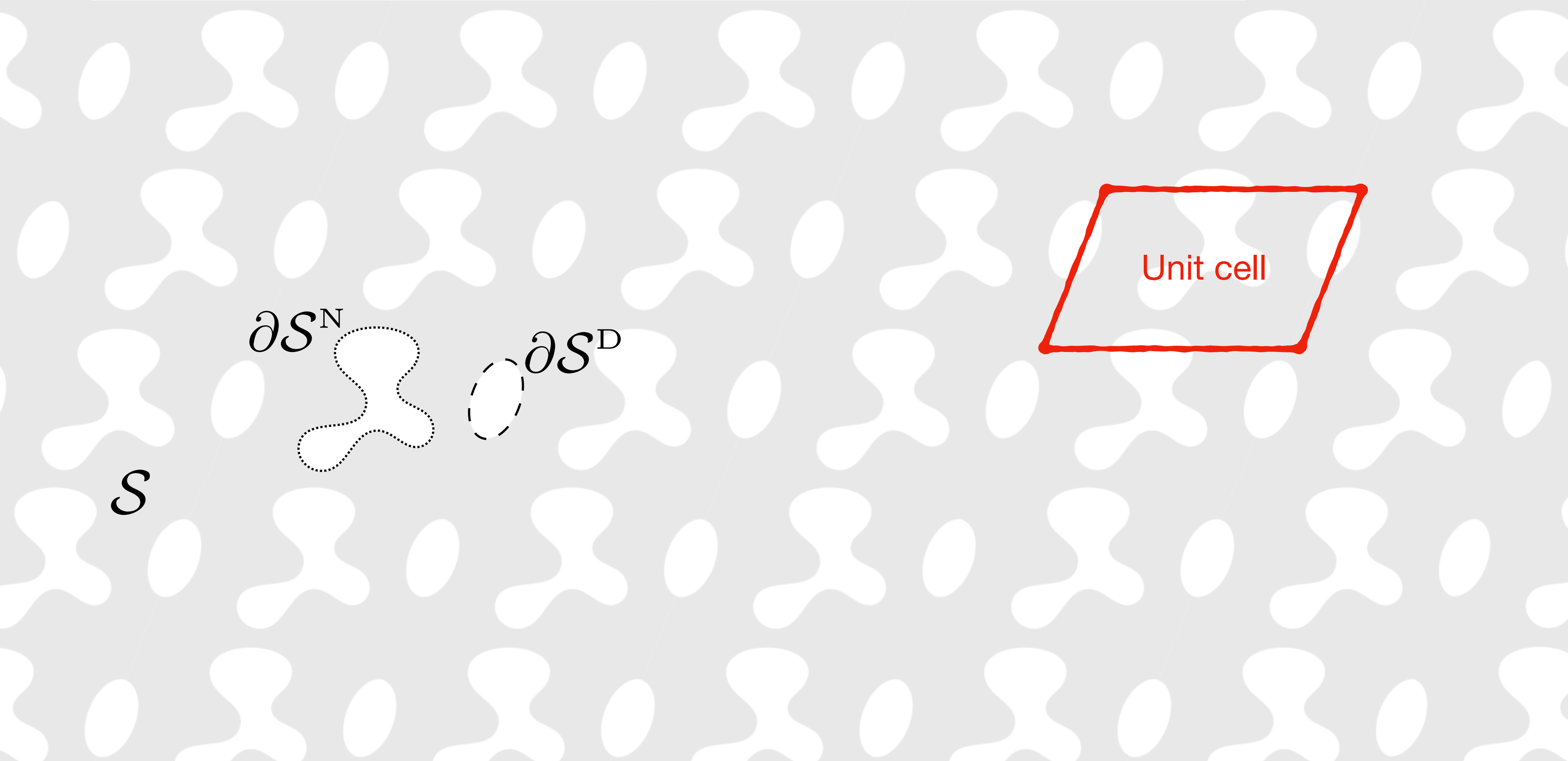
Part IV

Waves in Bravais lattices

Oudghiri-Idrissi/G/Meng

Geometry & boundary conditions

$$-\omega^2 \rho(\boldsymbol{x})\tilde{u} - \nabla_{\boldsymbol{k}} \cdot \big(G(\boldsymbol{x})\nabla_{\boldsymbol{k}}\tilde{u}\big) \;=\; \tilde{f}(\boldsymbol{x}) \quad \text{in} \;\; Y,$$



