Long-wave propagation in multi-layered and multi-component strongly inhomogeneous waveguides

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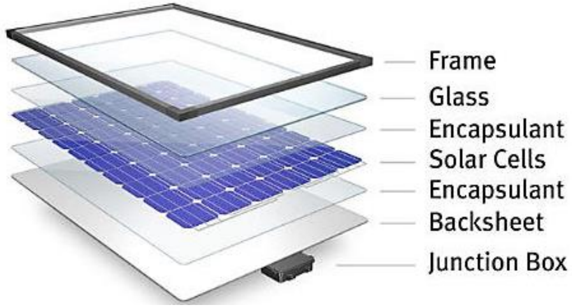
Outline

- Introduction
- Strongly inhomogeneous rods
- Anti-plane motion of circular layered cylinders
- Layered plates
- Coated half-space
- Periodic structures

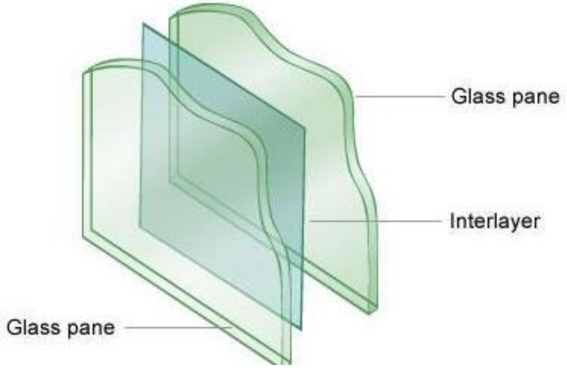
1. Introduction

High-contrast layered structures

- *photovoltaic panels*
- *laminated glass*



www.dupont.com

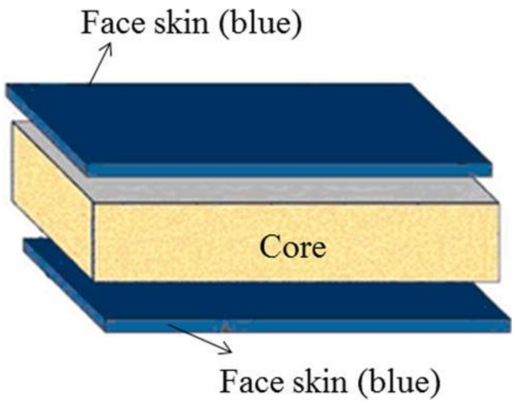


www.gscglass.com

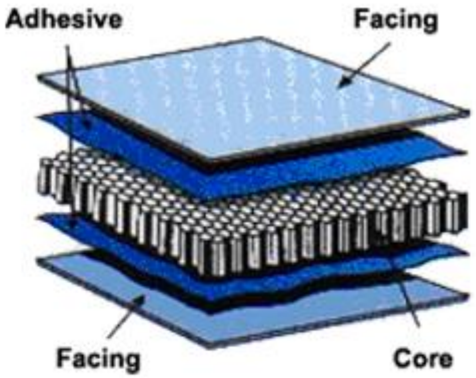
Introduction

Sandwich structures

- ***Classical sandwich plate***



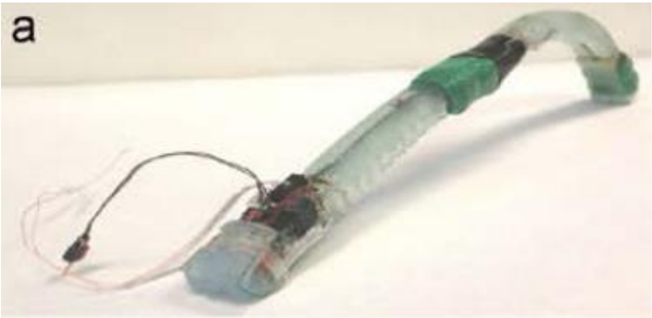
- ***Foam insulation panels***

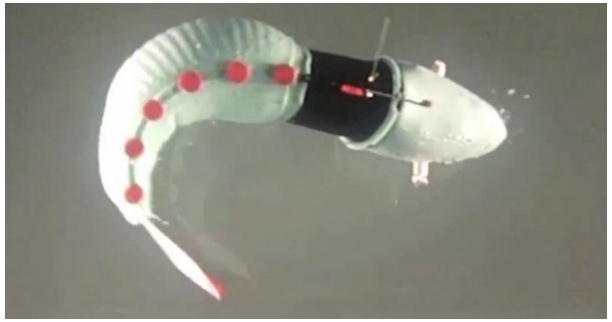


Soft robots

Rus & Tolley, 2015. Design, fabrication and control of soft robots. *Nature*, *521*(7553), 467.

Stokes et al. A hybrid combining hard and soft robots. *Soft Robotics* 1.1 (2014): 70-74





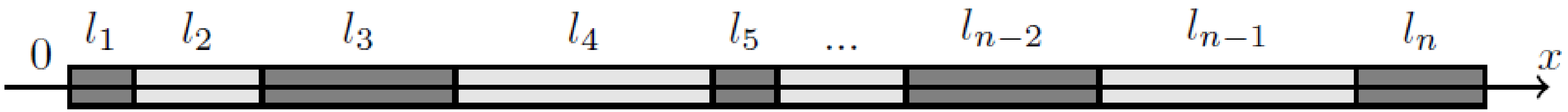
Bio-composites

- ***(Teeth, bones, etc., contain both soft protein/collagen matrix and hard mineral inclusions)***

Slesarenko et al., 2017. Understanding the strength of bioinspired soft composites. *Int. J. Mech. Sci*, *131*, 171-178.

2. Low-frequency vibrations of multi-component high-contrast elastic rods

J. Kaplunov et al., *J. Sound Vib.*, 445 (2019): 132-147



Contrast in

- *Stiffness*
- *Density*
- *Length*

$$\frac{E_i}{E_j}, \quad \frac{\rho_i}{\rho_j}, \quad \frac{l_i}{l_j}$$

Small parameters → asymptotic methods

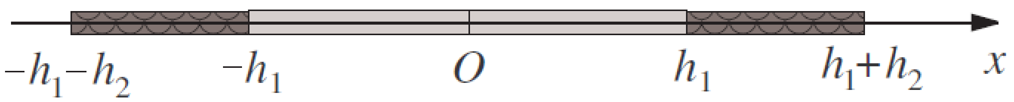
Physical intuition:

Strong components (free ends) - almost rigid body motions

Weak components (fixed b.c.) - almost homogeneous deformations

Toy problem: three-component rod (antisymmetric)

J. Kaplunov et al. *J. Sound Vib.* 366 (2016): 264-276



Equations of motion

$$E_i \frac{d^2 u}{dx^2} + \rho_i \omega^2 u = 0, \qquad i = 1, 2.$$

Free ends

$$u'|_{\pm(h_1+h_2)} = 0.$$

Continuity conditions

$$u|_{\pm(h_1+0)} = u|_{\pm(h_1-0)}, \qquad E_2 u|_{\pm(h_1+0)} = E_1 u|_{\pm(h_1-0)}.$$

Frequency equation

Scaling

$$E = \frac{E_1}{E_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad c = \frac{c_1}{c_2}, \quad h = \frac{h_1}{h_2},$$

and

$$\chi = \frac{x}{h_1}, \quad \lambda_i = \frac{\omega h_i}{c_i}, \qquad i = 1, 2.$$

Frequency equation

$$\tan \lambda_1 \tan \lambda_2 = \frac{E}{c}.$$

Low-frequency analysis in view of contrast:

- Global low-frequency behaviour ($\lambda_i \ll 1, \quad i = 1, 2$)
- Local low-frequency behaviour ($\lambda_i \ll 1, \lambda_k \gtrsim 1, \quad i \neq k$)

Global low-frequency behaviour

Conditions on material parameters

$$\lambda_1 \ll 1, \quad \lambda_2 \ll 1 \quad \Rightarrow$$

$$E \ll h \ll \rho^{-1}.$$

Approximate frequency equation

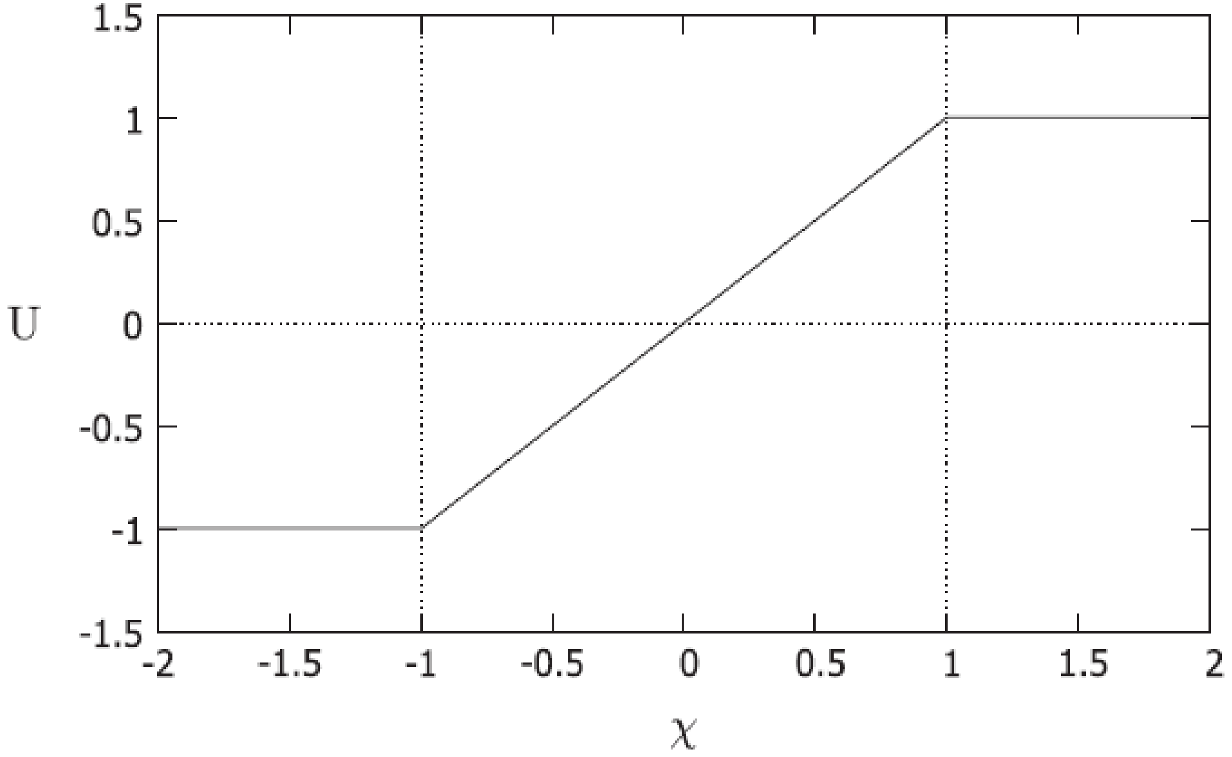
$$\lambda_1 \lambda_2 = \frac{E}{c} \quad \Rightarrow$$

$$\lambda_1 = \sqrt{E\rho}, \quad \lambda_2 = \sqrt{\frac{E}{h}}.$$

Global low-frequency behaviour

Approximate polynomial eigenform

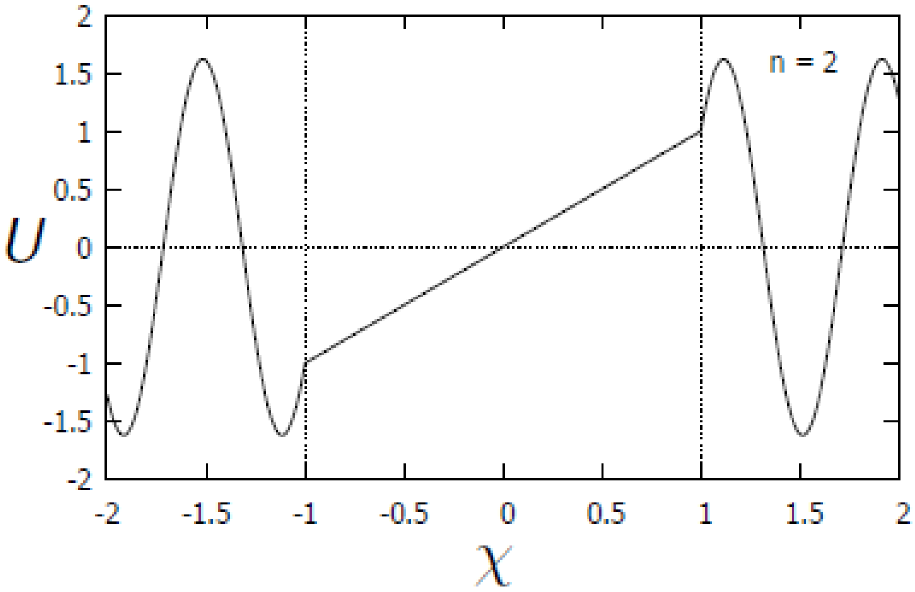
$$U = \begin{cases} 1, & |\chi| > 1; \\ \chi, & |\chi| \le 1. \end{cases}$$



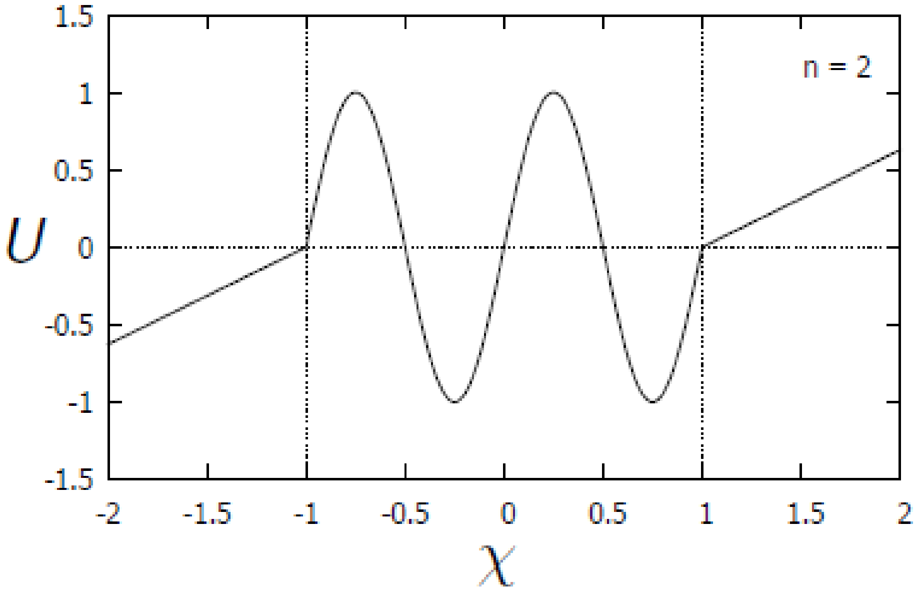
Local low-frequency behaviour

May occur for core or outer sections.

Approximate displacement profiles

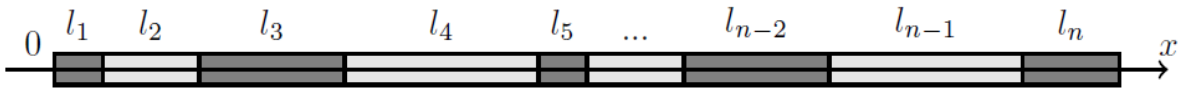


$$n \ll \min\left[\frac{E}{h}, \frac{c}{h}\right]$$



$$\rho h \ll n \ll \frac{h}{c}$$

Multi-component high-contrast elastic rods



Problem parameters

$$\varepsilon = \frac{E_2}{E_1} << 1, \qquad \rho_1 \sim \rho_2, \qquad L_i^j = \frac{l_j}{l_i}, \qquad c_m^2 = \frac{E_m}{\rho_m}.$$

Dimensionless scaling

$$X_i = \frac{x_i}{l_i}, \quad \Omega_i = \frac{\omega l_i}{c_m}, \quad b_i \le X_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1, n}; \quad m = 1,2.$$

Equations of motion

$$\frac{d^2 u_i}{dX_i^2} + \Omega_i^2 u_i = 0,$$

Boundary conditions

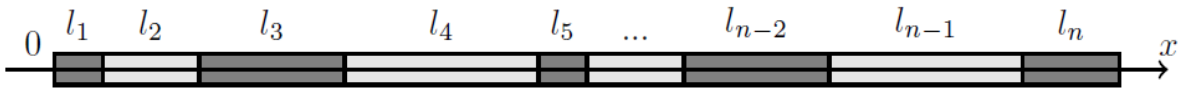
$$\left.\frac{du_1}{dX_1}\right|_{X_1=0} = \left.\frac{du_n}{dX_n}\right|_{X_n=b_n+1} = 0,$$

Continuity

$$u_i\big|_{X_i=b_i+1} = u_{i+1}\big|_{X_{i+1}=b_{i+1}}, \qquad \left.\frac{du_i}{dX_i}\right|_{X_i=b_i+1} = \varepsilon^j L_{i+1}^i \left.\frac{du_{i+1}}{dX_{i+1}}\right|_{X_{i+1}=b_{i+1}}.$$

($j=1$ or -1 for i^{th} component being stiff or soft, respectively)

Multi-component high-contrast elastic rods



Asymptotic expansions $$u_i = u_{i0} + \varepsilon u_{i1} + \ldots,$$

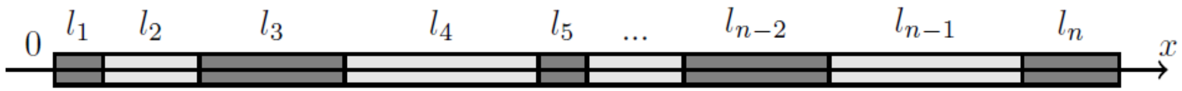
Global low-frequency regime $$\Omega_i^2 \sim \varepsilon, \quad \Omega_i^2 = \varepsilon\left(\Omega_{i0}^2 + \varepsilon\,\Omega_{i1}^2 + \ldots\right).$$

- Leading order problem for stiff components

$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad \left.\frac{du_{i0}}{dX_i}\right|_{X_i = b_i} = \left.\frac{du_{i0}}{dX_i}\right|_{X_i = b_i + 1} = 0.$$

(almost rigid body motion) $$u_{i0} = C_i = \text{const}$$

Multi-component high-contrast elastic rods



- Leading order problem for soft components

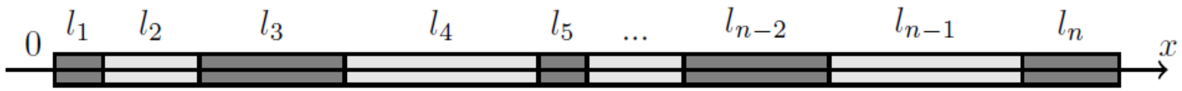
$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad u_i\big|_{X_i=b_i} = C_{i-1}, \quad u_i\big|_{X_i=b_i+1} = C_{i+1}.$$

$$\Downarrow$$

$$u_{i0} = C_{i-1} + \left(C_{i+1} - C_{i-1}\right)\left(X_i - b_i\right).$$

(almost homogeneous deformation)

Multi-component high-contrast elastic rods



From solvability of next order for stiff components

$$\begin{cases} \Omega_{10}^2 = L_2^1\left(1-\dfrac{C_3}{C_1}\right), \\ \vdots \\ \Omega_{i0}^2 = \left(L_{i-1}^i - L_{i+1}^i\right) - L_{i-1}^i\dfrac{C_{i-2}}{C_i} - L_{i+1}^i\dfrac{C_{i+2}}{C_i}, \\ \vdots \\ \Omega_{n0}^2 = L_{n-1}^n\left(1-\dfrac{C_{n-2}}{C_n}\right); \qquad i = 1,3,5,\ldots,n. \end{cases}$$

$\Rightarrow$ Polynomial equation for frequency!

