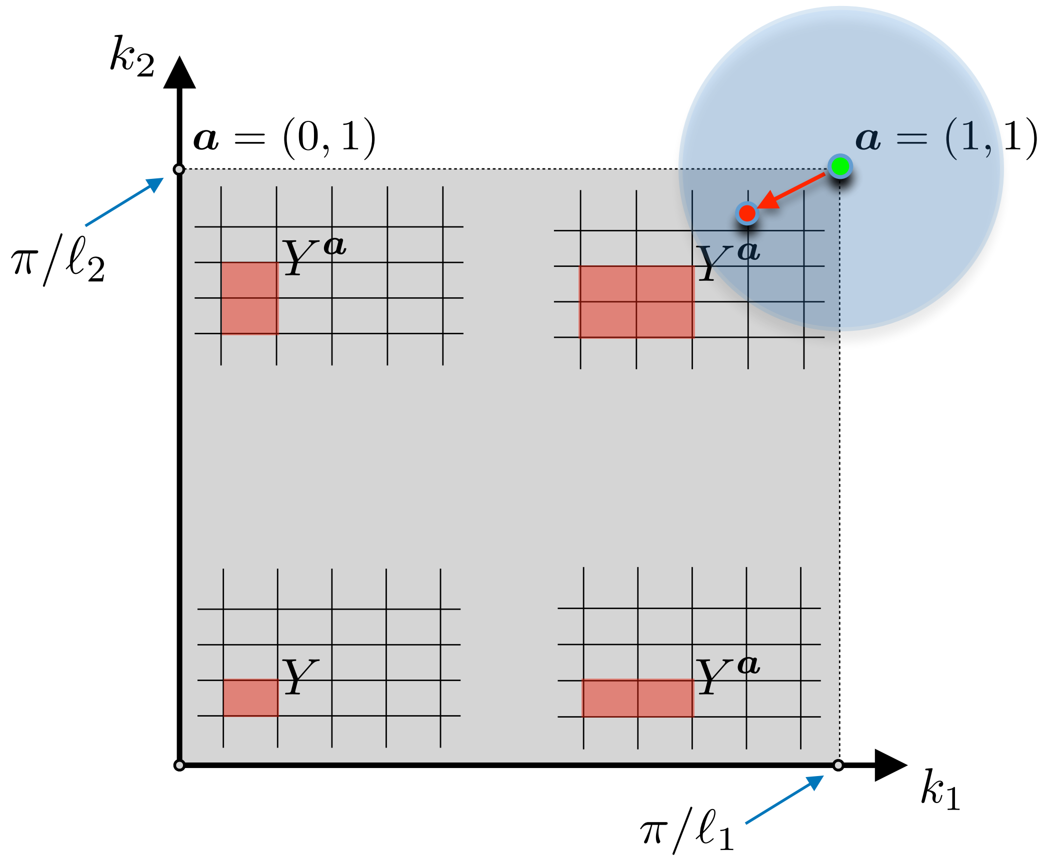
$$\tilde{w}_1(\boldsymbol{x}) = w_0\,\boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}} + w_1\,\tilde{\phi}_n^{\circ}(\boldsymbol{x})$$