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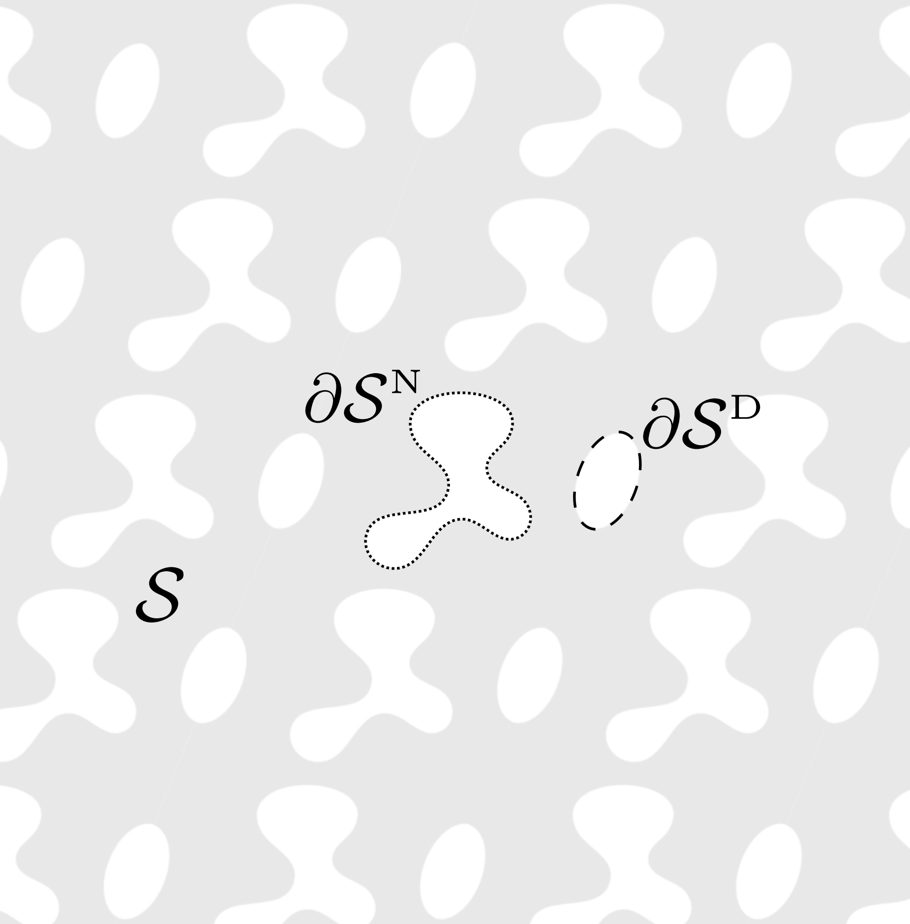
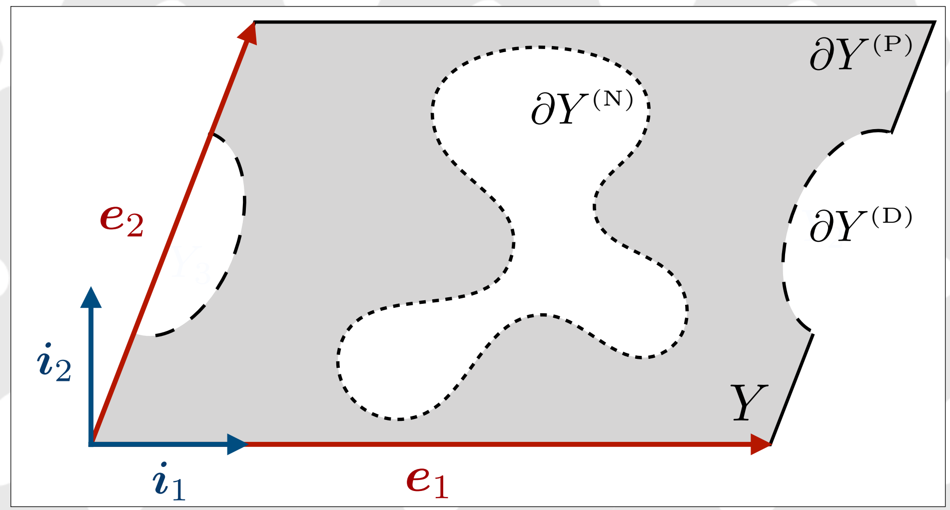