- Leading order eigenform – piecewise linear (constant for stiff parts)
- Next order – correction for frequencies, polynomial correction to the eigenform

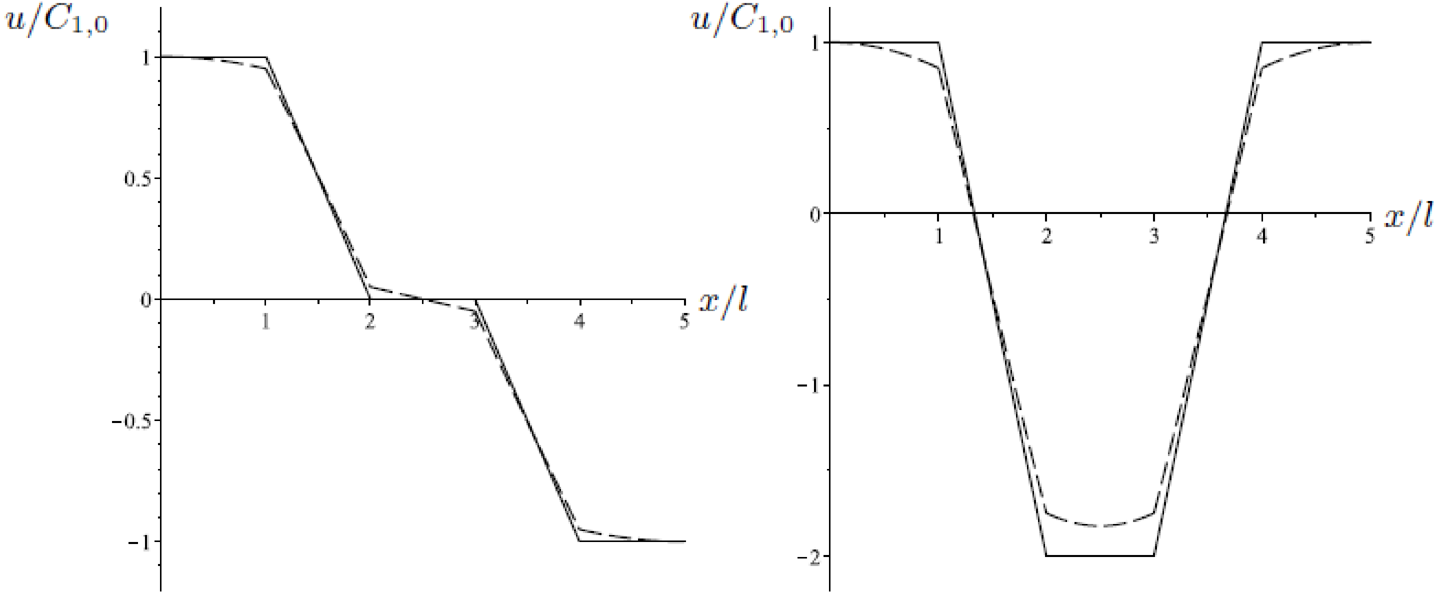
Example: Five-component rod *(free ends)*

Bicubic frequency equation

$$\Omega_{10}^6 + a_1\Omega_{10}^4 + a_2\Omega_{10}^2 = 0 \quad \Rightarrow \quad \Omega_{10}^2 = kL_2^1, \quad k = 0 \quad \text{or} \quad k = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2L_2^1}.$$

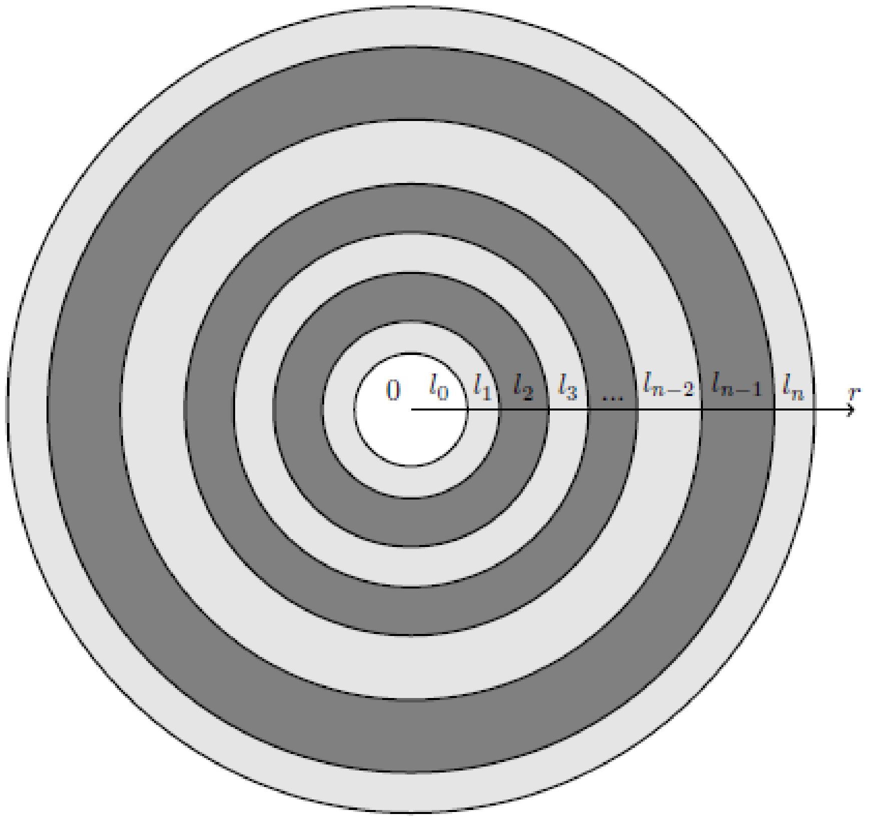
Approximate polynomial eigenform

- For $k = 0$ the solution is exact – rigid body motion!



(a) $k = 1$ (b) $k = 3$

3. Antiplane motion of concentric circular cylinders



Problem parameters

$$\varepsilon = \frac{\mu_1}{\mu_2} << 1, \quad \rho_1 \sim \rho_2, \quad L_i^j = \frac{l_j}{l_i}, \quad c_m^2 = \frac{\mu_m}{\rho_m}, \quad m = 1,2.$$

$$R_i = \frac{r_i}{l_i}, \quad \Omega_i = \frac{\omega l_i}{c_m}, \quad b_i \le R_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1, n}.$$

Equations of motion

$$R_i \frac{d^2 u_i}{dR_i^2} + \frac{du_i}{dR_i} + R_i \Omega_i^2 u_i = 0.$$

Boundary conditions

$$u_1\big|_{R_1 = b_1} = u_n\big|_{R_n = b_n + 1} = 0.$$

Continuity

$$u_i\big|_{R_i = b_i + 1} = u_{i+1}\big|_{R_{i+1} = b_{i+1}}, \qquad \left.\frac{du_i}{dR_i}\right|_{R_i = b_i + 1} = \varepsilon^j L_{i+1}^i \left.\frac{du_{i+1}}{dR_{i+1}}\right|_{R_{i+1} = b_{i+1}}.$$

($j = 1$ or -1 for i^{th} component being stiff or soft, respectively)

Antiplane motion of concentric circular cylinders

Summary of the approach

- Global low-frequency perturbation

↓

- Rigid body motions of stiffer components (at leading order)

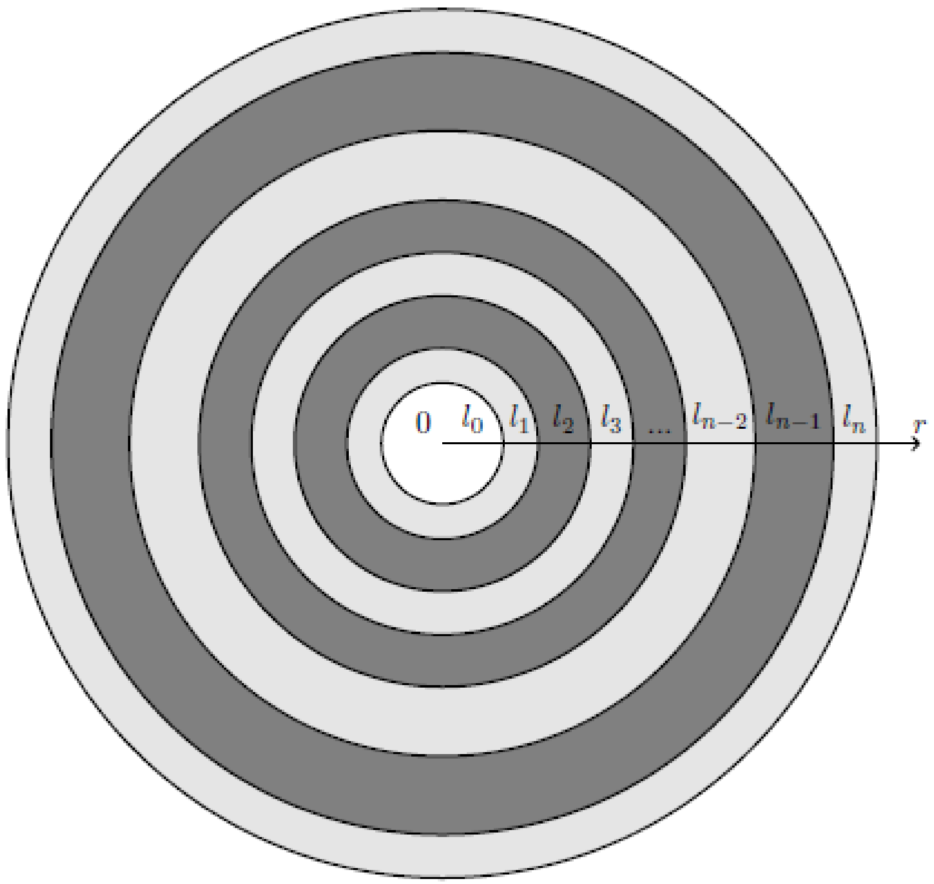
↓

- Leading order solution for softer components, involving logarithmic functions

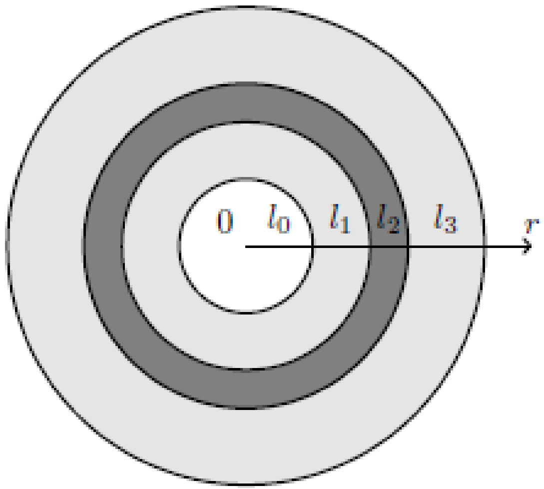
↓

- Solvability of the next order problem for stiffer components

→ Polynomial equation for frequency



Example: Three-layered cylinder



Frequency

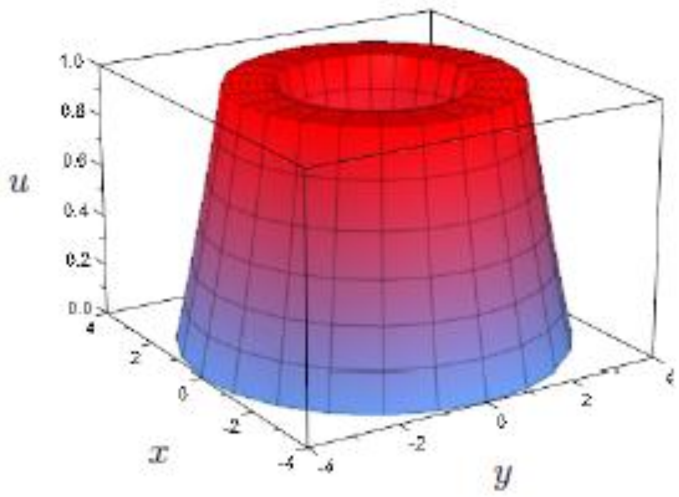
$$\Omega_{20}^2 = \frac{2(\delta_1 + \delta_3)}{2b_2 + 1}, \qquad \delta_i = \frac{1}{\ln\left(1 + b_i^{-1}\right)}.$$

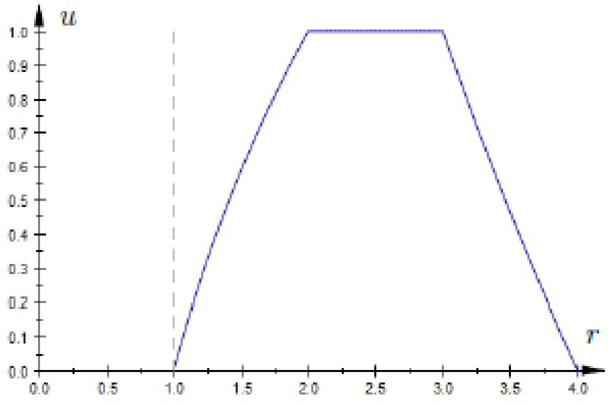
Eigenform

$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

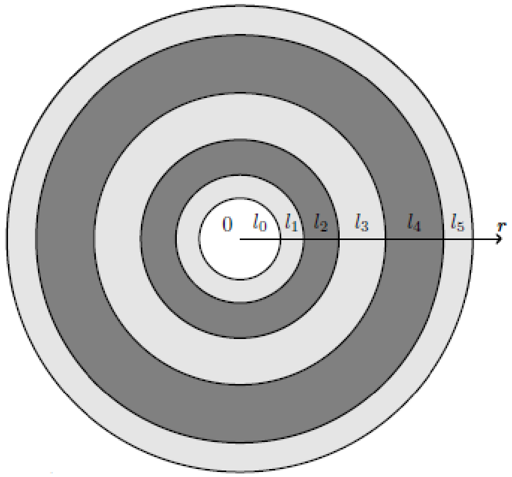
$$u_{20} = 1,$$

$$u_{30} = \delta_3 \ln\left(\frac{b_3 + 1}{R_3}\right).$$

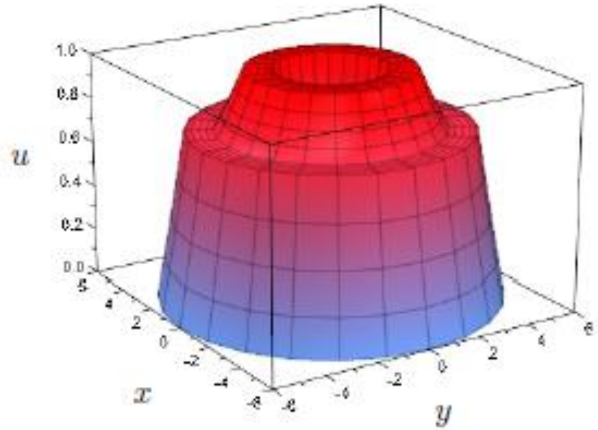


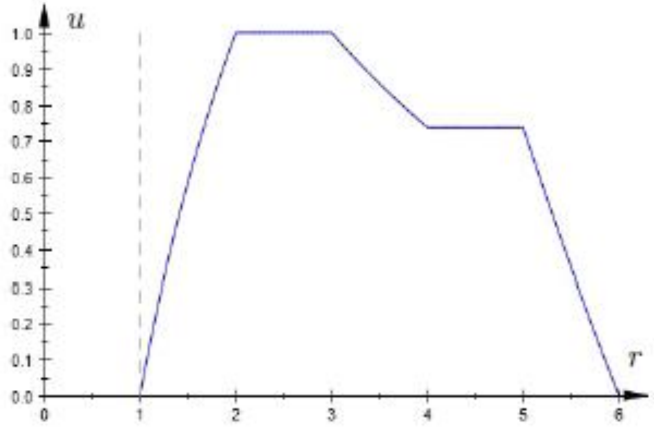


Example: Five-layered cylinder



Frequency $\Omega_{20}^2 = \dfrac{2}{2b_2+1}\left(\delta_1 + (1-k)\delta_3\right)$ (two roots for k)





Eigenform

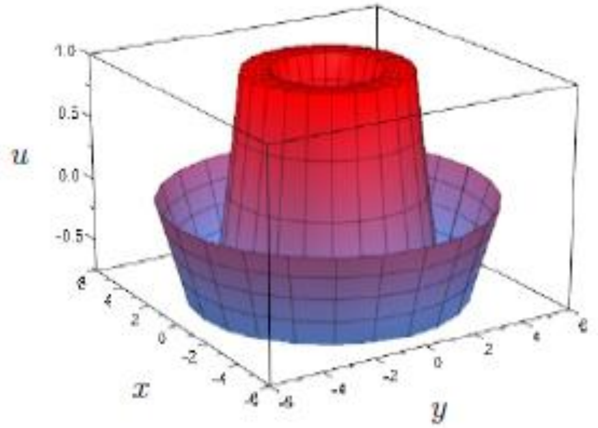
$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

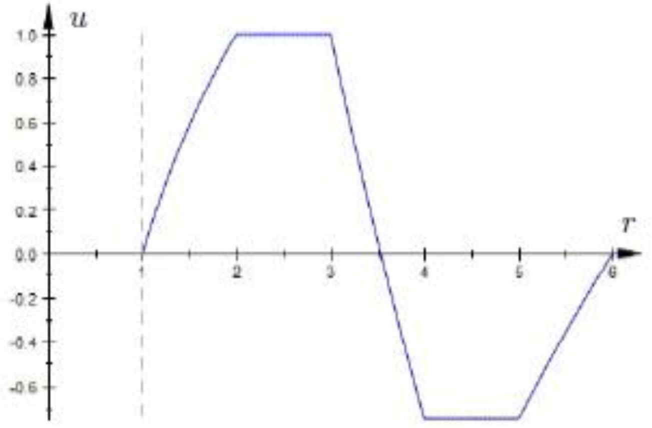
$$u_{20} = 1,$$

$$u_{30} = -\delta_3\left\{\ln\left(\frac{R_3}{b_3+1}\right) - k\ln\left(\frac{R_3}{b_3}\right)\right\},$$

$$u_{40} = k,$$

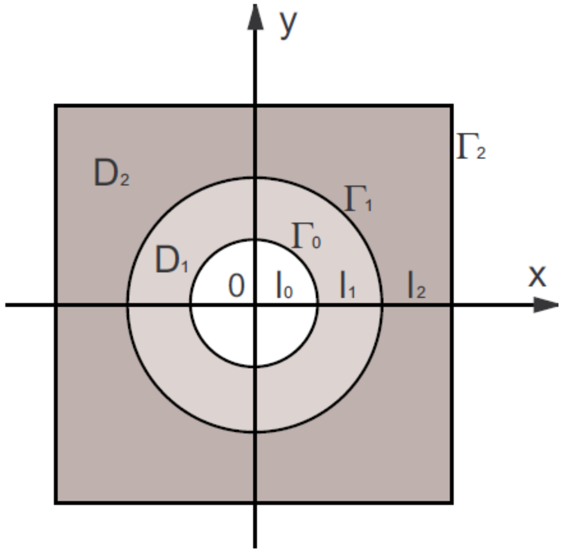
$$u_{50} = k\delta_5 \ln\left(\frac{b_5+1}{R_5}\right).$$





Example: square cylinder with a circular annular inclusion

J. Kaplunov et al. *Adv. Struct Mat.* 46 (2017): 265-277



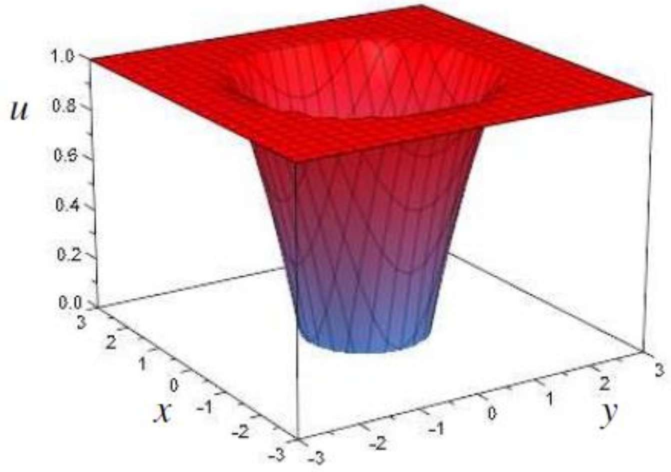
Frequency

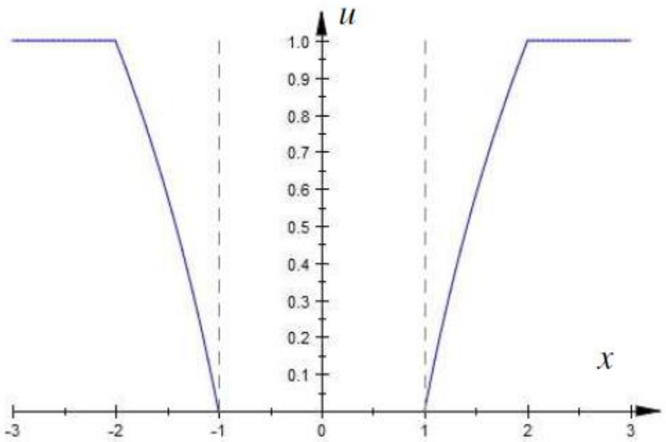
$$\Omega_{20}^2 = \frac{2\pi\delta_1 L_2^1}{4(b_2+1)^2 - \pi b_2^2}$$

Eigenform

$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

$$u_{20} = 1$$





4. High-contrast three-layered plates (antisymmetric)

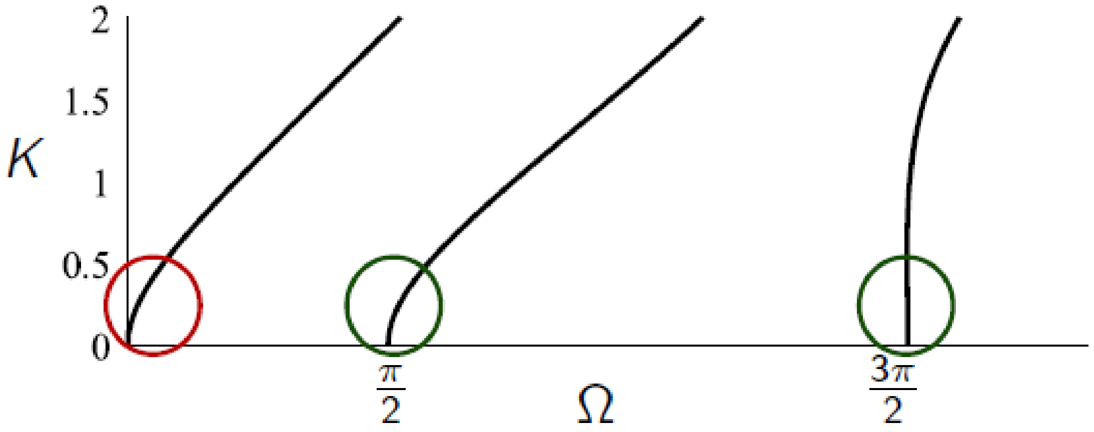
Preliminary remarks

- Rayleigh-Lamb dispersion relation for a single-layered plate

$$\gamma^4 \frac{\sinh \alpha}{\alpha} \cosh \beta - \beta^2 K^2 \cosh \alpha \frac{\sinh \beta}{\beta} = 0,$$

$$\alpha^2 = K^2 - \varkappa^2 \Omega^2, \quad \beta^2 = K^2 - \Omega^2, \quad \gamma^2 = K^2 - \frac{1}{2}\Omega^2, \quad \varkappa = \frac{c_2}{c_1}$$

$$K = kh, \quad \Omega = \frac{\omega h}{c_2}$$



NO CHANCE OF TWO-MODE APPROXIMATIONS!