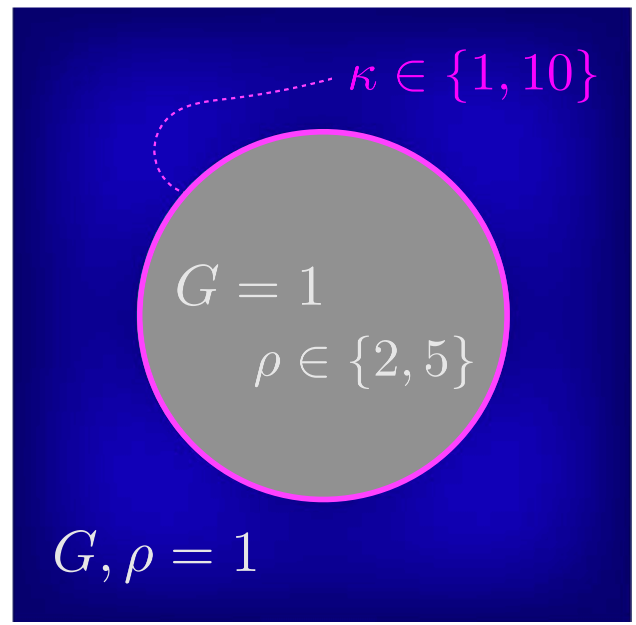
$$\tilde{w}_2(\boldsymbol{x}) = w_0\,\boldsymbol{\chi}^{(2)}(\boldsymbol{x}) : (i\hat{\boldsymbol{k}})^2 + w_1\,\boldsymbol{\chi}^{(1)}(\boldsymbol{x})\cdot i\hat{\boldsymbol{k}} + w_2\,\tilde{\phi}_n^{\circ}(\boldsymbol{x}) + \eta^{(0)}(\boldsymbol{x})$$

“Microscopic” behavior

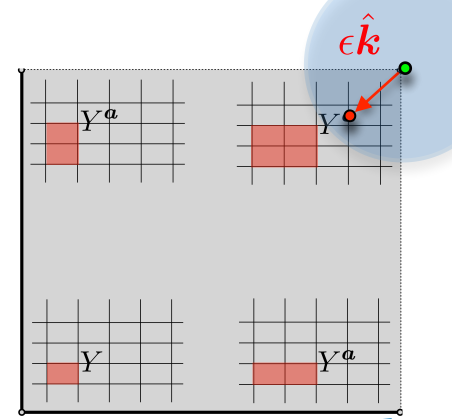
Neighborhood of $\hat{\boldsymbol{k}}^{\boldsymbol{a}}$

$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \epsilon\tilde{w}_3(\boldsymbol{x}) + \cdots$$





"Microscopic" behavior



Neighborhood of $\hat{\boldsymbol{k}}^{\boldsymbol{a}}$

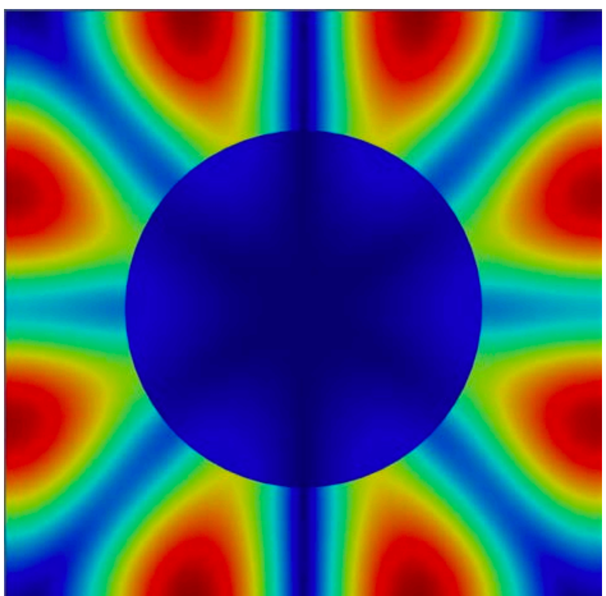
$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \cdots$$

$$\tilde{w}^{(\mathrm{m})}(\boldsymbol{x}) = \sum_{j=0}^{m} \epsilon^{j-2}\tilde{w}_j(\boldsymbol{x})$$