$\partial Y^{(\mathrm{P})}$ Periodic BCs

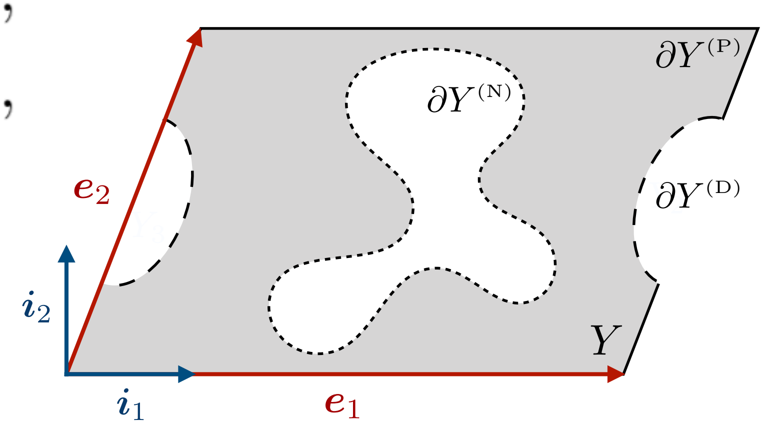
$\partial \mathcal{S}^{\mathrm{N}}, \partial Y^{(\mathrm{N})}$ Neumann BCs

$\partial \mathcal{S}^{\mathrm{D}}, \partial Y^{(\mathrm{D})}$ Dirichlet BCs

Bloch wave setup

Unit cell problem

$$
\begin{aligned}
-\omega^2 \rho(\boldsymbol{x}) \tilde{u} - \nabla_{\boldsymbol{k}} \cdot \left(G(\boldsymbol{x}) \nabla_{\boldsymbol{k}} \tilde{u} \right) &= \tilde{f}(\boldsymbol{x}) \quad \text{in } Y, \\
\tilde{u}|_{x^j=0} &= \tilde{u}|_{x^j=1}, \\
\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{u}|_{x^j=0} &= -\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{u}|_{x^j=1}, \\
\boldsymbol{\nu} \cdot G \nabla_{\boldsymbol{k}} \tilde{u}|_{\boldsymbol{x} \in \partial Y^{(\mathrm{N})}} &= 0, \\
\tilde{u}|_{\boldsymbol{x} \in \partial Y^{(\mathrm{D})}} &= 0,
\end{aligned}
$$



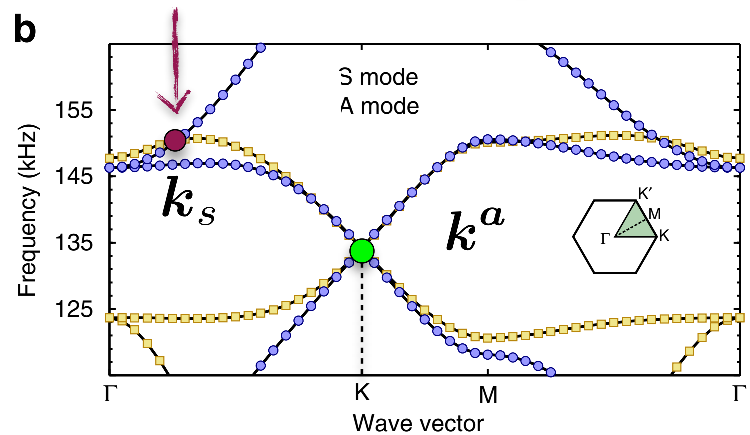
Source term

$$
f(\boldsymbol{x}) = \int_{\boldsymbol{k}_s + \mathcal{C}} \tilde{f}(\boldsymbol{x})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k}, \quad \boldsymbol{k}_s \in \overline{\mathcal{B}}, \quad \mathcal{C} \subseteq \mathcal{B}
$$

Asymptotic analysis

FF-FW regime

$$\boldsymbol{k} = \boldsymbol{k}_s + \epsilon\hat{\boldsymbol{k}}, \quad \omega^2 = \tilde{\lambda}_n + \epsilon\sigma\breve{\omega}^2 + \epsilon^2\sigma\hat{\omega}^2 \quad \boldsymbol{k}_s \in \mathcal{B}$$



Ansatz

$$\tilde{u}(\boldsymbol{x}) = \epsilon^{-2}\tilde{u}_0(\boldsymbol{x}) + \epsilon^{-1}\tilde{u}_1(\boldsymbol{x}) + \tilde{u}_2(\boldsymbol{x}) + \ldots$$