○ Low-frequency ($\Omega \ll 1$)

At the leading order $\Omega^2 \sim K^4$

○ High-frequency approximations near cut-off frequencies $\Omega_* \sim 1$ ($|\Omega - \Omega_*| \ll 1$)

At the leading order $K^2 \sim \Omega^2 - \Omega_*^2$

J. Kaplunov et al. *Dynamics of thin walled elastic bodies, Academic Press, 1998*

Composite (non-uniformly asymptotic) plate theories

Originate from Timoshenko-Reissner-Mindlin ad hoc theories.

$$\overbrace{D_a \frac{d^4 W}{d\xi^4} - \Omega^2 W}^{\text{low-frequency}} \underbrace{\vphantom{\frac{d^4W}{d\xi^4}} + B_a \Omega^2 \frac{d^2 W}{d\xi^2} + C_a \Omega^4 W}_{\text{high-frequency}} = 0,$$

Contributions for composite plate and shells theories

V.L. Berdichevsky. Variational principles of continuum mechanics: I. Fundamentals. Springer Science and Business Media, 2009

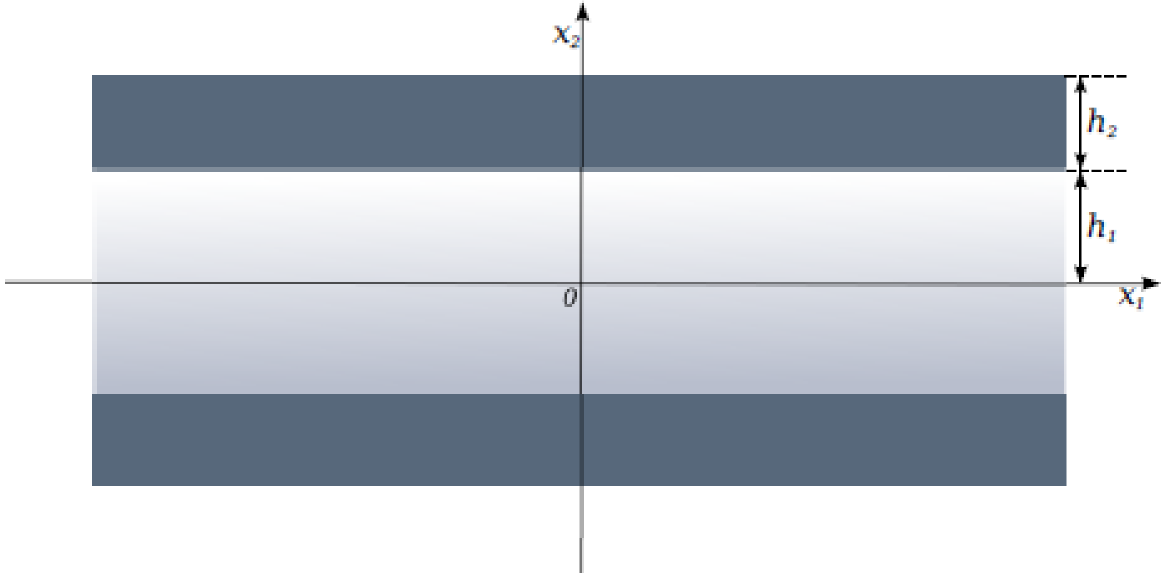
K.C. Le. Vibrations of shells and rods. Springer Science and Business Media, 2012

I.V. Andrianov, J. Awrejcewicz, L.I. Manevitch. Asymptotical mechanics of thin-walled structures. Springer Science and Business Media, 2013

Low-frequency vibrations of high-contrast three-layered plates

Kaplunov et al. *Int. J. Solids Struct.* 113 (2017): 169-179

Statement of the problem



Equations of motion

$$\sigma_{ji,j} = \rho \ddot{u}_i, \quad i = 1, 2 \quad \text{for layers I and II}$$

Boundary and continuity conditions

$$\sigma_{12}^{\mathrm{II}} = 0, \quad \sigma_{22}^{\mathrm{II}} = 0 \quad \text{at} \quad x_2 = h_1 + h_2$$

$$\sigma_{12}^{\mathrm{I}} = \sigma_{12}^{\mathrm{II}}, \quad \sigma_{22}^{\mathrm{I}} = \sigma_{22}^{\mathrm{II}} \quad \text{and} \quad u_1^{\mathrm{I}} = u_1^{\mathrm{II}}, \quad u_2^{\mathrm{I}} = u_2^{\mathrm{II}} \quad \text{at} \quad x_2 = h_1$$

Dispersion relation

Ustinov, Doklady Physics (1976); Lee, Chang, Journal of Elasticity (1979)

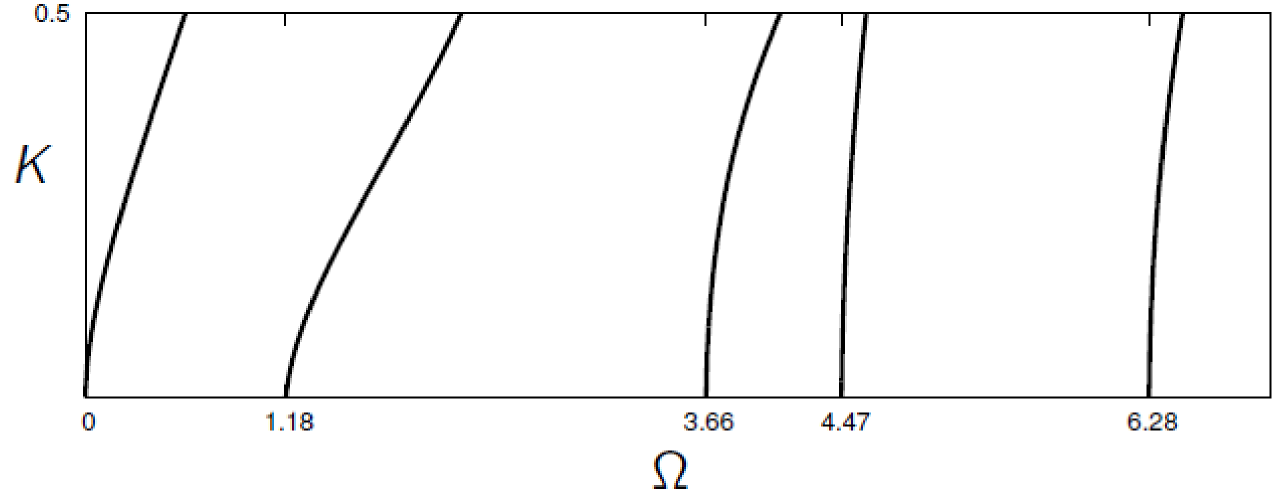
$$
\begin{aligned}
&4K^2h^3\alpha_2\beta_2F_4\left[F_1F_2C_{\beta_1}S_{\alpha_1}-2\alpha_1\beta_1(\varepsilon-1)F_3C_{\alpha_1}S_{\beta_1}\right]+\\
&h\alpha_2\beta_2C_{\alpha_2}C_{\beta_2}\left[4\alpha_1\beta_1K^2\left(h^4F_3^2+F_4{}^2(\varepsilon-1)^2\right)C_{\alpha_1}S_{\beta_1}-\right.\\
&\qquad\qquad\qquad\qquad\qquad\left.\left(4K^4h^4F_2^2+F_4{}^2F_1^2\right)S_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\beta_2}S_{\alpha_2}\varepsilon\beta_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_2^2\beta_1K^2h^2S_{\alpha_1}S_{\beta_1}-F_4{}^2\alpha_1C_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\alpha_2}S_{\beta_2}\varepsilon\alpha_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_1\beta_2^2K^2h^2C_{\alpha_1}C_{\beta_1}-F_4{}^2\beta_1S_{\alpha_1}S_{\beta_1}\right]+\\
&h^3S_{\alpha_2}S_{\beta_2}\left[\left(4\alpha_2^2\beta_2^2K^2F_1^2+K^2F_4{}^2F_2^2\right)C_{\beta_1}S_{\alpha_1}-\right.\\
&\qquad\qquad\left.\alpha_1\beta_1\left(16\alpha_2^2\beta_2^2(\varepsilon-1)^2K^4+F_4{}^2F_3^2\right)C_{\alpha_1}S_{\beta_1}\right]=0
\end{aligned}
$$

$\Omega = \dfrac{\omega h_1}{c_2^{\mathrm{I}}}$, $K = kh_1$, $C_{\alpha_j}, C_{\beta_j}, S_{\alpha_j}, S_{\beta_j}$ - hyperbolic functions

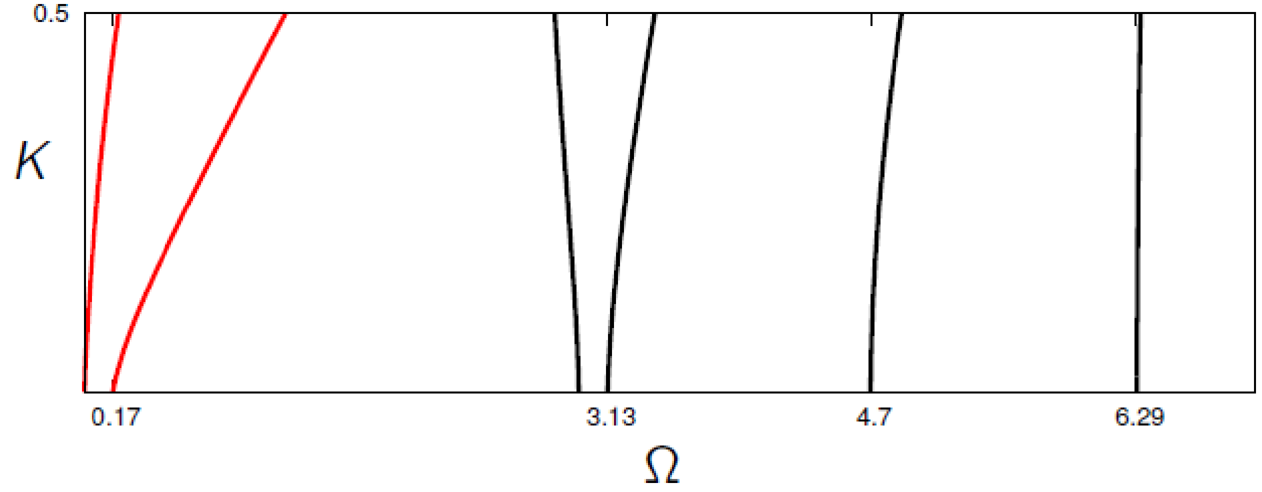
F_i, $i = 1..4$, α_j, β_j, $j = 1, 2$ - functions of Ω and K, $\varepsilon = \dfrac{\mu_1}{\mu_2}$

Dispersion curves

No contrast



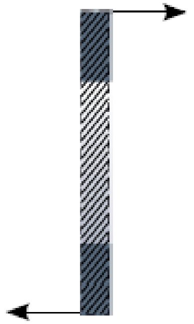
Effect of contrast

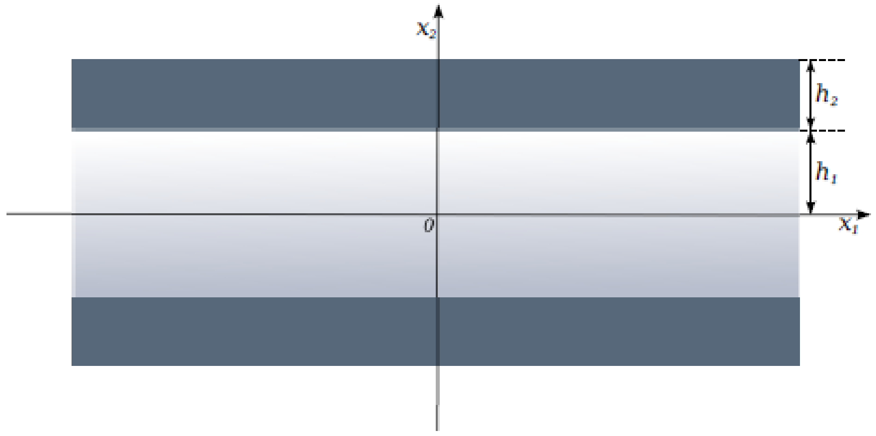


→ **NEED OF TWO-MODE MODELS!**

1D eigenvalue problem for shear cut-off

Flexural motion $\dfrac{\partial}{\partial x_1} = 0, \qquad u_2 = 0$





Frequency equation

$$\tan(\Omega)\tan\left(\sqrt{\frac{\varepsilon}{r}}h\,\Omega\right) = \sqrt{\varepsilon r}$$

Condition for a first shear cut-off frequency to be small

$$\boxed{r \ll h \ll \varepsilon^{-1}}$$

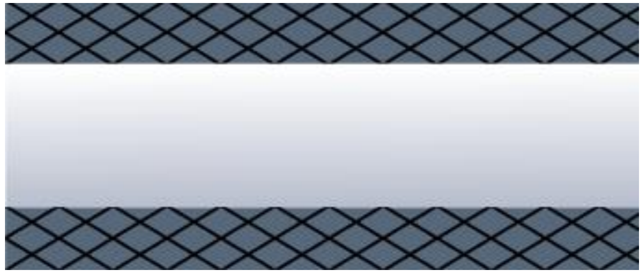
Frequency $\Omega^2 \sim \dfrac{r}{h}$

where $r = \dfrac{\rho_1}{\rho_2}, \quad h = \dfrac{h_2}{h_1}, \quad \varepsilon = \dfrac{\mu_1}{\mu_2}$

Some three-layered structures satisfying the condition $r \ll h \ll \varepsilon^{-1}$

A) Photovoltaic panels

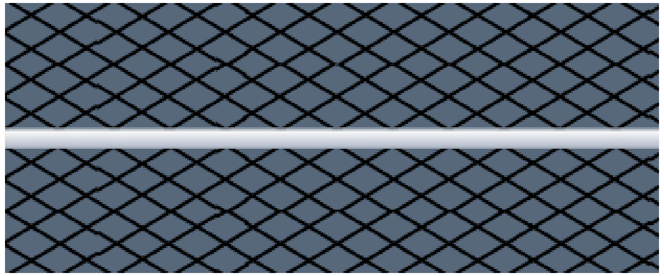
$$\varepsilon << 1,\ h \sim 1,\ \rho \sim \varepsilon$$



(stiff skin layers and light core layer)

B) Laminated glass

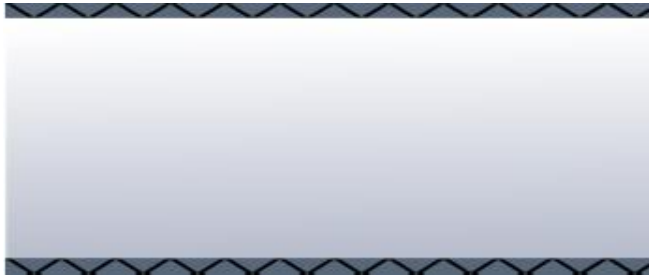
$$\varepsilon << 1,\ h \sim \varepsilon^{-1/4},\ \rho \sim 1$$



(stiff skin layers and light thin core layer)

C) Sandwich structure

$$\varepsilon << 1,\ h \sim \varepsilon,\ \rho \sim \varepsilon^{2}$$



(stiff thin skin layers and light core layer)

UNEXPECTEDLY LOW FIRST SHEAR CUT-OFF FREQUENCIES!

Long-wave low-frequency asymptotic approximation of the dispersion relation

For $K \ll 1$ and $\Omega \ll 1$

$$\gamma_1\Omega^2 + \gamma_2 K^4 + \gamma_3 K^2\Omega^2 + \gamma_4 K^6 + \gamma_5\Omega^4 + \gamma_6 K^4\Omega^2 + \gamma_7 K^8 + \gamma_8 K^2\Omega^4 + \gamma_9 K^2\Omega^6 + \gamma_{10}\Omega^6 + \ldots = 0$$

Multi-parametric analysis

$$\varepsilon \ll 1, \quad h \sim \varepsilon^a, \quad r \sim \varepsilon^b$$

Expanding coefficients

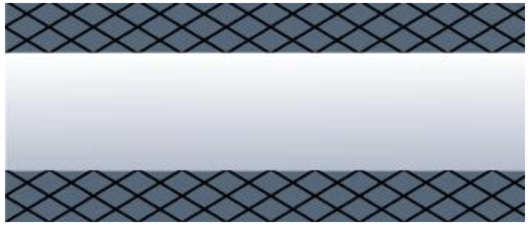
$$\gamma_i \to G_i \varepsilon^c$$

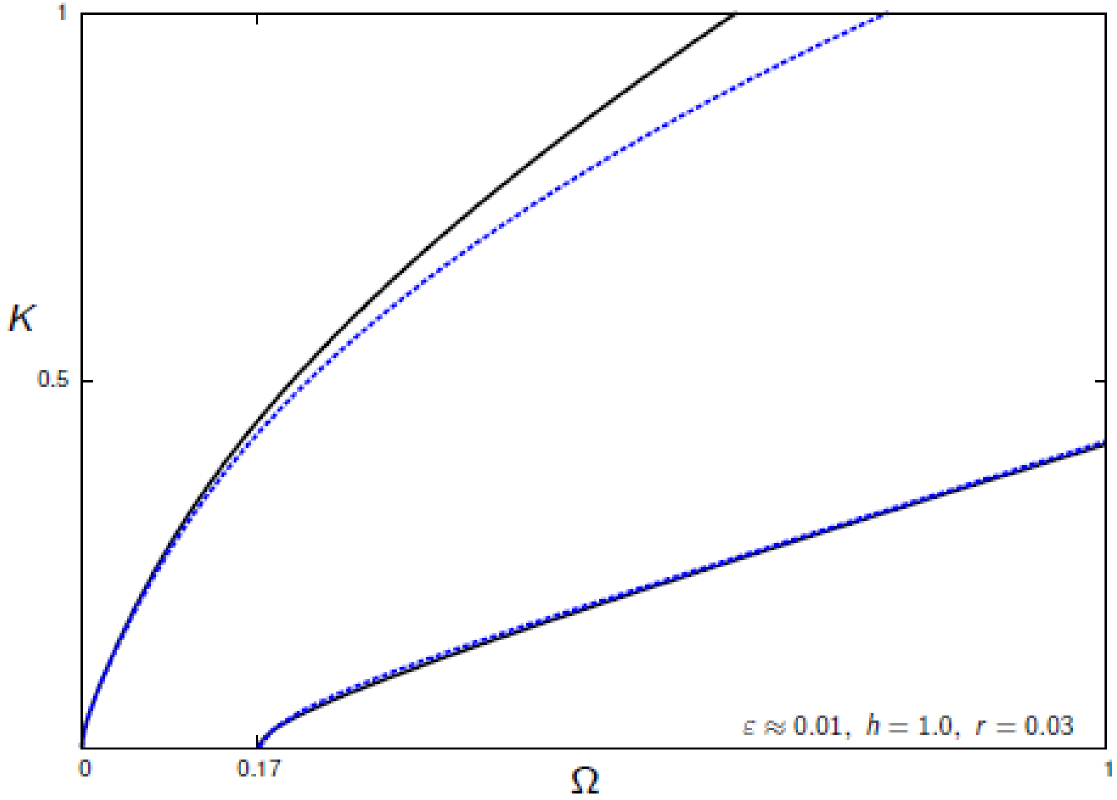
Low-frequency dispersion behaviour

A) Photovoltaic panels

(stiff skin layers and light core layer)

$$\varepsilon \ll 1, \quad h \sim 1, \quad r \sim \varepsilon$$





Retain leading order terms for both:

(i) fundamental mode ($\Omega \sim K^2$)

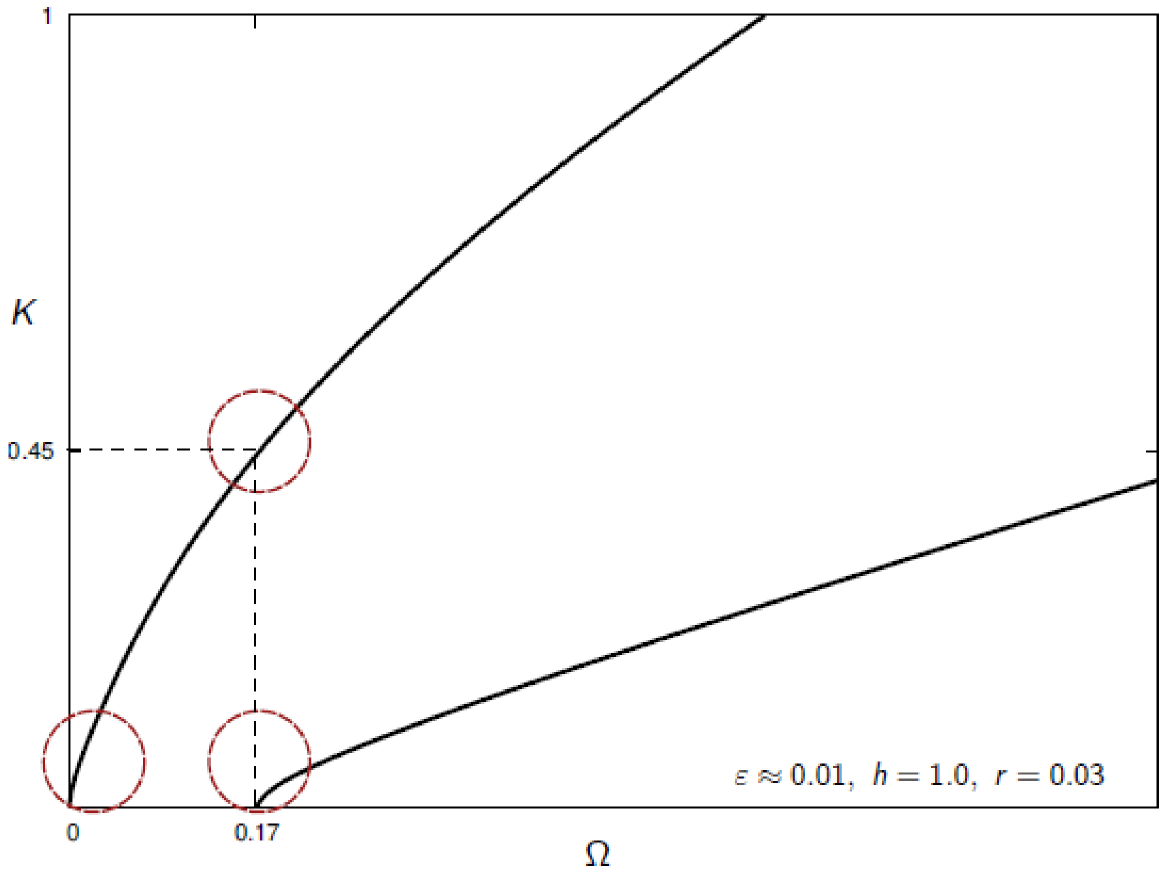
(ii) shear mode cut-off ($\Omega_{sh} \sim \sqrt{\varepsilon}$)