$$\epsilon = 0.10$$

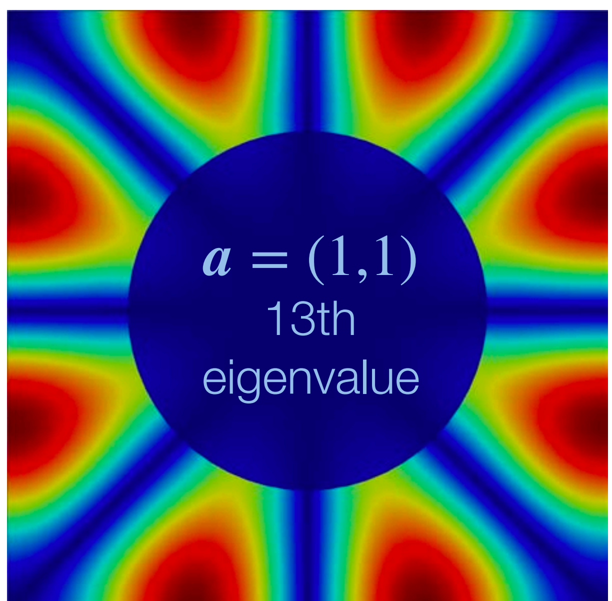
Exact

$\tilde{w}(\boldsymbol{x})$



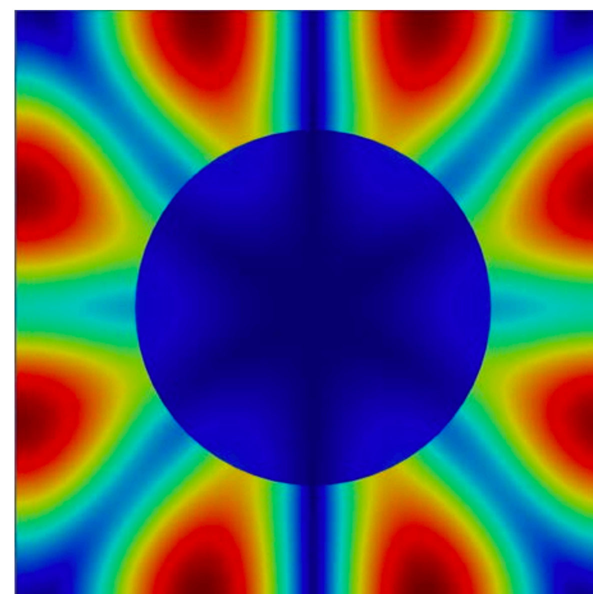
0th order

$\tilde{w}^{(0)}(\boldsymbol{x})$

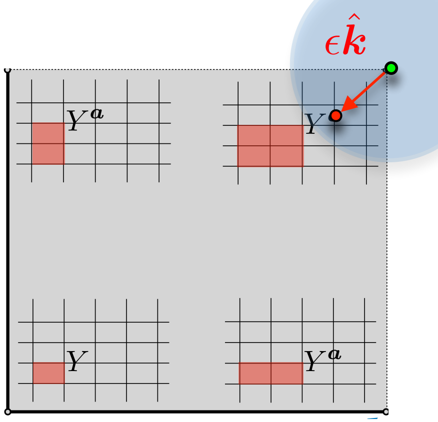


2nd order

$\tilde{w}^{(2)}(\boldsymbol{x})$



“Microscopic” behavior



Neighborhood of $\hat{\boldsymbol{k}}^{\boldsymbol{a}}$

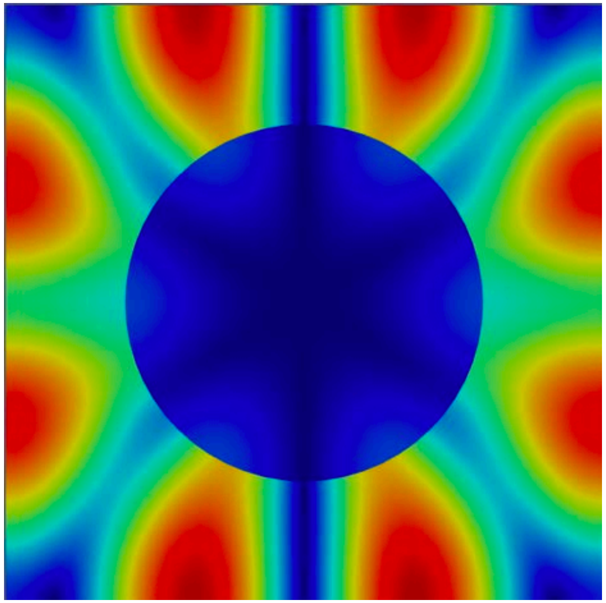
$$\tilde{w}(\boldsymbol{x}) = \epsilon^{-2}\tilde{w}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{w}_1(\boldsymbol{x}) + \tilde{w}_2(\boldsymbol{x}) + \cdots$$

$$\tilde{w}^{(\mathrm{m})}(\boldsymbol{x}) = \sum_{j=0}^{m} \epsilon^{j-2}\tilde{w}_j(\boldsymbol{x})$$