Source expansion

$$\tilde{f}(\boldsymbol{x}) = \tilde{f}_0(\boldsymbol{x}) + \epsilon\tilde{\boldsymbol{f}}_1(\boldsymbol{x})\cdot(i\hat{\boldsymbol{k}}) + \epsilon^2\tilde{\boldsymbol{f}}_2(\boldsymbol{x}):(i\hat{\boldsymbol{k}})^2 + \ldots$$

Cascade

$$O(\epsilon^{-2}): \quad -\tilde{\lambda}_n \rho \tilde{u}_0 - \nabla_{\boldsymbol{k}_s} \cdot \big(G \nabla_{\boldsymbol{k}_s} \tilde{u}_0\big) \;=\; 0$$

$$\rightarrow \quad \tilde{u}_0(\boldsymbol{x}) \;=\; u_0 \, \tilde{\phi}_n(\boldsymbol{x})$$

$$\boldsymbol{k} = \boldsymbol{k}_s \quad \Rightarrow \quad \arg(\tilde{\phi}_n(\boldsymbol{x})) \neq \text{const}$$

$$O(\epsilon^{-1}): \quad \rightarrow \quad \tilde{u}_1(\boldsymbol{x}) = u_0 \, \boldsymbol{\chi}^{(1)}(\boldsymbol{x}) \cdot i\hat{\boldsymbol{k}} + u_1 \tilde{\phi}_n(\boldsymbol{x})$$

$$\big\langle O(\epsilon^{-1}) \big\rangle : \; -\big(\boldsymbol{\theta}^{(0)} \cdot (i\hat{\boldsymbol{k}}) + \sigma \breve{\omega}^2 \rho^{(0)}\big) u_0 \;=\; 0$$

$$\boldsymbol{\theta}^{(0)} \;=\; \langle G \nabla_{\boldsymbol{k}_s} \tilde{\phi}_n \rangle - \langle G \nabla_{\boldsymbol{k}_s} \tilde{\phi}_n \rangle^* \in i\mathbb{R}^d, \qquad \rho^{(0)} \;=\; \langle \rho \tilde{\phi}_n \rangle \in \mathbb{R}_+$$

O(1) effective equation

Case $\boldsymbol{\theta}^{(0)} \neq \mathbf{0}$

$$-\left(\boldsymbol{\theta}^{(0)} \cdot (i\boldsymbol{k}) + \rho^{(0)} \sigma \breve{\omega}^2\right) \langle u \rangle \; = \; \epsilon \langle \tilde{f}_0 \rangle$$

Case $\boldsymbol{\theta}^{(0)} = \mathbf{0}$

$$-\left(\boldsymbol{\mu}^{(0)} : (i\boldsymbol{k})^2 + \rho^{(0)} \sigma \hat{\omega}^2\right) \langle u \rangle \; = \; \epsilon^2 \langle \tilde{f}_0 \rangle$$

$$\boldsymbol{\mu}^{(0)} = \left\langle G\{\nabla_{\boldsymbol{k}_s} \boldsymbol{\chi}^{(1)} + \tilde{\phi}_n \boldsymbol{I}\}\right\rangle - \left(G\{\boldsymbol{\chi}^{(1)} \otimes (\nabla_{\boldsymbol{k}_s} \tilde{\phi}_n)^*\}, 1\right) - \frac{1}{\rho^{(0)}} \{\boldsymbol{\theta}^{(0)} \otimes \boldsymbol{\rho}^{(1)}\}$$

$$\boldsymbol{\rho}^{(1)} = \langle \rho \boldsymbol{\chi}^{(1)} \rangle$$

$$\boldsymbol{\mu}^{(0)} \in \mathbb{R}^{d \times d}$$

Repeated eigenvalues

$O(\epsilon^{-2})$ equation $\Longrightarrow$ $$\tilde{u}_0 = \sum_{q=1}^{Q} u_{0q}\, \tilde{\phi}_{n_q}, \qquad u_{0q} \in \mathbb{C}$$