UNIFORM TWO-MODE APPROXIMATIONS

$$G_1 \varepsilon \Omega^2 + G_2 \varepsilon K^4 + G_3 K^2 \Omega^2 + G_4 K^6 + G_5 \Omega^4 = 0$$

Local approximations

Three local approximations

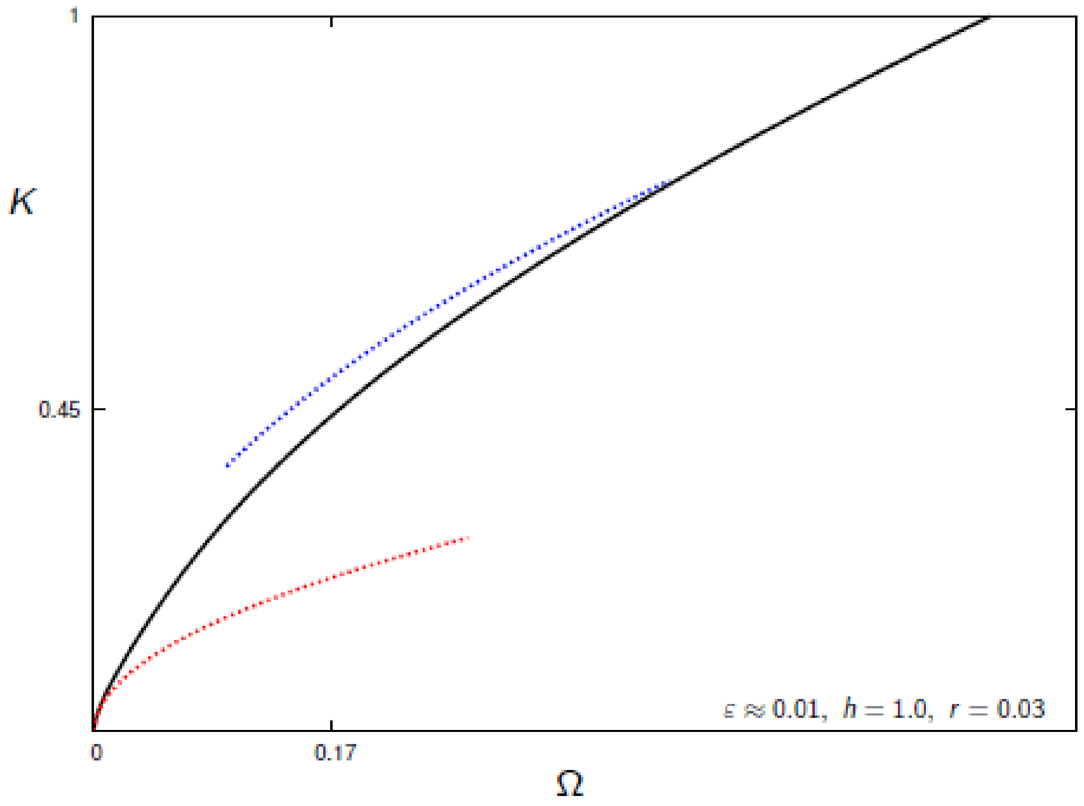


In the vicinity of zero frequency

$$G_1\Omega^2 + G_2K^4 = 0, \quad 0 < K \ll \sqrt{\varepsilon}, \quad \Omega \ll \varepsilon$$

At higher frequencies, including the vicinity of shear cut-off

$$G_3\Omega^2 + G_4K^4 = 0, \quad \sqrt{\varepsilon} \ll K \ll 1, \quad \varepsilon \ll \Omega \ll 1$$

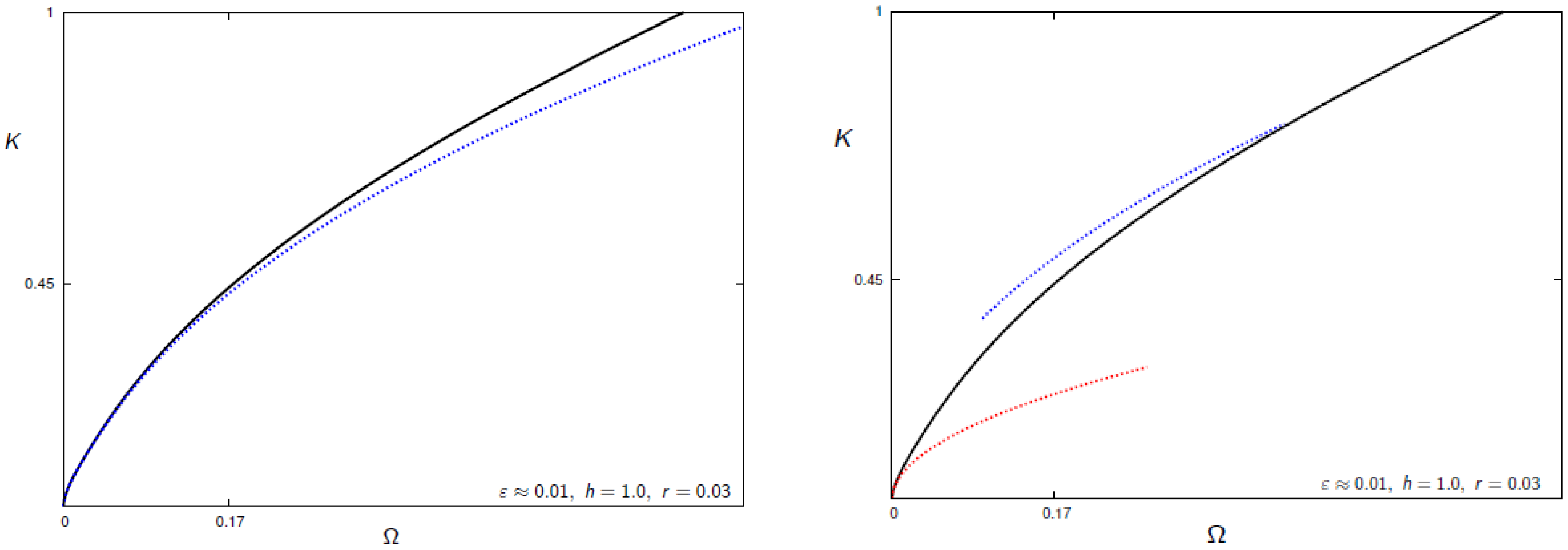


CLASSICAL KIRCHHOFF-TYPE THEORY IS NOT APPLICABLE!

Uniform approximation for the fundamental mode

Taking both local approximations we derive a uniform one:

$$G_1 \varepsilon \Omega^2 + G_2 \varepsilon K^4 + G_3 K^2 \Omega^2 + G_4 K^6 = 0$$

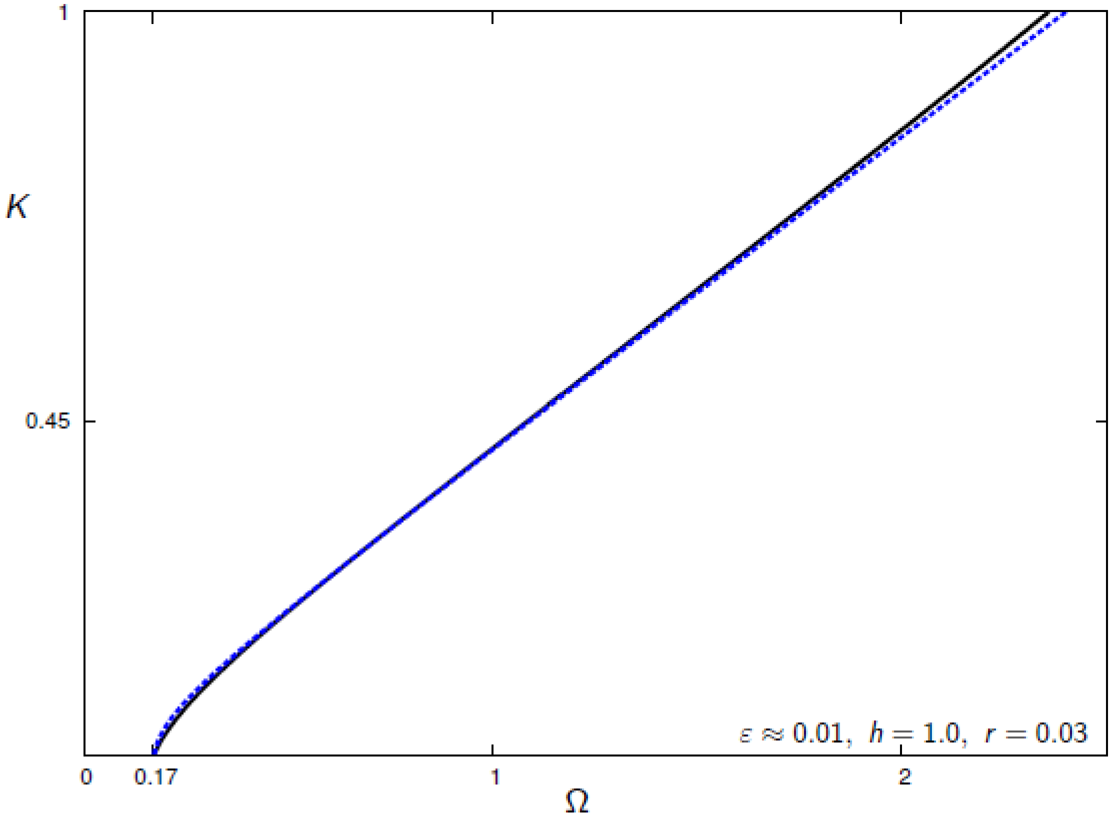


Also valid in the transition region $\Omega \sim \varepsilon, K \sim \sqrt{\varepsilon}$

Near shear cut-off approximation

For $\Omega \sim \sqrt{\varepsilon}$, $K \ll 1$

$$G_1 \varepsilon + G_3 K^2 + G_5 \Omega^2 = 0$$



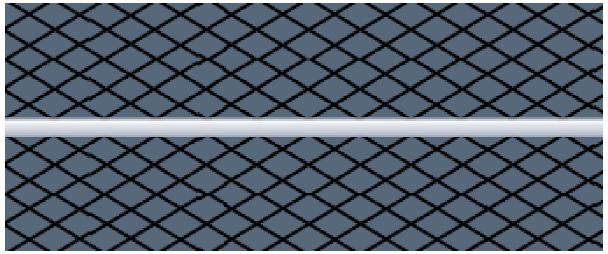
Low-frequency dispersion behaviour

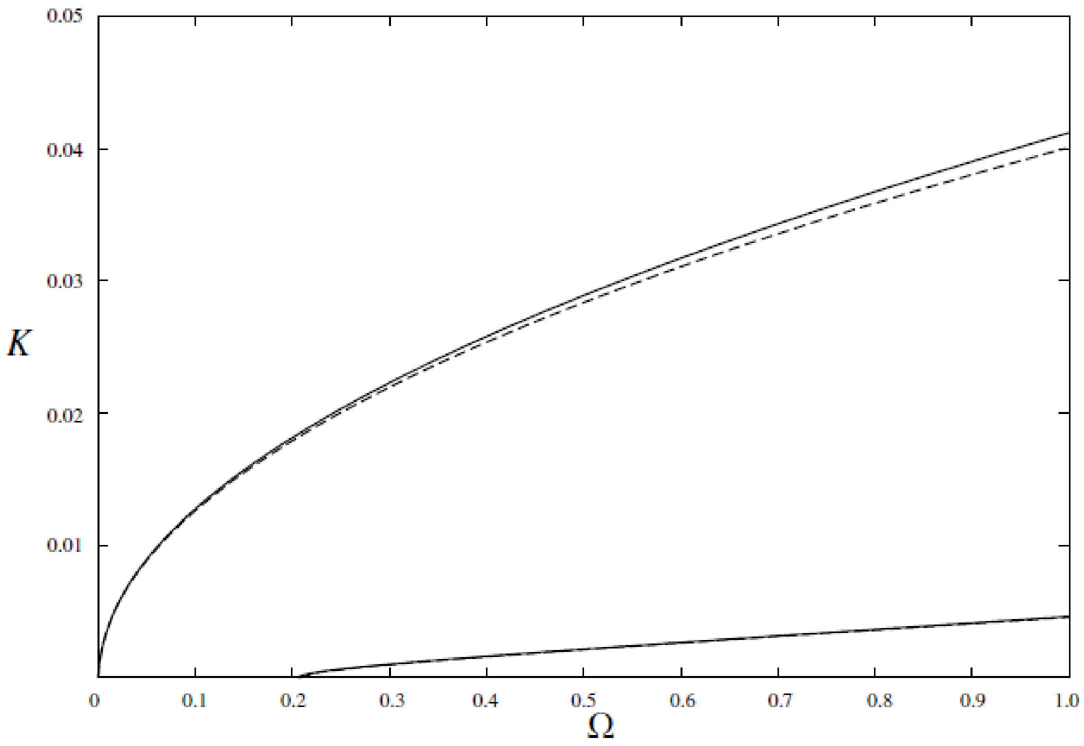
J. Kaplunov et al. *Proc. Eng.* 199 (2017): 1489-1494

B) Laminated glass

(stiff skin layers and light light core layer)

$\varepsilon << 1,\ h \sim \varepsilon^{-1/4},\ \rho \sim 1$





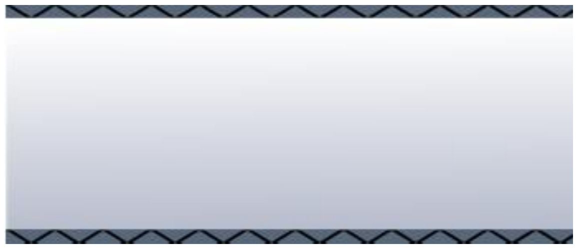
UNIFORM TWO-MODE APPROXIMATIONS

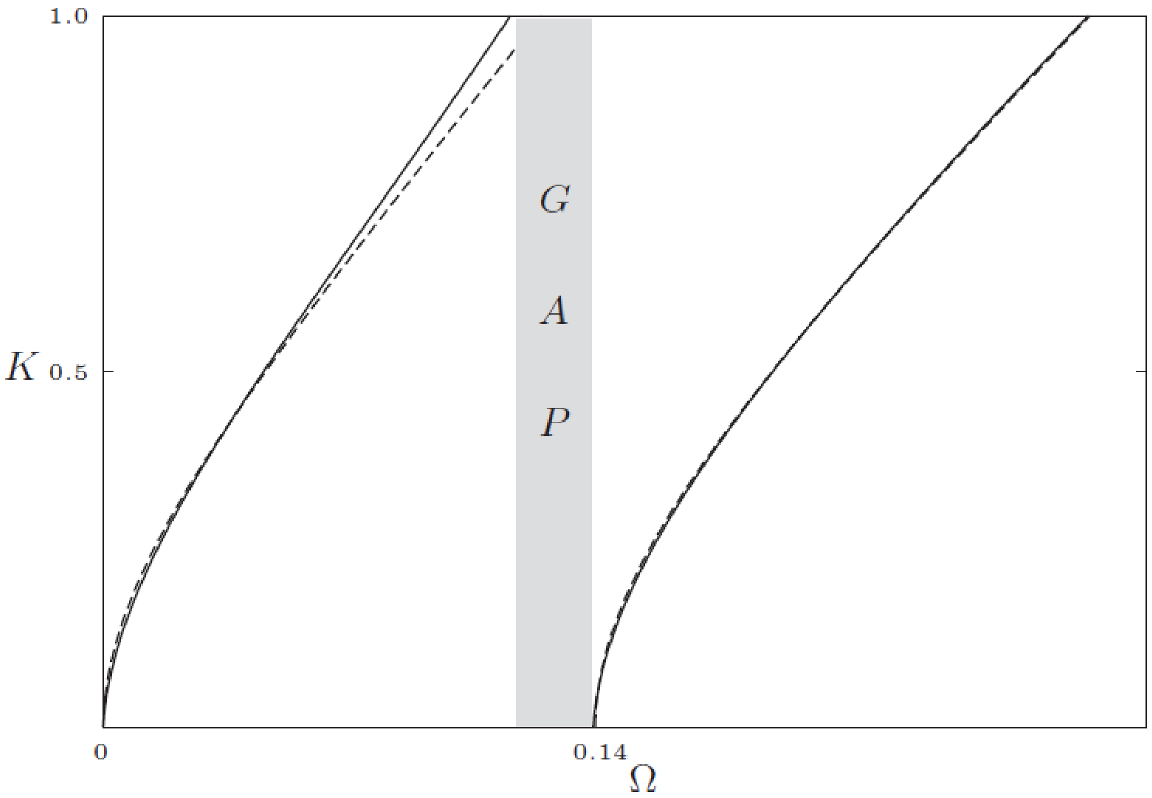
$$\varepsilon^{11/4} G_1 \Omega^2 + \varepsilon^{5/4} G_2 K^4 + \varepsilon^{3/2} G_3 K^2 \Omega^2 + G_4 K^6 + \varepsilon^{5/2} G_5 \Omega^4 = 0$$

Low-frequency dispersion behaviour

C) Sandwich structure $\varepsilon \ll 1, \quad h \sim \varepsilon, \quad r \sim \varepsilon^2$

(stiff thin skin layers and light core layer)

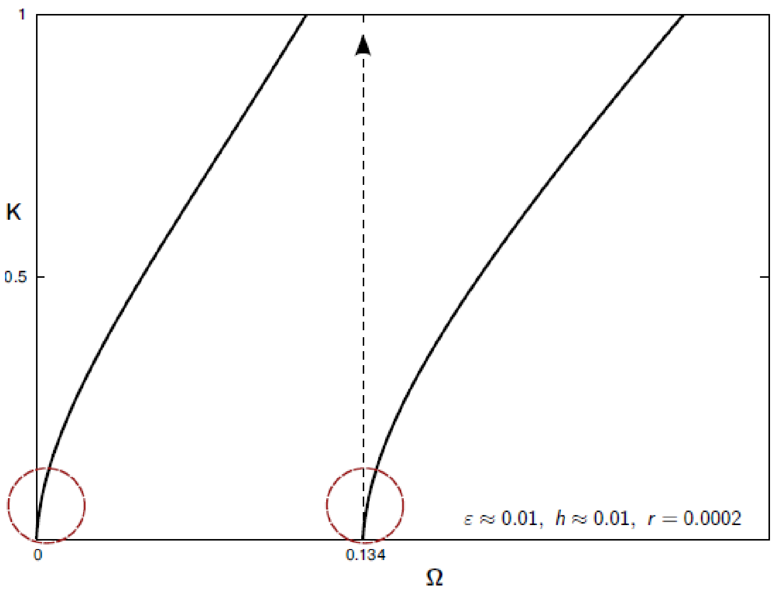


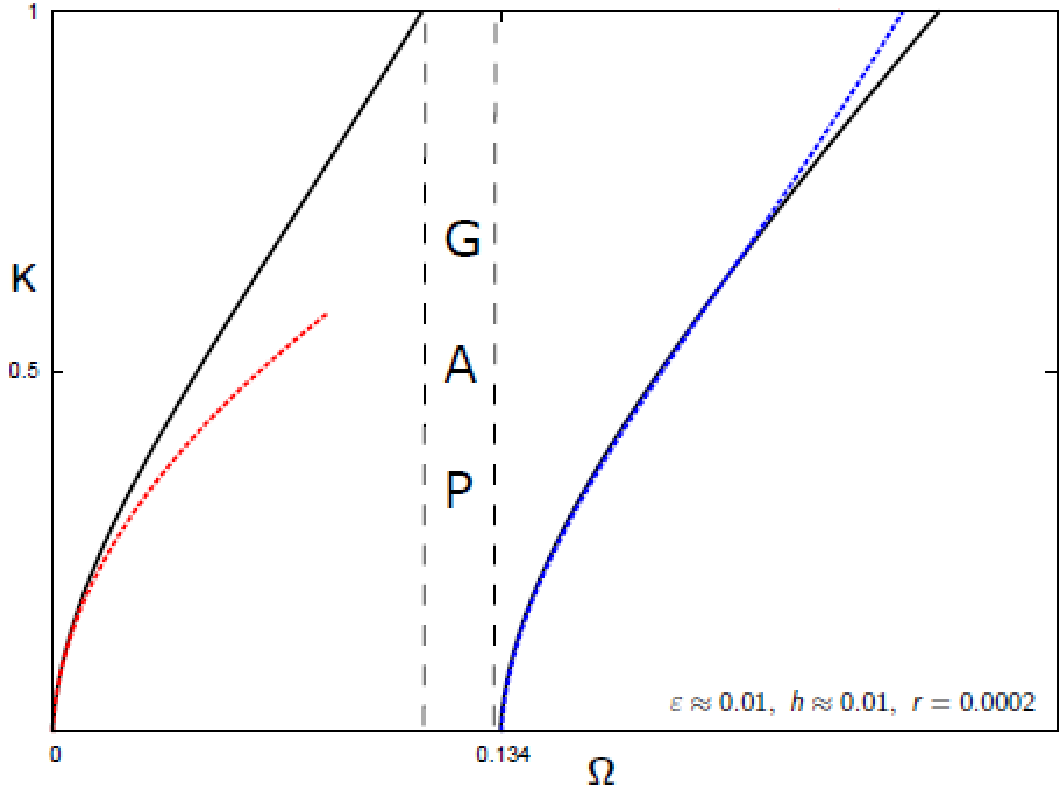


COMPOSITE APPROXIMATIONS

$$G_1 \varepsilon \Omega^2 + G_2 \varepsilon^2 K^4 + \varepsilon K^2 \Omega^2 \left(G_3 + \frac{r_0}{h_0} G_8 \right) + G_5 \Omega^4 = 0$$

Local approximations





NO OVERLAP REGION!