$$\epsilon = 0.20$$

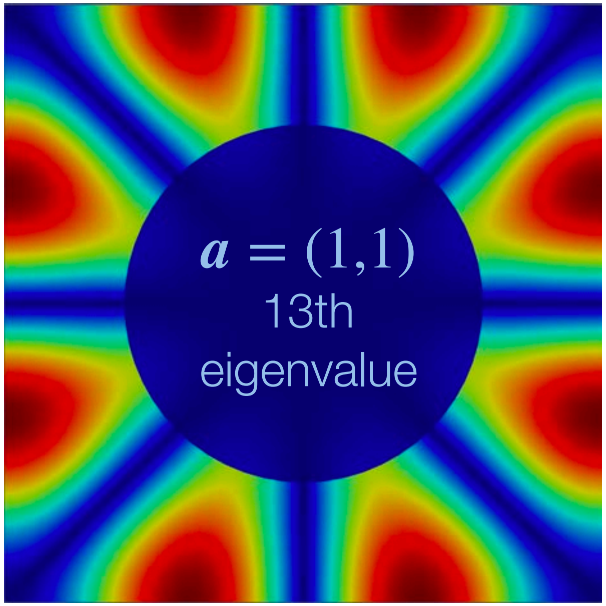
Exact

$\tilde{w}(\boldsymbol{x})$



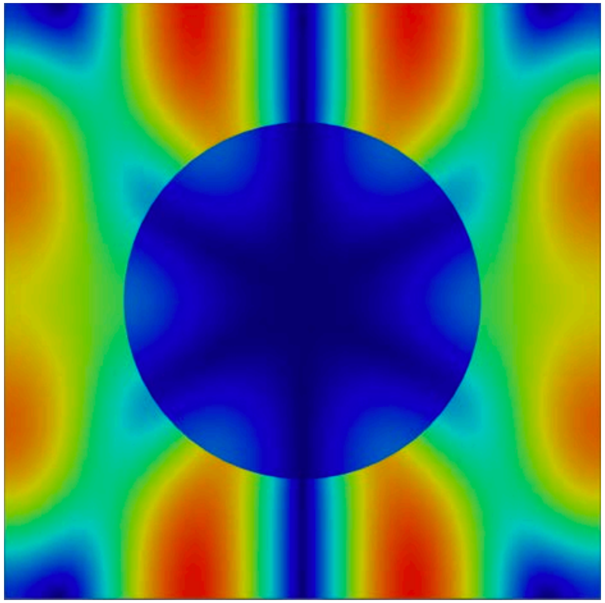
0th order

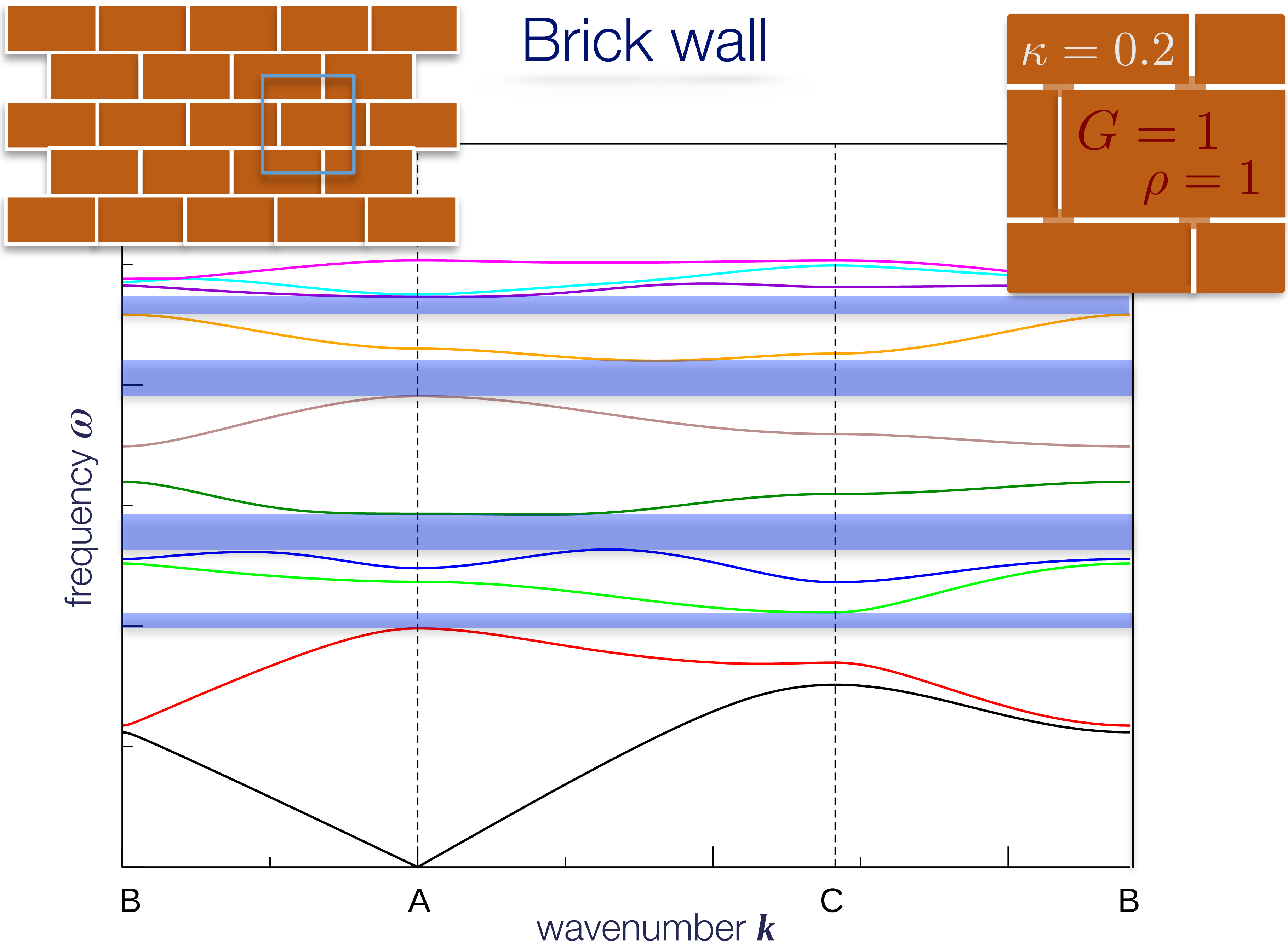
$\tilde{w}^{(0)}(\boldsymbol{x})$



2nd order

$\tilde{w}^{(2)}(\boldsymbol{x})$





Brick wall
$\kappa = 0.2$
$G = 1$
$\rho = 1$
frequency ω
B
A
C
B
wavenumber k