$O(\epsilon^{-1})$ equation $\Longrightarrow$ $$(\boldsymbol{A}^{(0)} + \sigma\breve{\omega}^2 \boldsymbol{D})\boldsymbol{u}_0 = \boldsymbol{0}$$

$$A^{(0)}_{pq} = \boldsymbol{\theta}^{(0)}_{pq} \cdot (i\boldsymbol{k}), \qquad D_{pq} = \rho^{(0)}_p \delta_{pq}$$

$$\boldsymbol{\theta}^{(0)}_{pq} = \langle G \nabla_{\boldsymbol{k}_s} \tilde{\phi}_{n_q} \rangle^{n_p} - \left(\langle G \nabla_{\boldsymbol{k}_s} \tilde{\phi}_{n_p} \rangle^{n_q} \right)^* \quad \text{and} \quad \rho^{(0)}_p = \langle \rho \tilde{\phi}_{n_p} \rangle^{n_p}$$

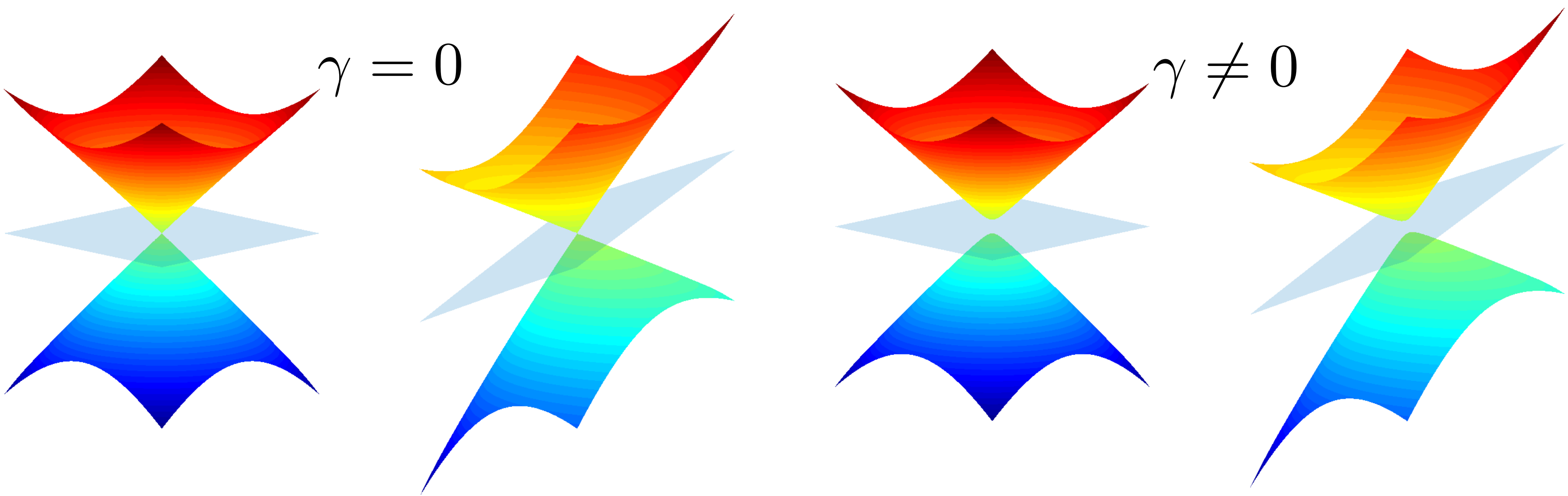
$O(1)$ equation

$\Longrightarrow$ $$-(\boldsymbol{B}^{(0)} + \sigma\hat{\omega}^2 \boldsymbol{D})\boldsymbol{u}_0 - (\boldsymbol{A}^{(0)} + \sigma\breve{\omega}^2 \boldsymbol{D})\boldsymbol{u}_1 = \boldsymbol{f}_0$$

$$B_{pq} = \boldsymbol{\mu}^{(0)}_{pq} : (i\hat{\boldsymbol{k}})^2, \qquad f_{0p} = \langle \tilde{f}_0 \rangle^{n_p}$$

Dirac cones in 2D

$$\omega^2_{n_{1/2}}(\boldsymbol{k}) = \omega^2_{n_1} - \frac{\epsilon}{2}\left(\gamma + \frac{\boldsymbol{\theta}^{(0)}_{11}\cdot \mathrm{i}\hat{\boldsymbol{k}}}{\rho^{(0)}_1} + \frac{\boldsymbol{\theta}^{(0)}_{22}\cdot \mathrm{i}\hat{\boldsymbol{k}}}{\rho^{(0)}_2} \pm \sqrt{\left(\gamma - \frac{\boldsymbol{\theta}^{(0)}_{11}\cdot \mathrm{i}\hat{\boldsymbol{k}}}{\rho^{(0)}_1} + \frac{\boldsymbol{\theta}^{(0)}_{22}\cdot \mathrm{i}\hat{\boldsymbol{k}}}{\rho^{(0)}_2}\right)^2 + \frac{4\{\boldsymbol{\theta}^{(0)}_{12}\otimes\boldsymbol{\theta}^{(0)*}_{12}\}:(\hat{\boldsymbol{k}})^2}{\rho^{(0)}_1\rho^{(0)}_2}}\right)$$