Fundamental mode

$$G_1\Omega^2 + G_2\varepsilon K^4 = 0,$$

$$K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}K^2 \ll \sqrt{\varepsilon}$$

Shear mode

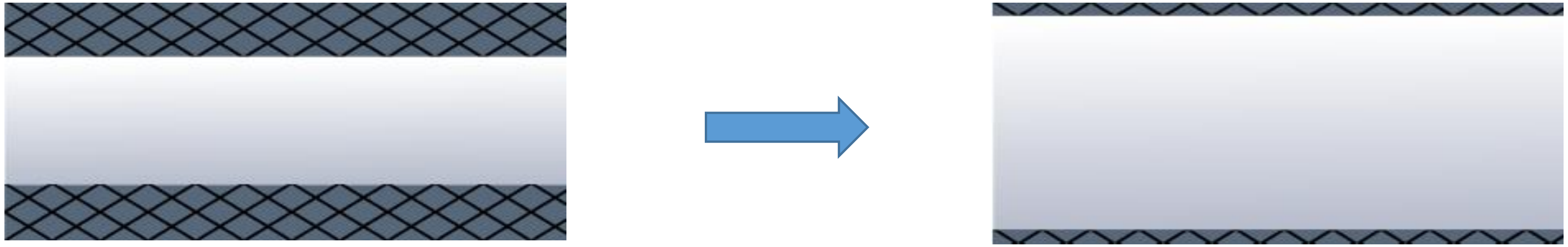
$$G_1\varepsilon + G_3\varepsilon K^2 + G_5\Omega^2 + G_8K^2\Omega^2 = 0,$$

$$\varepsilon^{\frac{1}{2}} \ll K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}$$

Lightweight structures

J. Kaplunov et al., to appear

Where is transition from uniform approximation to a composite one?



$$\rho = \frac{\rho_c}{\rho_s} \ll 1, \quad h = \frac{h_s}{h_c} \sim \rho^a, \quad 0 \le a < 1 \qquad \mu = \frac{\mu_c}{\mu_s} \sim \rho.$$

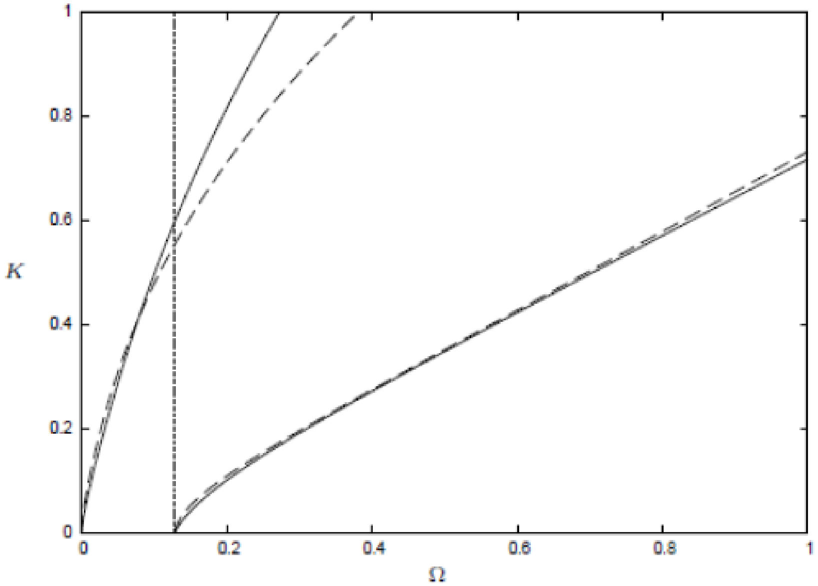
Low thickness shear cut-off frequency

$$\Omega_{sh} \approx \left(\frac{\rho}{h}\right)^{1/2} \sim \rho^{(1-a)/2} \ll 1$$

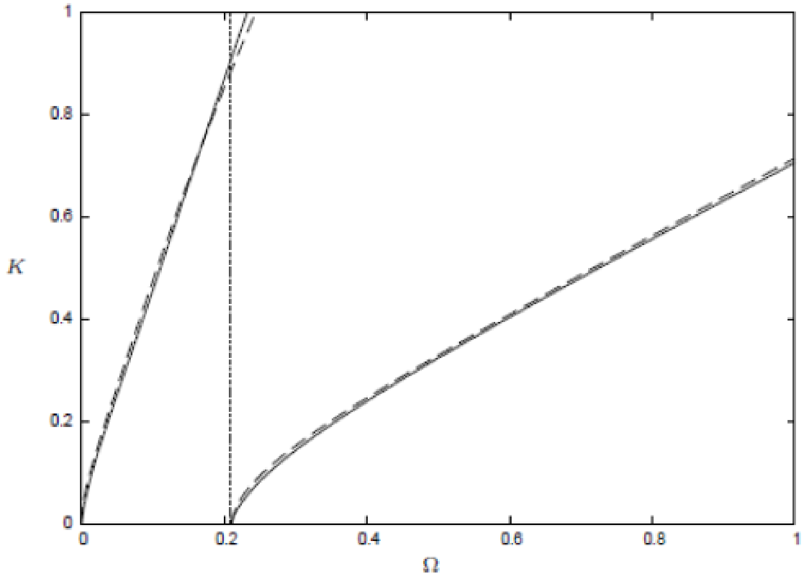
Uniform approximation $\left(0 \le a < \frac{1}{3}\right)$

Limiting case

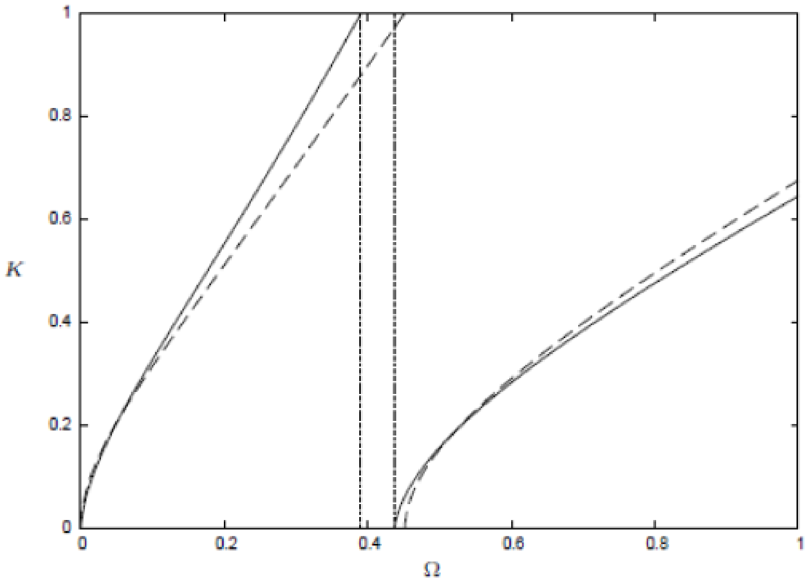
Composite approximation $\left(\frac{1}{3} < a < 1\right)$



$\left(a = \frac{1}{9}\right)$



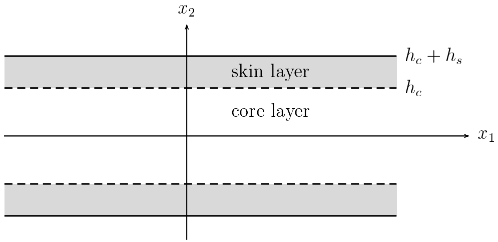
$\left(a = \frac{1}{3}\right)$



$\left(a = \frac{2}{3}\right)$

- **Two-mode uniform** $$\rho^{1-a} G_1 \Omega^2 + \rho^{1-a} G_2 K^4 + G_3 K^2 \Omega^2 + \frac{1}{3} \rho^{2a} G_2 K^6 + G_5 \Omega^4 = 0$$
- **Composite** $$\rho^{1-a} G_1 \Omega^2 + \rho^{1-a} G_2 K^4 + G_3 K^2 \Omega^2 + G_5 \Omega^4 = 0$$

Anti-plane antisymmetric motion



Equations of motion

$$\frac{\partial \sigma_{13}^{\mathrm{q}}}{\partial \mathrm{x}_1} + \frac{\partial \sigma_{23}^{\mathrm{q}}}{\partial \mathrm{x}_2} - \rho_{\mathrm{q}} \frac{\partial^2 \mathrm{u}_{\mathrm{q}}}{\partial \mathrm{t}^2} = 0, \qquad \mathrm{q} = \mathrm{c}, \mathrm{s},$$

with

$$\sigma_{\mathrm{i}3}^{\mathrm{q}} = \mu_{\mathrm{q}} \frac{\partial \mathrm{u}_{\mathrm{q}}}{\partial \mathrm{x}_{\mathrm{i}}}, \quad \mathrm{i} = 1, 2,$$

u_{q} are out of plane displacements, $\sigma_{\mathrm{i}3}^{\mathrm{q}}$ are shear stresses.

Dispersion relation

Continuity conditions along interfaces $x_2 = \pm h_c$

$$\sigma_{23}^{c} = \sigma_{23}^{s} \quad \text{and} \quad u_c = u_s.$$

Traction-free boundary conditions

$$\sigma_{23}^{s} = 0 \qquad \text{at} \qquad x_2 = \pm(h_c + h_s).$$

Equations of motion

$$\Delta u_q - \frac{1}{(c_2^q)^2}\frac{\partial^2 u_q}{\partial t^2} = 0, \quad q = c, s.$$

Dispersion relation

$$\mu\alpha_1 \cosh(\alpha_1)\cosh(\alpha_2 h) + \alpha_2 \sinh(\alpha_1)\sinh(\alpha_2 h) = 0,$$

with

$$\alpha_1 = \sqrt{K^2 - \Omega^2}, \qquad \alpha_2 = \sqrt{K^2 - \frac{\mu}{\rho}\Omega^2},$$

$$\Omega = \frac{\omega h_c}{c_2^c}, \quad K = k h_c, \quad h = \frac{h_s}{h_c}, \quad \mu = \frac{\mu_c}{\mu_s}, \quad \rho = \frac{\rho_c}{\rho_s}.$$

Exact solutions for displacements and stresses

$$\mathrm{u_c} = \mathrm{h_c}\frac{\mathsf{sinh}(\alpha_1\xi_{2\mathrm{c}})}{\alpha_1}, \quad \sigma_{13}^{\mathrm{c}} = \mathrm{i}\mu_\mathrm{c}\mathrm{K}\frac{\mathsf{sinh}(\alpha_1\xi_{2\mathrm{c}})}{\alpha_1}, \quad \sigma_{23}^{\mathrm{c}} = \mu_\mathrm{c}\,\mathsf{cosh}(\alpha_1\xi_{2\mathrm{c}}),$$

and

$$\mathrm{u_s} = \mathrm{h_c}\beta\left(\mathsf{cosh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right] - \mathsf{tanh}\left[\alpha_2(\mathrm{h}+1)\right]\mathsf{sinh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right]\right),$$
$$\sigma_{13}^{\mathrm{s}} = \mathrm{i}\mu_\mathrm{s}\mathrm{K}\beta\left(\mathsf{cosh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right] - \mathsf{tanh}\left[\alpha_2(\mathrm{h}+1)\right]\mathsf{sinh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right]\right),$$
$$\sigma_{23}^{\mathrm{s}} = \mu_\mathrm{s}\alpha_2\beta\left(\mathsf{sinh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right] - \mathsf{tanh}\left[\alpha_2(\mathrm{h}+1)\right]\mathsf{cosh}\left[\alpha_2(\mathrm{h}\xi_{2\mathrm{s}}+1)\right]\right),$$

where

$$\beta = \frac{\mathsf{sinh}\,\alpha_1}{\alpha_1\left(\mathsf{cosh}\,\alpha_2 - \mathsf{sinh}\,\alpha_2\,\mathsf{tanh}[\alpha_2(\mathrm{h}+1)]\right)}.$$

Dimensionless variables

$$\xi_{2\mathrm{c}} = \frac{\mathrm{x_2}}{\mathrm{h_c}}, \qquad 0 \le \mathrm{x_2} \le \mathrm{h_c},$$
$$\xi_{2\mathrm{s}} = \frac{\mathrm{x_2}-\mathrm{h_c}}{\mathrm{h_s}}, \quad \mathrm{h_c} \le \mathrm{x_2} \le \mathrm{h_c}+\mathrm{h_s}.$$

Long-wave low-frequency limit

Polynomial dispersion relation

$$\mu + \gamma_1 \mathrm{K}^2 + \gamma_2 \mathrm{K}^4 + \gamma_3 \mathrm{K}^2 \Omega^2 + \gamma_4 \Omega^2 + \gamma_5 \Omega^4 + \cdots = 0,$$

with

$$\gamma_1 = \frac{\mu}{2}\left(1 + \mathrm{h}^2\right) + \mathrm{h},$$

$$\gamma_2 = \frac{\mu}{24}\left(1 + 6\mathrm{h}^2 + \mathrm{h}^4\right) + \frac{\mathrm{h}}{6}(1 + \mathrm{h}^2),$$

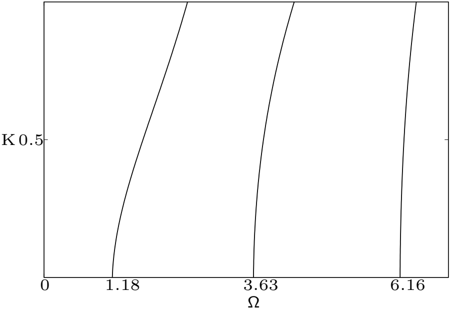
$$\gamma_3 = -\frac{\mu}{12}(1 + 3\mathrm{h}^2) - \frac{\mathrm{h}}{6} - \frac{\mu\mathrm{h}}{12\rho}(2 + 3\mu\mathrm{h}) - \frac{\mu\mathrm{h}^3}{12\rho}(4 + \mu\mathrm{h}),$$

$$\gamma_4 = -\frac{\mu}{2} - \frac{\mu\mathrm{h}}{\rho}\left(1 + \frac{\mu\mathrm{h}}{2}\right),$$

$$\gamma_5 = \frac{\mu}{24} + \frac{\mu\mathrm{h}}{12\rho}(2 + 3\mu\mathrm{h}) + \frac{\mu^2\mathrm{h}^3}{24\rho^2}(4 + \mu\mathrm{h}).$$

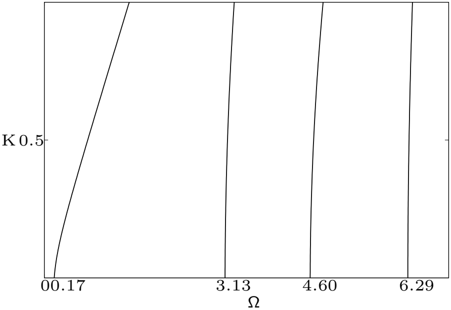
Dispersion curves

No contrast



$\mu = 0.232,\ \rho = 3.0,\ h = 1.0$

Effect of contrast



$\mu = 0.014,\ \rho = 0.03,\ h = 1.0$

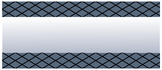
- No fundamental mode. It appears in case of symmetric motion.
- The lowest cut-off frequency in case of a contrast is $\Omega = 0.17$

Consider two setups of the contrast:
A. Photovoltaic panels and B. Sandwich structures

A. Photovoltaic panels. Shortened polynomial dispersion relation

Plate with stiff outer layers and light core

$\mu \ll 1, \quad \mathrm{h} \sim 1, \quad \rho \sim \mu$



$$\gamma_1 \sim \gamma_2 \sim \gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$$

Shortened dispersion relation

$$\frac{\mu}{\mathrm{h}} + \mathrm{K}^2 - \frac{1}{\rho_\mu}\Omega^2 = 0.$$

Scaled dimensionless frequency and wavenumber

$$\Omega^2 = \mu^\alpha \Omega_*^2 \quad \text{and} \quad \mathrm{K}^2 = \mu^\alpha \mathrm{K}_*^2,$$

where $\Omega_* \sim \mathrm{K}_* \sim 1$ and $0 < \alpha \leq 1$.

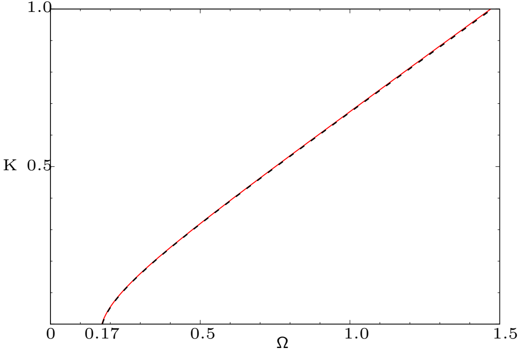
α covers the whole long-wave low-frequency band, given by $\Omega \ll 1$, and $\mathrm{K} \ll 1$.

Shortened polynomial dispersion relation

Dispersion relation expressed in Ω_* and K_* becomes

$$\Omega_*^2 = \rho_\mu \left(\mathrm{K}_*^2 + \frac{\mu^{1-\alpha}}{\mathrm{h}} \right).$$

At $\alpha < 1$ we have $\Omega_* \sim \sqrt{\rho_\mu}\mathrm{K}_*$ or $\omega \sim \mathrm{c}_2^{\mathrm{s}}\mathrm{k}$, corresponding to the short-wave limit for stiffer skin layers.



B. Sandwich structure. Shortened polynomial dispersion relation

Plate with stiff outer layers and light core

$\mu \ll 1, \quad \mathrm{h} \sim \mu, \quad \rho \sim \mu^2$

$$\gamma_1 \sim \gamma_2 \sim \mu \quad \text{and} \quad \gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$$

Approximate dispersion relation

$$\mu + \mu\left(\frac{1}{2} + \mathrm{h}_\mu\right)\mathrm{K}^2 - \frac{\mathrm{h}_\mu}{6\rho_\mu}\mathrm{K}^2\Omega^2 - \left(\frac{\mu}{2} + \frac{\mathrm{h}_\mu}{\rho_\mu}\right)\Omega^2 + \frac{\mathrm{h}_\mu}{6\rho_\mu}\Omega^4 = 0.$$

Normalized wavenumber and frequency

$$\mathrm{K}^2 = \mu\mathrm{K}_*^2 \quad \text{and} \quad \Omega^2 = \mu\Omega_*^2,$$

we obtain

$$1 + \mu\left(\frac{1}{2} + \mathrm{h}_\mu\right)\mathrm{K}_*^2 - \mu\frac{\mathrm{h}_\mu}{6\rho_\mu}\mathrm{K}_*^2\Omega_*^2 - \left(\frac{\mu}{2} + \frac{\mathrm{h}_\mu}{\rho_\mu}\right)\Omega_*^2 + \mu\frac{\mathrm{h}_\mu}{6\rho_\mu}\Omega_*^4 = 0.$$

Shortened polynomial dispersion relation

Adapt a near cut-off asymptotic expansion in the form

$$\Omega_*^2 = \Omega_0^2 + \mu\Omega_1^2 + \cdots$$

where

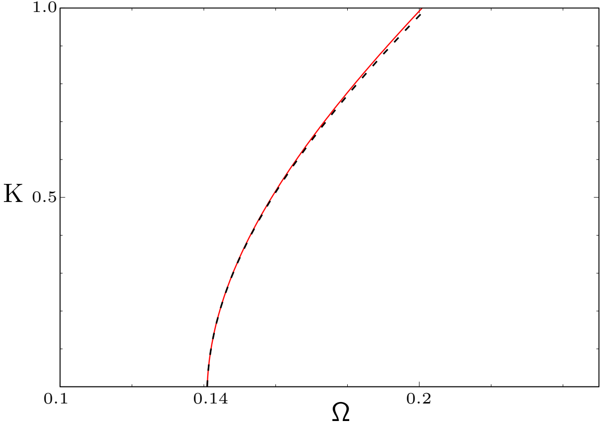
$$\Omega_0^2 = \frac{\rho_\mu}{\mathrm{h}_\mu} \quad \text{and} \quad \Omega_1^2 = \frac{\rho_\mu}{\mathrm{h}_\mu}\left(\frac{1}{3} + \mathrm{h}_\mu\right)\mathrm{K}_*^2 - \frac{1}{3}\frac{\rho_\mu^2}{\mathrm{h}_\mu^2},$$

leading to the optimal shortened dispersion relation

$$\left(\mathrm{h}_\mu + \frac{1}{3}\right)\mathrm{K}^2 - \frac{1}{\mu}\frac{\mathrm{h}_\mu}{\rho_\mu}\Omega^2 + \left(1 - \frac{\mu\rho_\mu}{3\mathrm{h}_\mu}\right) = 0.$$

Valid only over a narrow vicinity of the cut-off frequency!