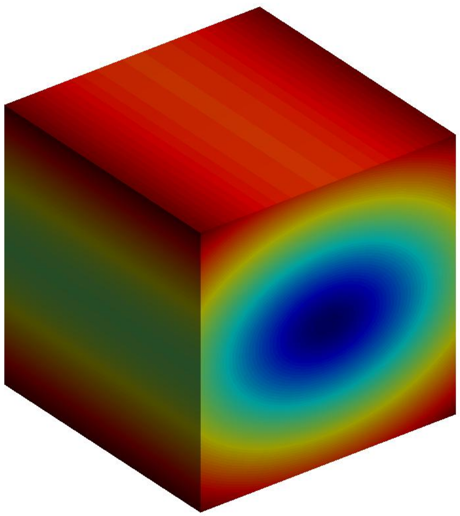
Dirac cones | Tilted Dirac cones | Blunted Dirac cones | Blunted and tilted Dirac cones

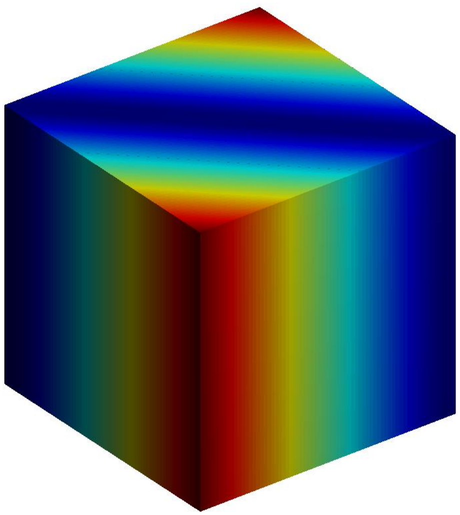
Hyperconones in 3D

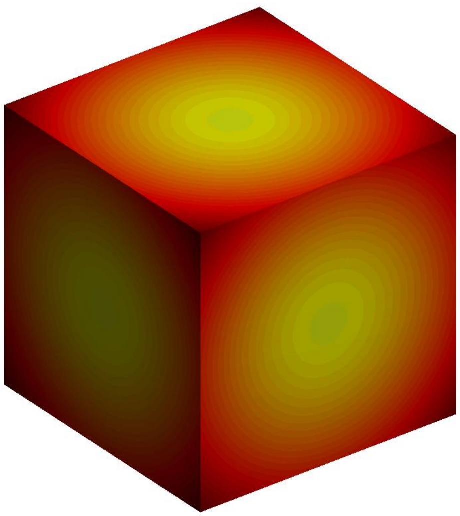
$$\omega^2_{n_{1/2}}(\boldsymbol{k}) = \omega^2_{n_1} - \frac{\epsilon}{2}\left(\gamma \pm \sqrt{\gamma^2 + \frac{4\{\boldsymbol{\theta}^{(0)}_{12} \otimes \boldsymbol{\theta}^{(0)*}_{12}\} : (\hat{\boldsymbol{k}})^2}{\rho^{(0)}_1 \rho^{(0)}_2}}\right)$$

$$\omega^2_{n_{1/2}}(\boldsymbol{k}) = \omega^2_{n_1} \mp \frac{\|\boldsymbol{\theta}^{(0)}_{11}\|}{\rho^{(0)}_1} \|\epsilon\hat{\boldsymbol{k}}\|$$

$$\gamma \neq 0$$



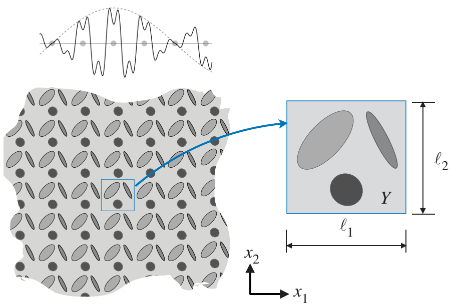


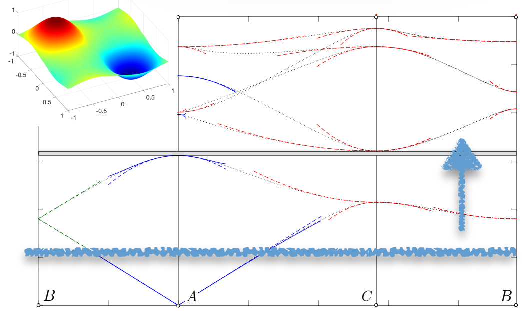


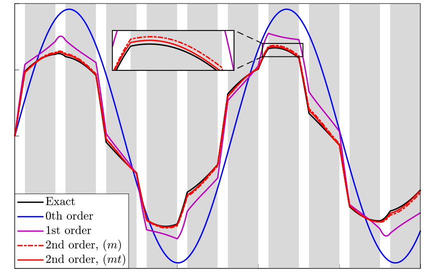
$\mathrm{Re}(\boldsymbol{\theta}^{(0)}_{12}) \times \mathrm{Im}(\boldsymbol{\theta}^{(0)}_{12}) \neq \boldsymbol{0}$

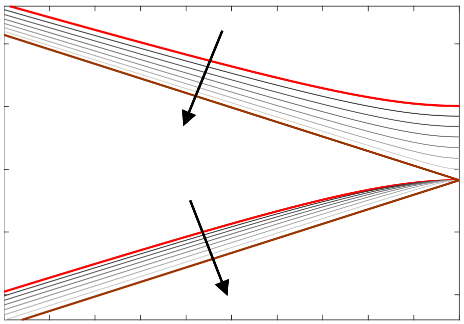
$\mathrm{Re}(\boldsymbol{\theta}^{(0)}_{12}) \times \mathrm{Im}(\boldsymbol{\theta}^{(0)}_{12}) = \boldsymbol{0}$

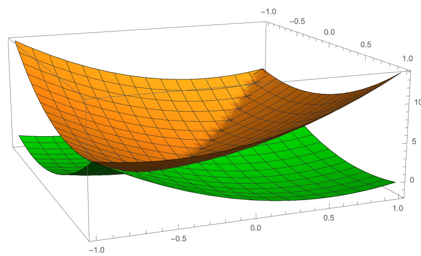
Thanks

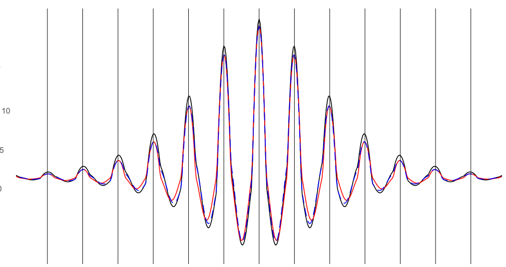












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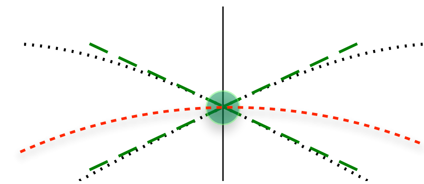
Change of the eigenfunction basis

T_{pq} invertible, real-valued matrix

$$\widetilde{\tilde{\varphi}^{\boldsymbol{a}}_{np}} = \sum_q T_{pq}\,\tilde{\varphi}^{\boldsymbol{a}}_{nq}, \qquad p = \overline{1, Q}$$



Effective coefficients

$$\widetilde{\rho^{(0)}_p} = \sum_r T_{pr}\,\rho^{(0)}_r\,T^T_{rp}, \qquad \widetilde{\boldsymbol{\theta}^{(0)}_{pq}} = \sum_{r,s} T_{pr}\,\boldsymbol{\theta}^{(0)}_{rs}\,T^T_{sq}, \qquad \widetilde{\boldsymbol{\mu}^{(0)}_{pq}} = \sum_{r,s} T_{pr}\,\boldsymbol{\mu}^{(0)}_{rs}\,T^T_{sq},$$

Effective system of eqs.

$$-\sum_q A_{pq}\,w_{1q} - \sigma\breve{\omega}^2 \sum_q D_{pq}\,w_{1q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}}$$