Numerical illustration



$\mu = 0.014$, $\rho = 0.03$, and h = 1.0

Asymptotic formulae for displacements and stresses (setup A)

Leading order displacements and stresses

$$\mathrm{u_c} = \mathrm{h_c}\xi_{2\mathrm{c}},$$
$$\sigma_{13}^{\mathrm{c}} = \mathrm{i}\mu_{\mathrm{c}}\sqrt{\mu}\,\mathrm{K}_*\xi_{2\mathrm{c}},$$
$$\sigma_{23}^{\mathrm{c}} = \mu_{\mathrm{c}},$$

and

$$\mathrm{u_s} = \mathrm{h_c},$$
$$\sigma_{13}^{\mathrm{s}} = \mathrm{i}\mu_{\mathrm{s}}\sqrt{\mu}\mathrm{K}_*,$$
$$\sigma_{23}^{\mathrm{s}} = \mu_{\mathrm{c}}\mathrm{h}\left(\mathrm{K}_*^2 - \frac{\Omega_*^2}{\rho_\mu}\right)(\xi_{2\mathrm{s}} - 1)\,.$$

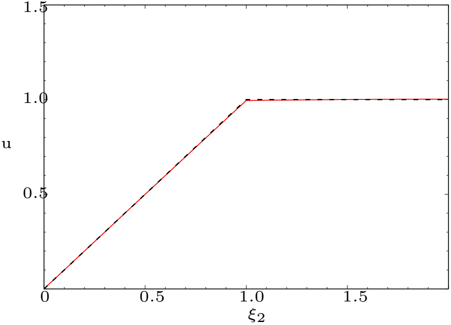
We obtain

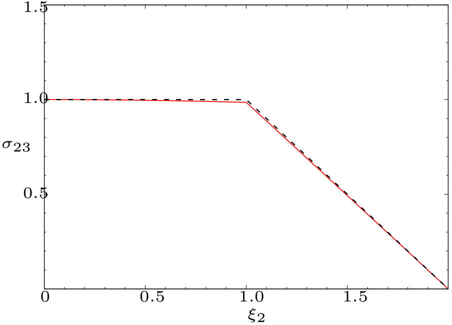
$$\frac{\mathrm{u_q}}{\mathrm{h_c}} \sim \frac{\sigma_{23}^{\mathrm{q}}}{\mu_{\mathrm{c}}} \sim \frac{\sigma_{13}^{\mathrm{q}}}{\mu_{\mathrm{q}}\sqrt{\mu}}, \quad \mathrm{q} = \mathrm{c}, \mathrm{s}.$$

Normalised displacement and stress σ_{23} (setup A)

$$\xi_2 = \xi_{2c},\ \mathrm{u} = \frac{\mathrm{u_c}}{\mathrm{h_c}},\ \text{and}\ \sigma_{23} = \frac{\sigma_{23}^{\mathrm{c}}}{\mu_{\mathrm{c}}},\ (0 < \xi_2 \leq 1)$$

$$\text{or}\ \xi_2 = 1 + \xi_{2s},\ \mathrm{u} = \frac{\mathrm{u_s}}{\mathrm{h_c}},\ \text{and}\ \sigma_{23} = \frac{\sigma_{23}^{\mathrm{s}}}{\mu_{\mathrm{c}}},\ (1 < \xi_2 \leq 2)$$





Model construction (setup A)

Scaled longitudinal coordinate and time

$$x_1 = \frac{h_c}{\sqrt{\mu}}\xi_1 \quad \text{and} \quad t = \frac{h_c}{c_{2c}\sqrt{\mu}}\tau,$$

Normalised displacement and stresses

$$u^q = h_c v^q, \quad \sigma_{13}^q = \mu_q\sqrt{\mu}S_{13}^q, \quad \sigma_{23}^q = \mu_c S_{23}^q, \quad q = c, s.$$

with all dimensionless quantities assumed to be of order unity.

Core layer

$$\mu\frac{\partial S_{13}^c}{\partial \xi_1} + \frac{\partial S_{23}^c}{\partial \xi_{2c}} - \mu\frac{\partial^2 v^c}{\partial \tau^2} = 0,$$

$$S_{13}^c = \frac{\partial v^c}{\partial \xi_1}, \quad S_{23}^c = \frac{\partial v^c}{\partial \xi_{2c}}.$$

Skin layer

$$\frac{\partial S_{13}^s}{\partial \xi_1} + \frac{1}{h}\frac{\partial S_{23}^s}{\partial \xi_{2s}} - \frac{1}{\rho_\mu}\frac{\partial^2 v^s}{\partial \tau^2} = 0,$$

$$S_{13}^s = \frac{\partial v^s}{\partial \xi_1}, \quad \mu h S_{23}^s = \frac{\partial v^s}{\partial \xi_{2s}}.$$

Derivation of a shortened equation (setup A)

Continuity and boundary conditions

$$\begin{aligned} \mathrm{v}^{\mathrm{c}}\big|_{\xi_{2\mathrm{c}}=1} &= \mathrm{v}^{\mathrm{s}}\big|_{\xi_{2\mathrm{s}}=0}, \\ \mathrm{S}_{23}^{\mathrm{c}}\big|_{\xi_{2\mathrm{c}}=1} &= \mathrm{S}_{23}^{\mathrm{s}}\big|_{\xi_{2\mathrm{s}}=0}, \end{aligned}$$

and

$$\mathrm{S}_{23}^{\mathrm{s}}\big|_{\xi_{2\mathrm{s}}=1} = 0.$$

Expand displacements and stresses into asymptotic series as

$$\begin{aligned} \mathrm{v}^{\mathrm{q}} &= \mathrm{v}_0^{\mathrm{q}} + \mu \mathrm{v}_1^{\mathrm{q}} + \cdots, \\ \mathrm{S}_{\mathrm{j}3}^{\mathrm{q}} &= \mathrm{S}_{\mathrm{j}3,0}^{\mathrm{q}} + \mu \mathrm{S}_{\mathrm{j}3,1}^{\mathrm{q}} + \cdots, \quad \mathrm{q} = \mathrm{c}, \mathrm{s} \quad \text{and} \quad \mathrm{j} = 1, 2. \end{aligned}$$

Leading order problem

$$\mathrm{S}^{\mathrm{c}}_{13,0} = \frac{\partial \mathrm{v}^{\mathrm{c}}_0}{\partial \xi_1}, \quad \frac{\partial \mathrm{S}^{\mathrm{c}}_{23,0}}{\partial \xi_{2\mathrm{c}}} = 0, \quad \mathrm{S}^{\mathrm{c}}_{23,0} = \frac{\partial \mathrm{v}^{\mathrm{c}}_0}{\partial \xi_{2\mathrm{c}}},$$

and

$$\frac{\partial \mathrm{S}^{\mathrm{s}}_{13,0}}{\partial \xi_1} + \frac{1}{\mathrm{h}} \frac{\partial \mathrm{S}^{\mathrm{s}}_{23,0}}{\partial \xi_{2\mathrm{s}}} - \frac{1}{\rho_\mu} \frac{\partial^2 \mathrm{v}^{\mathrm{s}}_0}{\partial \tau^2} = 0,$$

$$\mathrm{S}^{\mathrm{s}}_{13,0} = \frac{\partial \mathrm{v}^{\mathrm{s}}_0}{\partial \xi_1}, \quad \frac{\partial \mathrm{v}^{\mathrm{s}}_0}{\partial \xi_{2\mathrm{s}}} = 0,$$

with

$$\mathrm{v}^{\mathrm{c}}_0\Big|_{\xi_{2\mathrm{c}}=1} = \mathrm{v}^{\mathrm{s}}_0\Big|_{\xi_{2\mathrm{s}}=0},$$

$$\mathrm{S}^{\mathrm{c}}_{23,0}\Big|_{\xi_{2\mathrm{c}}=1} = \mathrm{S}^{\mathrm{s}}_{23,0}\Big|_{\xi_{2\mathrm{s}}=0},$$

and

$$\mathrm{S}^{\mathrm{s}}_{23}\Big|_{\xi_{2\mathrm{s}}=1} = 0.$$

Leading order solution

$$\mathrm{v}_0^{\mathrm{s}} = \mathrm{w}(\xi_1, \tau).$$

The rest of the quantities are expressed in terms of w as

$$\begin{aligned} \mathrm{S}_{13,0}^{\mathrm{c}} =& \xi_{2\mathrm{c}} \frac{\partial \mathrm{w}}{\partial \xi_1}, \qquad \mathrm{S}_{23,0}^{\mathrm{c}} = \mathrm{w}, \qquad \mathrm{v}_0^{\mathrm{c}} = \xi_{2\mathrm{c}} \mathrm{w}, \\ \mathrm{S}_{13,0}^{\mathrm{s}} =& \frac{\partial \mathrm{w}}{\partial \xi_1}, \qquad \mathrm{S}_{23,0}^{\mathrm{s}} = \mathrm{w}(1 - \xi_{2\mathrm{s}}), \end{aligned}$$

with w satisfying the 1D equation

$$\frac{\partial^2 \mathrm{w}}{\partial \xi_1^2} - \frac{1}{\rho_\mu} \frac{\partial^2 \mathrm{w}}{\partial \tau^2} - \frac{1}{\mathrm{h}} \mathrm{w} = 0,$$

which may be presented in the original variables as

$$\frac{\partial^2 \mathrm{u_s}}{\partial \mathrm{x}_1^2} - \frac{\rho_\mathrm{s}}{\mu_\mathrm{s}} \frac{\partial^2 \mathrm{u_s}}{\partial \mathrm{t}^2} - \frac{\mu_\mathrm{c}}{\mu_\mathrm{s} \mathrm{h_c} \mathrm{h_s}} \mathrm{u_s} = 0,$$

where $\mathrm{u_s}(\mathrm{x}_1, \mathrm{t}) \approx \mathrm{w}(\mathrm{x}_1, \mathrm{t})$.

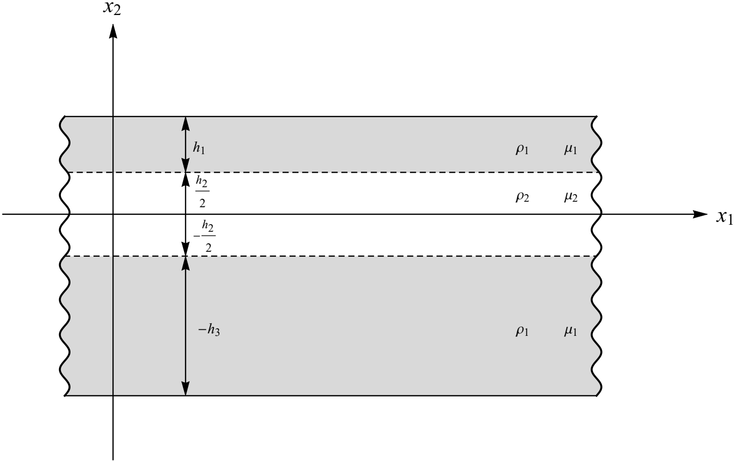
Justification of the model

Insert ansatz $u_s = \mathsf{exp}\{i(kx_1 - \omega t)\}$ into the last equation. As a result, we have the dispersion relation

$$k^2 - \frac{\rho_s}{\mu_s}\omega^2 + \frac{\mu_c}{\mu_s h_c h_s} = 0.$$

Coincides with the shortened dispersion relation for setup A!

Anti-plane shear of three-layered asymmetric plates



More sophisticated dispersion relation

$$\mu\alpha_1\alpha_2 \,\mathsf{tanh}(\mathrm{h}\alpha_1) + \mu^2{\alpha_2}^2 \,\mathsf{tanh}(\alpha_2) + \\ \mu\alpha_1\alpha_2 \,\mathsf{tanh}(\mathrm{h}^*\alpha_1) + {\alpha_1}^2 \,\mathsf{tanh}(\mathrm{h}^*\alpha_1)\,\mathsf{tanh}(\alpha_2)\,\mathsf{tanh}(\mathrm{h}\alpha_1) = 0,$$

where

$$\alpha_1 = \sqrt{\mathrm{K}^2 - \frac{\mu}{\rho}\Omega^2}, \quad \alpha_2 = \sqrt{\mathrm{K}^2 - \Omega^2},$$

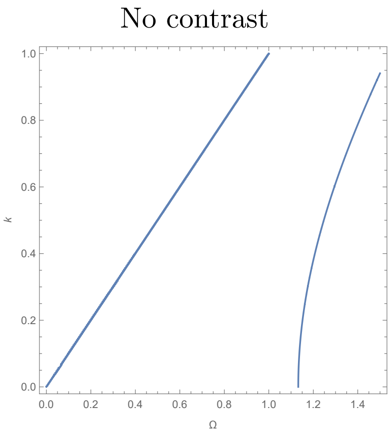
with

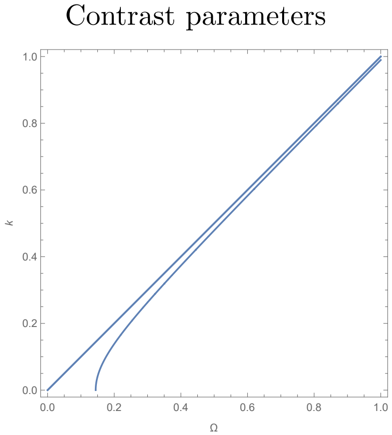
$$\Omega = \frac{\omega \mathrm{h}_2}{\mathrm{c}_2^{(2)}}, \quad \mathrm{K} = \mathrm{k}\mathrm{h}_2,$$

and

$$\mathrm{h} = \frac{\mathrm{h}_1}{\mathrm{h}_2}, \quad \mathrm{h}^* = \frac{\mathrm{h}_3}{\mathrm{h}_2}, \quad \mu = \frac{\mu_2}{\mu_1}, \quad \rho = \frac{\rho_2}{\rho_1}, \quad \mathrm{c}_2^{(\mathrm{i})} = \sqrt{\frac{\mu_\mathrm{i}}{\rho_\mathrm{i}}}, \quad \mathrm{i} = 1, 2$$

Effect of contrast





Two modes in case of high contrast for a scalar problem!

Cut-off frequencies

Frequency equation

$$\sqrt{\mu\rho}\left(\tan\left(\mathrm{h}\sqrt{\frac{\mu}{\rho}}\Omega\right)+\tan\left(\mathrm{h}^*\sqrt{\frac{\mu}{\rho}}\Omega\right)\right)$$
$$+\mu\rho\tan(\Omega)-\tan\left(\mathrm{h}\sqrt{\frac{\mu}{\rho}}\Omega\right)\tan(\Omega)\tan\left(\mathrm{h}^*\sqrt{\frac{\mu}{\rho}}\Omega\right)=0.$$

Lowest cut-off

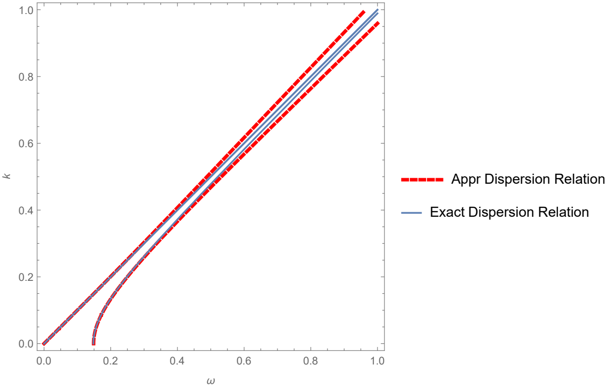
$$\Omega\approx\sqrt{\frac{\mu\rho(\mathrm{h}+\mathrm{h}^*+\rho)}{\mathrm{h}\mathrm{h}^*\mu}}$$

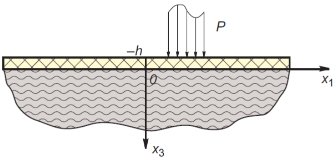
A. Photovoltaic panels. Two-mode approximation

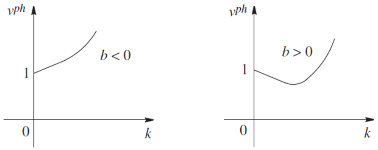
Shortened polynomial dispersion relation for two modes

$$G_1K^2+G_2\Omega^2+G_3K^4+G_4K^2\Omega^2+G_5\Omega^4+G_6K^4\Omega^2++G_7K^2\Omega^4=0$$

Out[∘]=







Typical long-wave behaviour.

H.-H. Dai, J. Kaplunov, D.A. Prikazchikov. A long-wave model for the surface elastic wave in a coated half-space. Proc. Roy. Soc. A, 466, 3097–3116 (2010)

Singularly perturbed hyperbolic equation on the interface $x_3 = 0$

$$\varphi_{,11} - c_R^{-2}\varphi_{,tt} - bh\overline{\varphi}_{,111} = A_R P,$$

isotropic coating $$b = \frac{\mu_0}{2\mu B}\left(1-\beta_R^2\right)\left(\frac{c_R^2}{c_{20}^2}(\alpha_R+\beta_R) - 4\beta_R\left(1-\frac{c_{20}^2}{c_{10}^2}\right)\right)$$

orthorhombic coating $$b = \frac{c_{66}^0}{2\mu B}\left(1-\beta_R^2\right)\left(\frac{c_R^2}{c_{60}^2}(\alpha_R+\beta_R) - 4\beta_R\frac{c_{c0}^2}{c_{60}^2}\right),$$

with $c_{60}^2 = c_{66}^0/\rho_0, \quad c_{c0}^2 = \left(c_{11}^0 c_{22}^0 - (c_{12}^0)^2\right)/\rho_0.$

Hyperbolic-elliptic model for the Rayleigh wave on an isotropic half-space

- Formulation of the problem: $(-\infty < x_1 < \infty, \quad 0 \le x_3 < \infty)$

Equations of motion $\quad c_1^2 \Delta \varphi - \varphi_{,tt} = 0, \qquad c_2^2 \Delta \psi - \psi_{,tt} = 0.$

Boundary conditions $\quad \sigma_{33}\big|_{x_3=0} = P(x_1, t), \qquad \sigma_{31}\big|_{x_3=0} = Q(x_1, t).$

- Asymptotic model:

Elliptic equations $\quad \varphi_{,33} + \alpha_R^2 \varphi_{,11} = 0, \qquad \psi_{,33} + \beta_R^2 \psi_{,11} = 0,$

where $\quad \alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}}.$

Potentials are related $\quad \varphi(x_1 - c_R t, \alpha_R x_3) = \gamma \overline{\psi}(x_1 - c_R t, \alpha_R x_3).$

Hyperbolic equations on the surface $\; x_3 = 0$

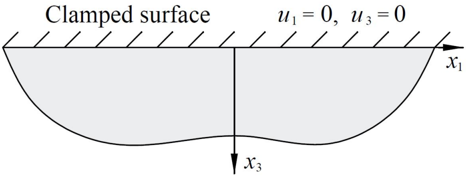
$$\varphi_{,11} - c_R^{-2} \varphi_{,tt} = A_R P, \qquad \psi_{,11} - c_R^{-2} \psi_{,tt} = -A_R Q.$$

where $\quad A_R = \frac{1 + \beta_R^2}{2\mu B}, \qquad B = \frac{\alpha_R}{\beta_R}(1 - \beta_R^2) + \frac{\beta_R}{\alpha_R}(1 - \alpha_R^2) - 1 + \beta_R^4.$

Kaplunov, J., Prikazchikov, D.A.: Asymptotic theory for Rayleigh and Rayleigh-type waves. Advances in Applied Mechanics **50**, 1–106 (2017)

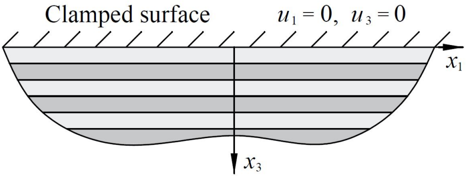
Clamped surface

Homogeneous half-space



$\Rightarrow$ No surface wave

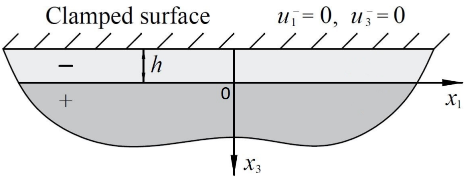
Inhomogeneous half-space



Rigorous mathematical analysis proves a possibility of localized waves

K.D. Cherednichenko, S. Cooper. On the existence of high-frequency boundary resonances in layered elastic media. Proc. R. Soc. A, 471(2178), 20140878 (2015)

Coated half-space with fixed surface. Problem statement



Equations of motion

$$\sigma^{\pm}_{i1,1} + \sigma^{\pm}_{i3,3} - \rho^{\pm} u^{\pm}_{i,tt} = 0, \qquad i = 1, 3.$$

Constitutive relations

$$\sigma^{\pm}_{ij} = \lambda^{\pm}\delta_{ij}(u^{\pm}_{1,1} + u^{\pm}_{2,2} + u^{\pm}_{3,3}) + \mu^{\pm}(u^{\pm}_{i,j} + u^{\pm}_{j,i}), \qquad j = 1, 3.$$

Boundary conditions at $x_3 = -h$

$$u_i^- = 0.$$

– Novelty!

Continuity conditions at $x_3 = 0$

$$u_i^- = u_i^+, \qquad \sigma^-_{i3} = \sigma^+_{i3}.$$

Coated half-space with fixed surface. Exact solution

Elastic wave potentials

$$u_1^{\pm} = \frac{\partial \varphi^{\pm}}{\partial x_1} - \frac{\partial \psi^{\pm}}{\partial x_3}, \qquad u_3^{\pm} = \frac{\partial \varphi^{\pm}}{\partial x_3} + \frac{\partial \psi^{\pm}}{\partial x_1}.$$

Wave equations

$$\Delta \varphi^{\pm} - \frac{1}{(c_1^{\pm})^2} \frac{\partial^2 \varphi^{\pm}}{\partial t^2} = 0, \qquad \Delta \psi^{\pm} - \frac{1}{(c_2^{\pm})^2} \frac{\partial^2 \psi^{\pm}}{\partial t^2} = 0,$$

where $c_1^{\pm} = \sqrt{\dfrac{\lambda^{\pm} + 2\mu^{\pm}}{\rho^{\pm}}}, \quad c_2^{\pm} = \sqrt{\dfrac{\mu^{\pm}}{\rho^{\pm}}}$ and $\Delta = \dfrac{\partial^2}{\partial x_1} + \dfrac{\partial^2}{\partial x_3}$.

The sought for wave potentials

$$\begin{aligned}
\varphi^- &= [A_1 \cos(\alpha^- k x_3) + A_2 \sin(\alpha^- k x_3)] \mathrm{e}^{ik(x_1 - ct)}, \\
\psi^- &= [A_3 \cos(\beta^- k x_3) + A_4 \sin(\beta^- k x_3)] \mathrm{e}^{ik(x_1 - ct)}, \\
\varphi^+ &= A_5 \mathrm{e}^{ik(x_1 - ct) - \alpha^+ k x_3}, \qquad \psi^+ = A_6 \mathrm{e}^{ik(x_1 - ct) - \beta^+ k x_3},
\end{aligned}$$

where

$$\alpha^- = \sqrt{\frac{c^2}{(c_1^-)^2} - 1}, \quad \alpha^+ = \sqrt{1 - \frac{c^2}{(c_1^+)^2}}, \quad \beta^- = \sqrt{\frac{c^2}{(c_2^-)^2} - 1}, \quad \beta^+ = \sqrt{1 - \frac{c^2}{(c_2^+)^2}}.$$

Analysis of full dispersion relation

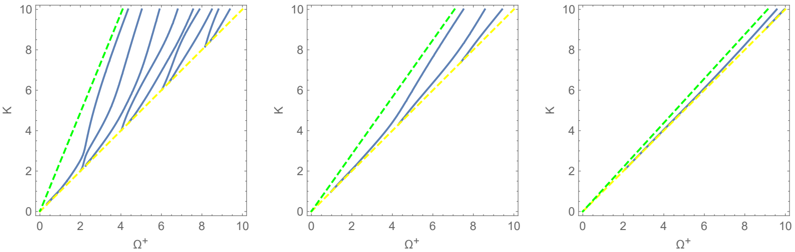
Dispersion relation $\det \mathbf{A} = 0$, where **A** is a 6×6 matrix.