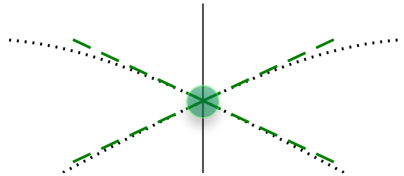
$$-\sum_q B_{pq}\,w_{0q} - \sigma\hat{\omega}^2 \sum_q D_{pq}\,w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}}$$

$$w_{jq} \to \widetilde{w_{jq}} = \sum_s T^{-T}_{qs}\,w_{js}, \qquad M_{pq} \to \widetilde{M_{pq}} = \sum_{r,s} T_{pr}\,M_{rs}\,T^T_{sq},$$

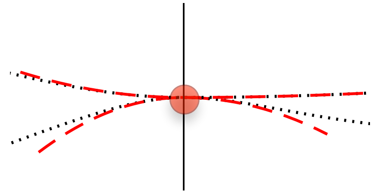
$$\langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\varphi}_{\boldsymbol{a}} \to \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{p\widetilde{\varphi}}_{\boldsymbol{a}} = \sum_s T_{qs}\,\langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}}\rangle^{s\varphi}_{\boldsymbol{a}}, \qquad j \in \{0, 1\}, \quad M \in \{A, B, D\}$$

Uncoupling

Dirac/wave system



$$-\sum_q A_{pq} w_{1q} - \sigma \breve{\omega}^2 \sum_q D_{pq} w_{1q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}} \rangle_{\boldsymbol{a}}^{p\varphi}$$



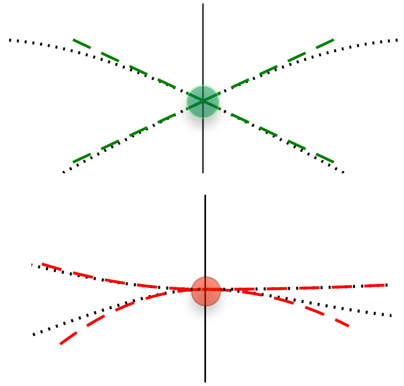
$$-\sum_q B_{pq} w_{0q} - \sigma \hat{\omega}^2 \sum_q D_{pq} w_{0q} = \langle e^{i\boldsymbol{k}^{\boldsymbol{a}}\cdot\boldsymbol{x}} \rangle_{\boldsymbol{a}}^{p\varphi}$$

→ fixed direction $\hat{\boldsymbol{k}}/\|\hat{\boldsymbol{k}}\|$

Uncoupling

Average effective density (energy): invariant

$$\rho^{(0)} = \frac{1}{Q} \sum_q \rho_q^{(0)} \longrightarrow D_{pq}^{-1/2} = \delta_{pq} \left(\rho_q^{(0)}\right)^{-1/2}$$



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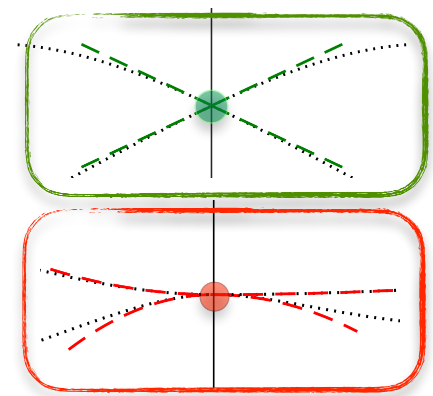

→ fixed direction $\hat{\boldsymbol{k}}/||\hat{\boldsymbol{k}}||$

Transformation ($\boldsymbol{R}_{ps}$ orthogonal, real-valued)

$$T_{pq} = \sqrt{\rho^{(0)}}\sum_s R^T_{ps}\,D^{-1/2}_{sq}$$

$$\widetilde{\tilde{\varphi}^{\boldsymbol{a}}_{np}} = \sum_q T_{pq}\,\tilde{\varphi}^{\boldsymbol{a}}_{nq}$$

$$\begin{bmatrix} 0 & A_{12} & 0 & 0 \\ -A_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} \\ 0 & 0 & -A_{34} & 0 \end{bmatrix}$$

$$\widetilde{A_{pq}} = \widetilde{\boldsymbol{\theta}^{(0)}_{pq}}\cdot(i\hat{\boldsymbol{k}})$$ 2x2 block-diagonal

$$\widetilde{D_{pq}} = \delta_{pq}\,\rho^{(0)}$$ ∝ identity

$$\widetilde{B_{pq}} = \widetilde{\boldsymbol{\mu}^{(0)}_{pq}} : (i\hat{\boldsymbol{k}})^2$$ diagonal