Material parameter (relative stiffness) $\mu = \dfrac{\mu^-}{\mu^+}$.

Dimensionless variables $\zeta = \dfrac{c}{c_2^+}, \qquad \Omega^+ = \dfrac{\omega h}{c_2^+}, \qquad K = kh.$

Numerical values $\rho^- = 150, \qquad \rho^+ = 250, \qquad \nu^- = 0.3, \qquad \nu^+ = 0.25.$

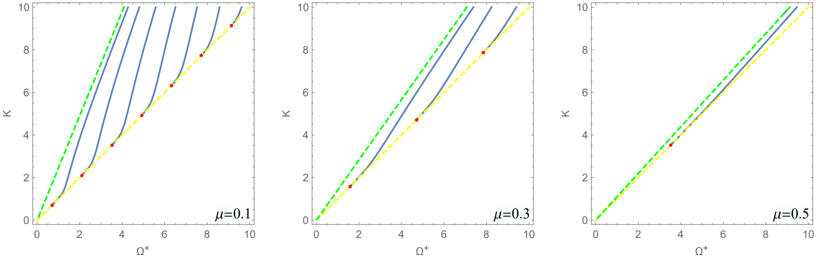
$\mu = 0.1$ $\qquad\qquad$ $\mu = 0.3$ $\qquad\qquad$ $\mu = 0.5$



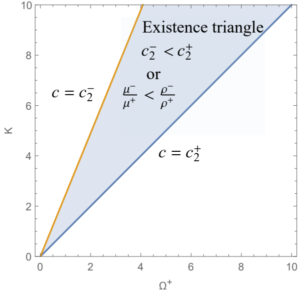
Love waves

Dispersion relation

$$\tan(K\beta^-) + \frac{\mu\beta^-}{\beta^+} = 0.$$



Existence of surface waves



Initial points

$$c = c_2^+ \Rightarrow \beta^+ \to 0 \Rightarrow \tan(K\beta^-) \to \infty$$

$$\Omega_{in}^+ = K_{in} = \frac{\pi n}{2\sqrt{\frac{\rho}{\mu} - 1}}, \quad n = 1, 3, \ldots$$

Approximations for $K \gg 1$

$$c \to c_2^- \Rightarrow \beta^- \to 0 \Rightarrow \tan(K\beta^-) \to 0$$

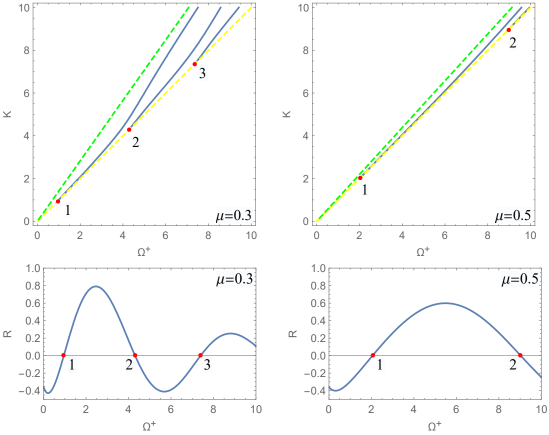
$$K\beta^- = \pi m, \quad m = 1, 2, \ldots$$

Analysis of full dispersion relation

Initial points

$$R(\Omega^+) = 0,$$

where $R(\Omega^+) = \det\mathbf{A}$ at $c = c_2^+$ or $K = \Omega^+$.

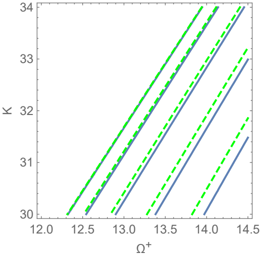


Approximations for $K \gg 1$

Similarly to the Love waves

$$\Omega^+ = \frac{c_2^-}{c_2^+}\sqrt{\pi^2 m^2 + K^2}$$

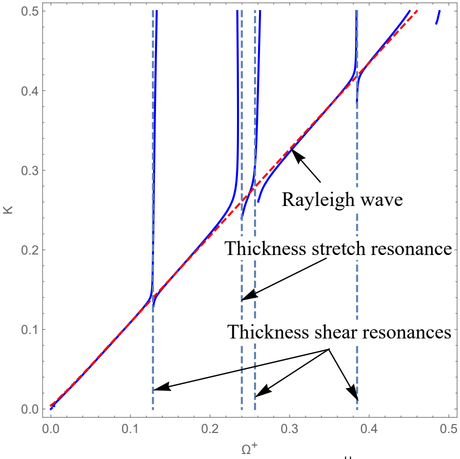
$$m = 1, 2, \ldots$$



Analysis of full dispersion relation

High contrast (soft layer, stiff half-space)

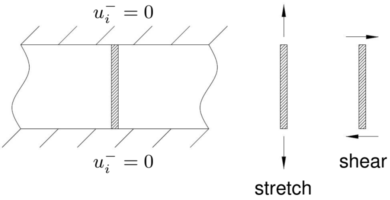
$\mu = 0.001$



The thickness resonances are eigenvalues of the problems

$$u^-_{3,33} + \left(\frac{\Omega^-}{\kappa^-}\right)^2 u^-_3 = 0,$$
$$u^-_{1,33} + (\Omega^-)^2 u^-_1 = 0$$

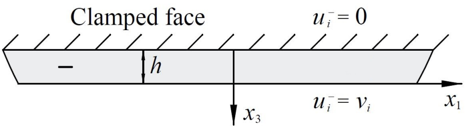
with $u^-_i = 0$, $i = 1, 3$ at the faces.



⇓

Inspiration for using the asymptotic model for Rayleigh-type waves

Thin coating



$\Rightarrow$ Effective boundary conditions for stresses at $x_3 = 0$

Dirichlet boundary conditions

$$u_i^- = 0, \qquad x_3 = -h, \qquad u_i^- = v_i, \qquad x_3 = 0,$$

where v_i are prescribed values.

Small parameter (long-wave approximation)

$$\varepsilon = \frac{h}{l} \ll 1.$$

Dimensionless variables

$$\xi_1 = \frac{x_1}{l}, \qquad \xi_3 = \frac{x_3 + h}{h}, \qquad \Omega^- = \frac{\omega h}{c_2^-}$$

Therefore $\Omega^- = \Omega^+ \dfrac{c_2^+}{c_2^-} = \Omega^+ \sqrt{\dfrac{\rho}{\mu}}$ with $\rho = \dfrac{\rho^-}{\rho^+}$.

Thin coating. Asymptotic procedure

Method of direct asymptotic integration of 3D equations in linear elasticity

Goldenveizer, A.L., Kaplunov, J.D., Nolde, E.V.: On Timoshenko-Reissner type theories of plates and shells. Int. J. Solids Struct., **30**, 675-694 (1993)

Kaplunov, J., Prikazchikov, D., Sultanova, L.: Justification and refinement of Winkler–Fuss hypothesis. Z. Angew. Math. Phys. **69**(3), 80 (2018)

Scaling of the displacements and stresses

$$u_i^- = h u_i^{*-}, \qquad \sigma_{ij}^- = \mu^- \sigma_{ij}^{*-}, \qquad \Omega^- \sim 1, \qquad v_i = h v_i^*.$$

Note that $\Omega^- \sim 1$ is associated with high-frequency localized wave.

Governing equations

$$\varepsilon \sigma_{i1,1}^{*-} + \sigma_{i3,3}^{*-} + (\Omega^-)^2 u_i^{*-} = 0, \qquad \sigma_{11}^{*-} = \varepsilon (\kappa^-)^2 u_{1,1}^{*-} + ((\kappa^-)^2 - 2) u_{3,3}^{*-},$$
$$\sigma_{33}^{*-} = \varepsilon ((\kappa^-)^2 - 2) u_{1,1}^{*-} + (\kappa^-)^2 u_{3,3}^{*-}, \qquad \sigma_{13}^{*-} = u_{1,3}^{*-} + \varepsilon u_{3,1}^{*-},$$

where $\kappa^\pm = \dfrac{c_1^\pm}{c_2^\pm} = \sqrt{\dfrac{2 - 2\nu^\pm}{1 - 2\nu^\pm}}$.

Boundary conditions $\quad u_i^{*-} = 0, \qquad \xi_3 = 0, \qquad u_i^{*-} = v_i^*, \qquad \xi_3 = 1.$

Thin coating. Leading order

Asymptotic series
$$\begin{pmatrix} u_i^{*-} \\ \sigma_{ij}^{*-} \end{pmatrix} = \begin{pmatrix} u_i^{-(0)} \\ \sigma_{ij}^{-(0)} \end{pmatrix} + \varepsilon \begin{pmatrix} u_i^{-(1)} \\ \sigma_{ij}^{-(1)} \end{pmatrix} + \dots$$

Leading order equations

$$\sigma_{i3,3}^{-(0)} + (\Omega^-)^2 u_i^{-(0)} = 0, \qquad \sigma_{11}^{-(0)} = ((\kappa^-)^2 - 2) u_{3,3}^{-(0)},$$
$$\sigma_{33}^{-(0)} = (\kappa^-)^2 u_{3,3}^{-(0)}, \qquad \sigma_{13}^{-(0)} = u_{1,3}^{-(0)}.$$

Leading order boundary conditions

$$u_i^{-(0)} = 0, \qquad \xi_3 = 0, \qquad u_i^{-(0)} = v_i^*, \qquad \xi_3 = 1.$$

Leading order stresses

$$\sigma_{33}^{-(0)} = \frac{\kappa^- \Omega^- v_3^*}{\sin\left(\dfrac{\Omega^-}{\kappa^-}\right)} \cos\left(\frac{\Omega^-}{\kappa^-}\xi_3\right), \qquad \sigma_{13}^{-(0)} = \frac{\Omega^- v_1^*}{\sin(\Omega^-)} \cos(\Omega^- \xi_3).$$

Sinusoidal, not polynomial thickness variation!

Thin coating. Asymptotic effective boundary conditions

Asymptotic dynamic effective boundary conditions at the interface $x_3 = 0$

$$\sigma_{33}^{+} = \mu^{-}\kappa^{-}\Omega^{+}\sqrt{\frac{\rho}{\mu}}\frac{v_3}{h}\cot\left(\frac{\Omega^{+}}{\kappa^{-}}\sqrt{\frac{\rho}{\mu}}\right),$$

$$\sigma_{13}^{+} = \mu^{-}\Omega^{+}\sqrt{\frac{\rho}{\mu}}\frac{v_1}{h}\cot\left(\Omega^{+}\sqrt{\frac{\rho}{\mu}}\right).$$

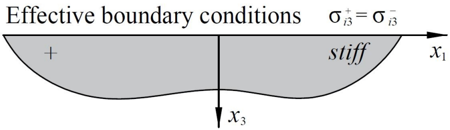
Non-traditional effective boundary conditions, corresponding to high-frequency long-wave phenomena failing at thickness resonances

$$\left(\sin\left(\frac{\Omega^{+}}{\kappa^{-}}\sqrt{\frac{\rho}{\mu}}\right) = \sin\left(\Omega^{+}\sqrt{\frac{\rho}{\mu}}\right) = 0\right).$$

Kaplunov, J.D., Kossovitch, L.Yu., Nolde, E.V.: Dynamics of thin walled elastic bodies. Academic Press (1998)

Kaplunov, J., Krynkin, A.: Resonance vibrations of an elastic interfacial layer. J. Sound Vib. **294**(4-5), 663–677 (2006)

Model for Rayleigh-type waves on a stiff half-space



Kaplunov, J., Prikazchikov, D.A.: Asymptotic theory for Rayleigh and Rayleigh-type waves. Advances in Applied Mechanics **50**, 1–106 (2017)

Displacements at the surface $x_3 = 0$

$$v_1 = \frac{1 - \beta_R^2}{2}\,\varphi_{,1}, \qquad v_3 = -\frac{1 - \beta_R^2}{1 + \beta_R^2}\,\varphi_{,3}.$$

Hyperbolic equation for the volume wave potential at the interface

$$\varphi_{,11} - \frac{1}{c_R^2}\varphi_{,tt} = A(P + \vartheta\,\overline{Q}) = \tilde{A}\,\overline{\varphi}_{,1}.$$

Here $P(x_1, t) = \sigma_{33}^{+}$ and $Q(x_1, t) = \sigma_{13}^{+}$, and the bar may be interpreted in a sense of the Hilbert transform.

Effect of contrast (soft coating)

Considering at the boundary $x_3 = 0$ the potential $\varphi = Be^{ik(x_1 - ct)}$ with $k > 0$

$$\frac{\zeta^2}{\zeta_R^2} - \sqrt{\mu\rho}\gamma\zeta - 1 = 0,$$

where $\quad \zeta = \frac{c}{c_2^+}, \qquad \zeta_R = \frac{c_R}{c_2^+},$ and

$$\gamma = K_R \left[\cot\left(\Omega^+\sqrt{\frac{\rho}{\mu}}\right) + \kappa^- \cot\left(\frac{\Omega^+}{\kappa^-}\sqrt{\frac{\rho}{\mu}}\right)\right],$$

with $\quad \Omega^+ = \frac{\omega h}{c_2^+}.$

Asymptotic series $\quad \zeta = \zeta^{(0)} + \sqrt{\mu}\zeta^{(1)} + \ldots$

Small parameter for soft coating $\quad \left(\mu = \frac{\mu^-}{\mu^+} \ll 1\right)$

At leading order $\quad \zeta^{(0)} = \zeta_R.$

Correction $\quad \frac{2\zeta^{(0)}\zeta^{(1)}}{\zeta_R^2} - \sqrt{\rho}\gamma\zeta^{(0)} = 0 \qquad \Rightarrow \qquad \zeta^{(1)} = \frac{1}{2}\zeta_R^2\sqrt{\rho}\gamma.$

Effect of high contrast (soft coating)

Non-uniform approximation for dimensionless velocity

$$\zeta = \zeta_R + \sqrt{\mu}\left(\frac{\sqrt{\rho}}{2}\,\gamma\,\zeta_R^2\right) + \ldots$$

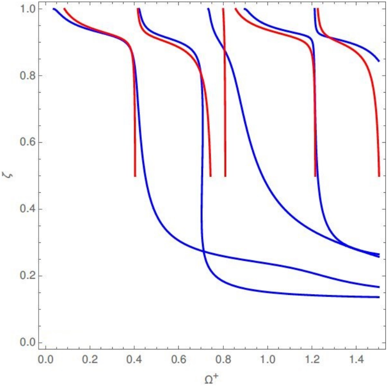
Note, γ is the source of non-uniformity, since $\gamma \to \infty$ at

$$\Omega^+ = \sqrt{\frac{\mu}{\rho}}\pi n \quad \text{and} \quad \Omega^+ = \kappa^- \sqrt{\frac{\mu}{\rho}}\pi n, \quad n = 0, 1, 2, \ldots$$

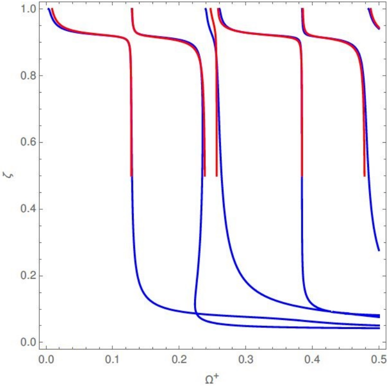
Numerical results

Numerical values $\rho^- = 150, \quad \rho^+ = 250, \quad \nu^- = 0.3, \quad \nu^+ = 0.25.$

$\mu = 0.01$



$\mu = 0.001$

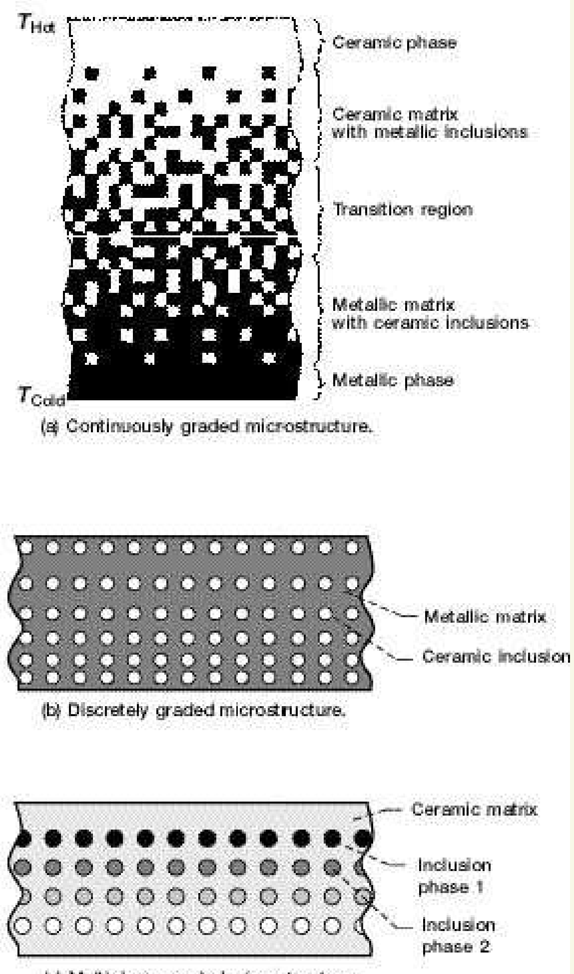


6. Periodic structures

R. Craster, L. Joseph, J. Kaplunov, *Wave Motion*, 2014, 51, 581-588

- Similarity between asymptotic procedures for thin and periodic structures
- Knowledge transfer from theory of plates and shells to homogenisation
- High-frequency homogenisation

Functionally graded microstructures



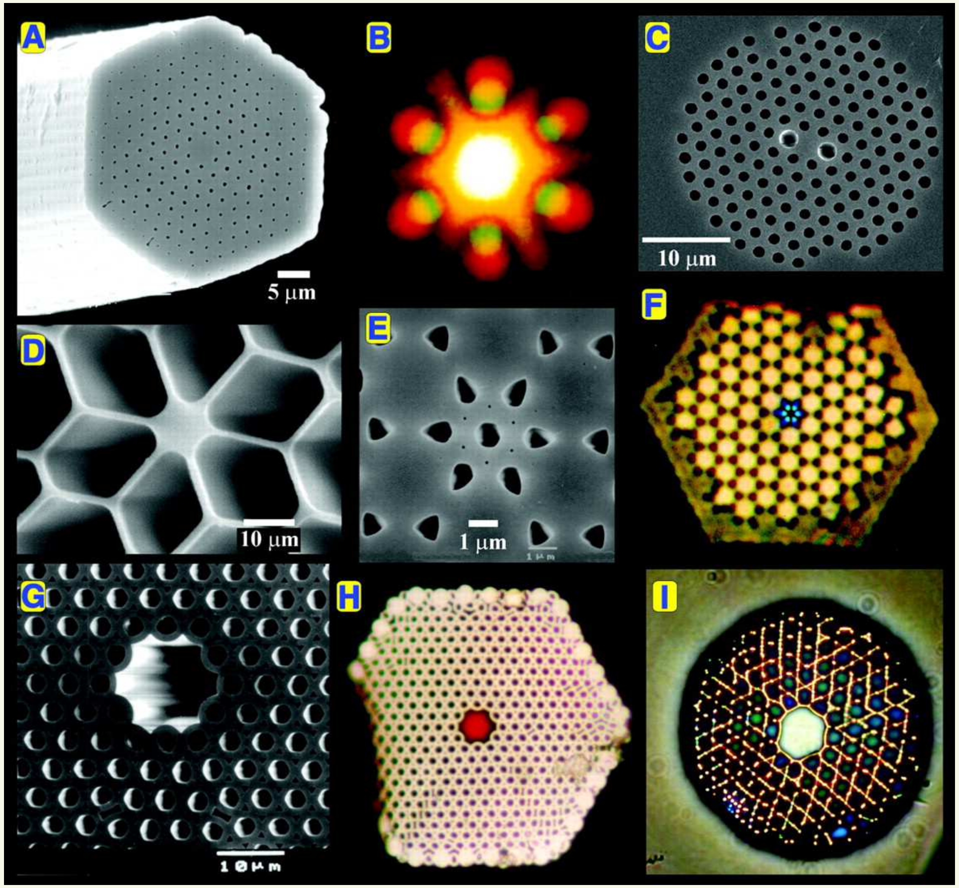
(a) Continuously graded microstructure.
(b) Discretely graded microstructure.
(c) Multiphase graded microstructure.

the picture is taken from NASA webpage

Civil structures

the picture is taken from http://www.wikipedia.org

Photonic crystals

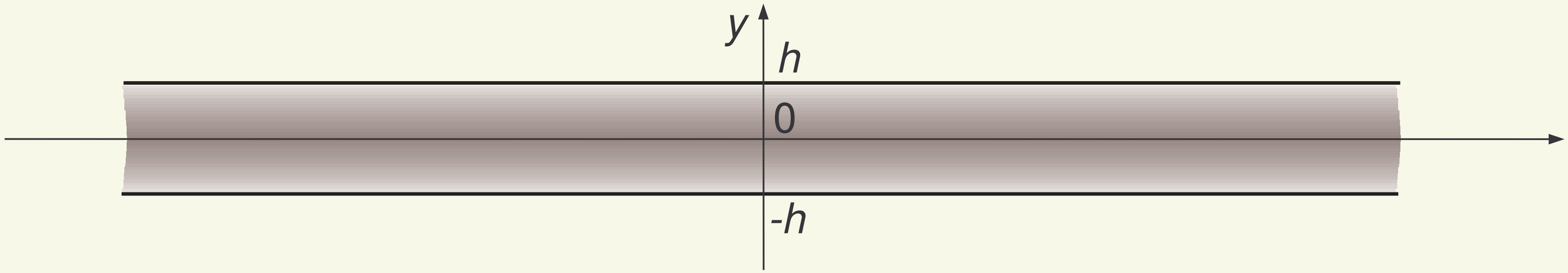


the picture is taken from the review of P Russell, Science 2003

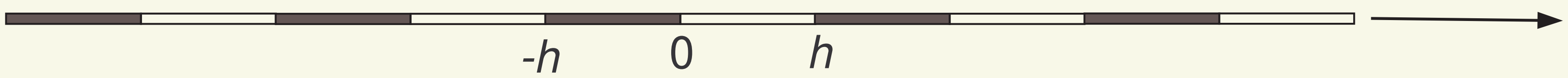
Two toy problems

The goal is to demonstrate the similarity of the homogenization procedures for 2D thin functionally graded structures and 1D periodic structures, see R.V. Craster, L.M. Joseph & J. Kaplunov in Wave Motion 2014.

(A) SH waves in a functionally graded layer (2D problem)



(B) Longitudinal waves in a periodic rod (1D problem)



Dynamic homogenization

Problem A

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\omega^2}{c^2(y)} u = 0$$

where $u = u(x, y)$

traction free faces

$\partial u / \partial y|_{y=\pm h} = 0$

Problem B

$$\frac{\mathrm{d}^2 u}{\mathrm{d} x^2} + \frac{\omega^2}{c^2(x)} u = 0$$

where $u = u(x)$

periodicity

$c(x) = c(x + 2h)$

Small parameter

$\epsilon = h/L \ll 1$ (L is typical wavelength along x-axis)

Scaling

$X = x/L, \qquad \xi = \alpha/h,$ where

$\alpha = y$

$\alpha = x$

Dynamic homogenization

Dimensionless equations in $u(X,\xi)$

$$u_{\xi\xi} + \epsilon^2 u_{XX} + \frac{\lambda^2}{C^2(\xi)} u = 0$$

$$u_{\xi\xi} + \underbrace{2\epsilon u_{X\xi}}_{\text{the only difference}} + \epsilon^2 u_{XX} + \frac{\lambda^2}{C^2(\xi)} u = 0$$

with $\lambda = \frac{\omega h}{c_0}$ and $C(\xi) = \frac{c(\xi)}{c_0}$

Classical low frequency limit ($\lambda \sim \epsilon$)

$$u(X,\xi) = u_0(X,\xi) + \epsilon u_1(X,\xi) + \epsilon^2 u_2(X,\xi) + \ldots$$

and

$$\lambda^2 = \epsilon^2 \left(\lambda_0^2 + \epsilon\lambda_1^2 + \epsilon^2\lambda_2^2 + \ldots\right)$$

with Neumann boundary conditions

$$u_{i\xi}\big|_{\xi=\pm 1} = 0$$

with periodicity conditions

$$u_i(X,1) = u_i(X,-1),$$
$$u_{i\xi}(X,1) = u_{i\xi}(X,-1)$$

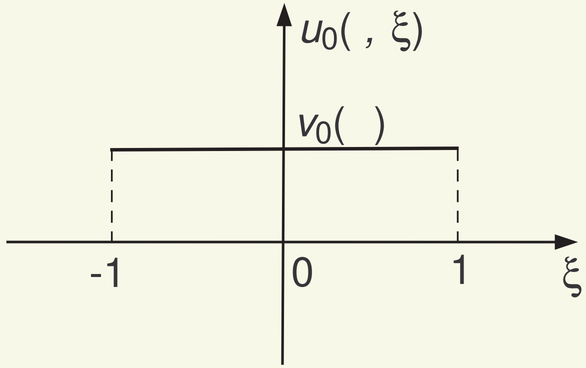
Dynamic homogenization

At leading order we get over a microscale

$$u_{0\xi\xi} = 0$$

resulting in uniform static variation along thickness or cell

$$u_0(X, \xi) = v_0(X).$$



Proceeding to higher orders

$$u_1(X, \xi) = 0, \quad \lambda_1 = 0 \text{ and } u_{2\xi\xi} = -v_{0XX} - \frac{\lambda_2^2}{C^2(\xi)} v_0$$

Finally, we arrive at 1D homogenized equation

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}x^2} + \frac{\omega^2}{\langle c \rangle^2} v_0 = 0, \text{ with } \langle c \rangle = \left[\frac{1}{2h} \int_{-h}^{h} c^{-2}(z) \mathrm{d}z \right]^{-1/2}$$

Non-classical high frequency limit $(\lambda \sim 1)$

The so-called high frequency long wave theory for thin elastic structures established some time ago (*e.g. see J.D.Kaplunov, L.Yu.Kossovich & E.V.Nolde, Dynamics of Thin Walled Elastic Bodies, Academic Press, N.-Y. 1998*) inspired a more recently developed high frequency homogenization procedure (*see R.V.Craster, J.Kaplunov & A.V.Pichugin in Proc R Soc A 2010, J.Kaplunov & A.Nobili in Math Meth Appl Sci 2017, and D.J.Colquitt, V. Danishevsky & J.Kaplunov in Math Mech Solids 2018)*

At leading order $u_0(X,\xi) = v_0(X)U_0(\xi)$ and $U_{0\xi\xi} + \dfrac{\lambda_0^2}{C^2(\xi)}U_0 = 0$

with Neumann boundary conditions

$U_{0\xi}|_{\xi=\pm 1} = 0$

with periodicity conditions
$U_0(X,1) = U_0(X,-1)$,
$U_{0\xi}(X,1) = U_{0\xi}(X,-1)$

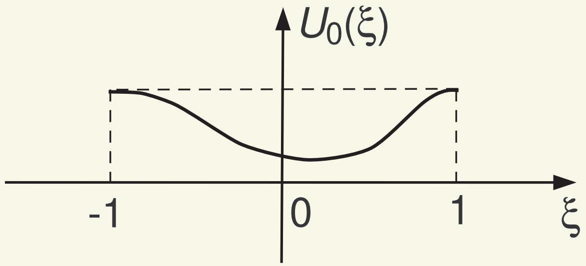
or antiperiodicity conditions (leading to periodicity with a **double** period)
$U_0(X,1) = -U_0(X,-1)$,
$U_{0\xi}(X,1) = -U_{0\xi}(X,-1)$

Dynamic homogenization

Eigenvalues λ_0 correspond to

thickness resonances

cell resonances



The sought for 1D homogenized equation is

$$h^2 T v_{0xx} + (\lambda^2 - \lambda_0^2) v_0 = 0$$

$$T = \frac{\int_{-h}^{h} U_0^2(z) C^{-2}(z) \mathrm{d}z}{\int_{-h}^{h} U_0^2(z) \mathrm{d}z}$$

T takes slightly more complicated form

(see R.V.Craster, J.Kaplunov & A.V.Pichugin in Proc R Soc A 2010)

Floquet-Bloch waves

$$\begin{bmatrix} u(-1) \\ u_\xi(-1) \end{bmatrix} = \exp(i2\kappa\varepsilon) \begin{bmatrix} u(1) \\ u_\xi(1) \end{bmatrix}$$

where κ - Bloch parameter.

Bloch spectra $\lambda(\kappa)$ near edges of stop bands

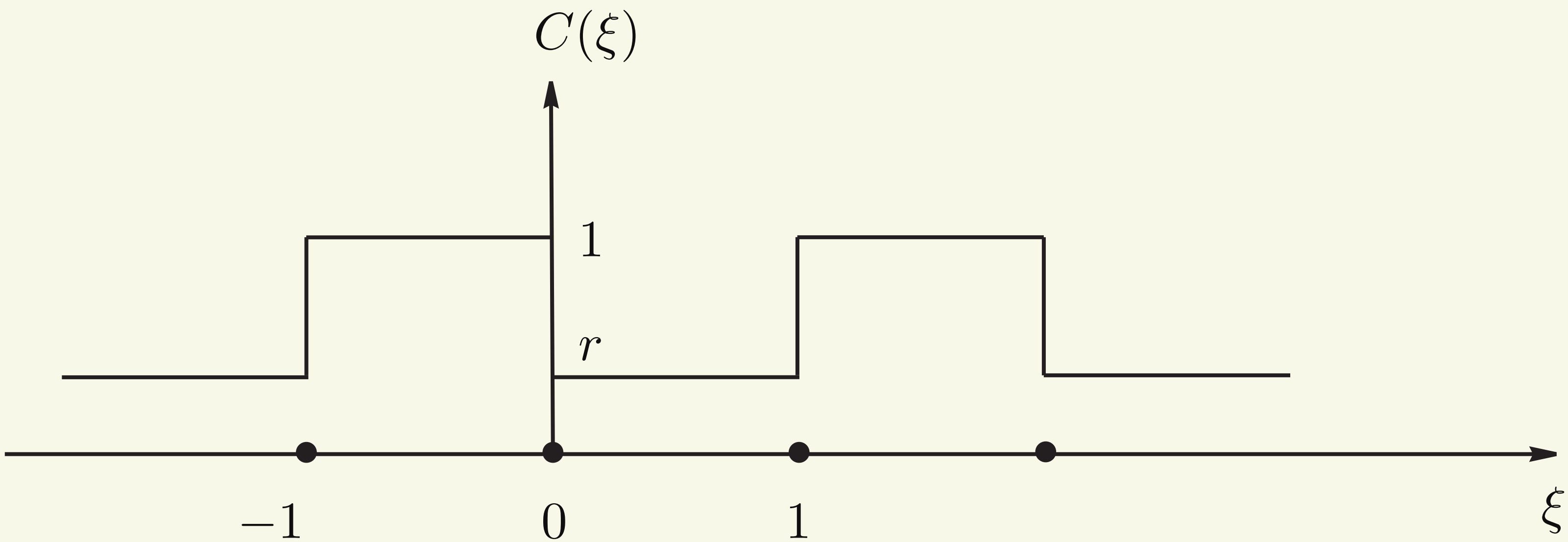
$$\kappa = 0 \qquad\qquad \kappa = \frac{\pi}{2\varepsilon}$$

almost periodic solutions

almost anti-periodic solutions

Dynamic homogenization

a) piecewise uniform sound speed (constant coefficients)

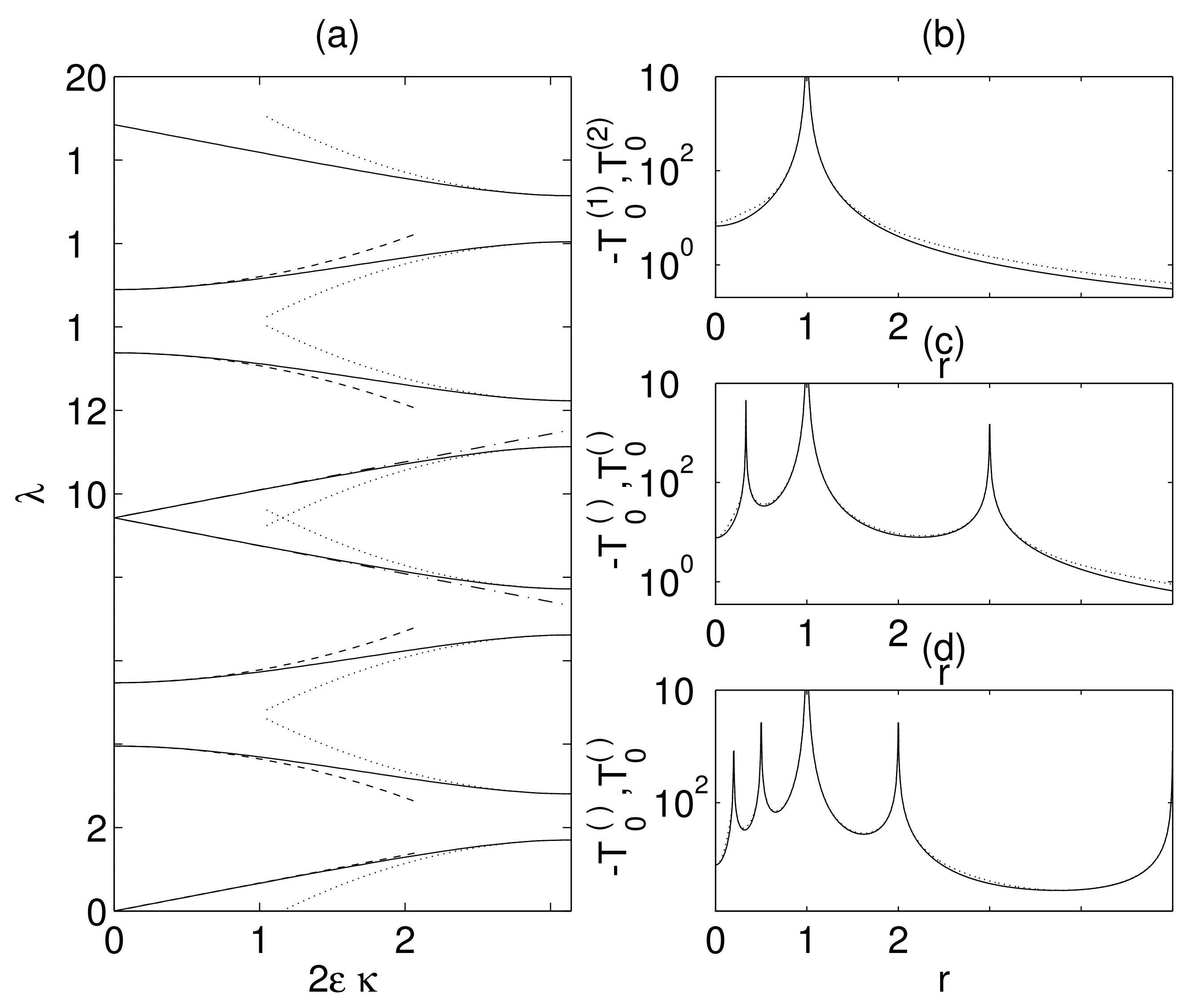


b) Mathieu's equation (variable coefficients)

$$C^{-2}(\xi) = \alpha - 2\theta \cos \xi$$

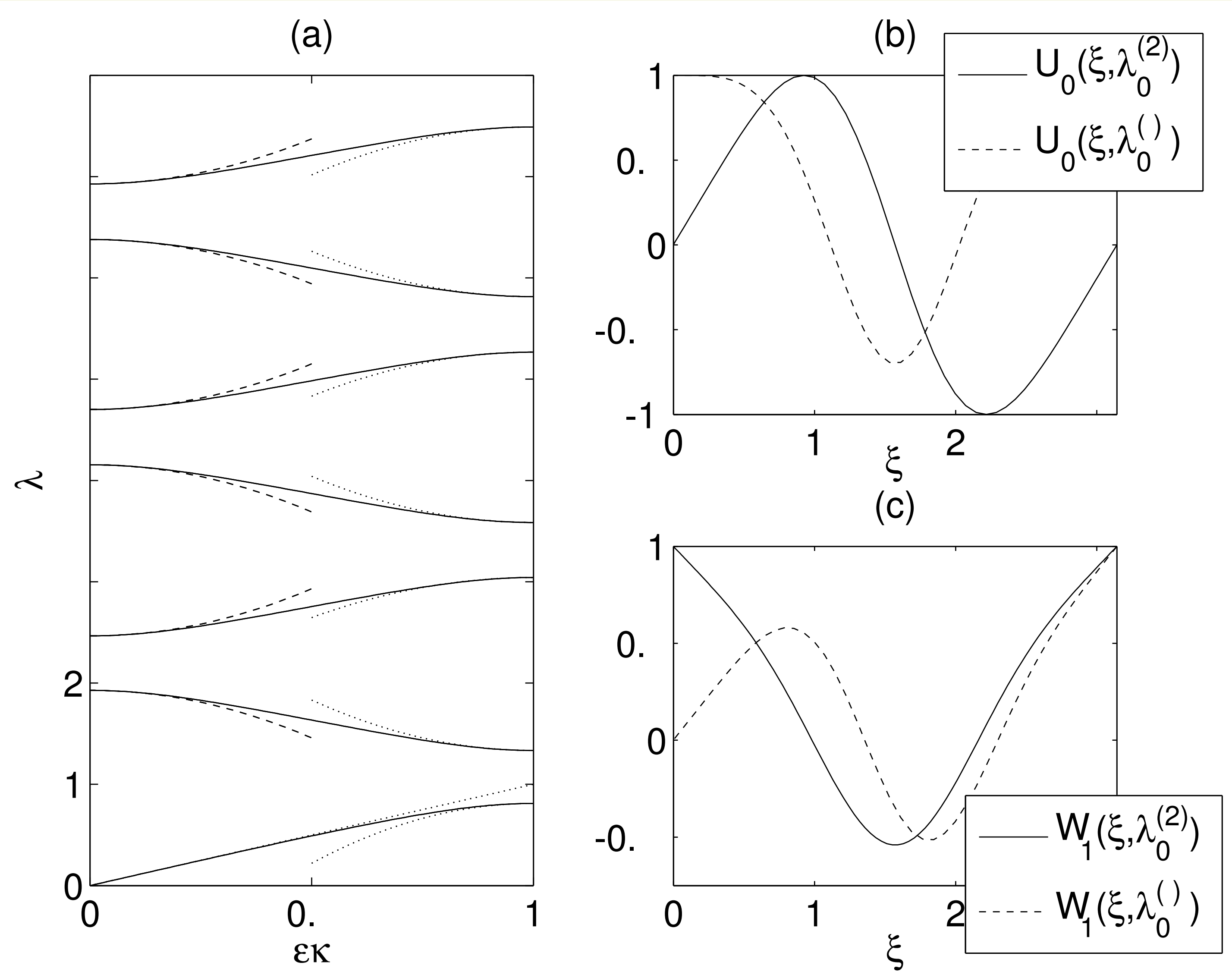
Dynamic homogenization

Piecewise uniform rod $(r = 1/3)$



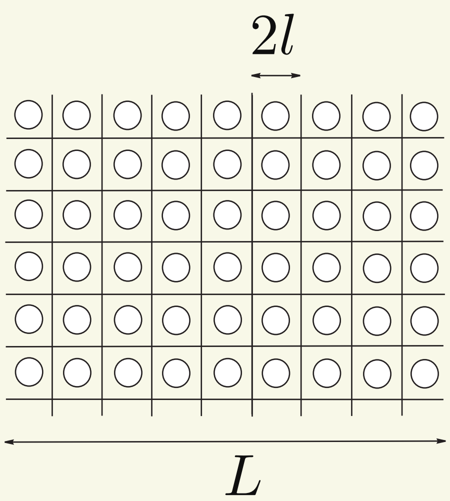
Dynamic homogenization

Mathieu's equation $(\alpha = 1, \theta = 1/2)$



High frequency homogenization in 2D

(see R.V.Craster, J.Kaplunov & A.V.Pichugin in Proc R Soc A 2010)



$$\nabla_x \cdot [a(\mathbf{x})\nabla_x u(\mathbf{x})] + \omega^2 \rho(\mathbf{x}) u(\mathbf{x}) = 0$$

with double periodic $a(\mathbf{x})$ and $\rho(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)$

Small parameter $\epsilon = l/L \ll 1$

Scaling $\mathbf{X} = \dfrac{\mathbf{x}}{L}, \quad \boldsymbol{\xi} = \dfrac{\mathbf{x}}{l}$

Dynamic homogenization

Asymptotic series

$$u(\mathbf{X},\boldsymbol{\xi}) = u_0(\mathbf{X},\boldsymbol{\xi}) + \varepsilon u_1(\mathbf{X},\boldsymbol{\xi}) + \varepsilon^2 u_2(\mathbf{X},\boldsymbol{\xi}) + \dots$$

and

$$\lambda^2 = \lambda_0^2 + \varepsilon\lambda_1^2 + \varepsilon^2\lambda_2^2 + \dots, \qquad \text{where } \lambda = \frac{\omega l}{c_0}$$

Double periodicity - antiperiodicity conditions

$$\begin{aligned} u_i\left(\mathbf{X}; -1, \xi_2\right) &= \pm u_i\left(\mathbf{X}; 1, \xi_2\right) \\ u_{i\xi_1}\left(\mathbf{X}; -1, \xi_2\right) &= \pm u_{i\xi_1}\left(\mathbf{X}; 1, \xi_2\right) \\ u_i\left(\mathbf{X}; \xi_1, -1\right) &= \pm u_i\left(\mathbf{X}; \xi_1, 1\right) \\ u_{i\xi_2}\left(\mathbf{X}; \xi_1, -1\right) &= \pm u_{i\xi_2}\left(\mathbf{X}; \xi_1, 1\right) \end{aligned}$$

At leading order

$$u_0\left(\mathbf{X},\boldsymbol{\xi}\right) = v_0\left(\mathbf{X}\right) U_0\left(\boldsymbol{\xi}\right)$$

and

$$\nabla_\xi \cdot [a(\boldsymbol{\xi})\nabla_\xi U_0] + \lambda_0^2 c_0^2 \rho(\boldsymbol{\xi}) U_0 = 0 \qquad (*)$$

with periodicity - antiperiodicity conditions on cell contour

Dynamic homogenization

The final macroscale equation becomes

$$l^2 T_{ij} \frac{\partial^2 v_0}{\partial x_i \partial x_j} + \left(\lambda^2 - \lambda_0^2\right) v_0 = 0 \quad (i, j = 1, 2) \qquad (**)$$

with T_{ij} expressed through double integrals over the domain $-1 \leq \xi_1, \xi_2 \leq 1$ containing double periodic eigenfunction $U_0\left(\boldsymbol{\xi}\right)$ and a pair of single periodic functions $V_i\left(\boldsymbol{\xi}\right)$, calculated from non-homogeneous boundary value problems for the equation (*).

Remarks.

(i) The equation (**) is valid near edges of stop bands.

(ii) The type of the equation (**) depends on problem parameters.

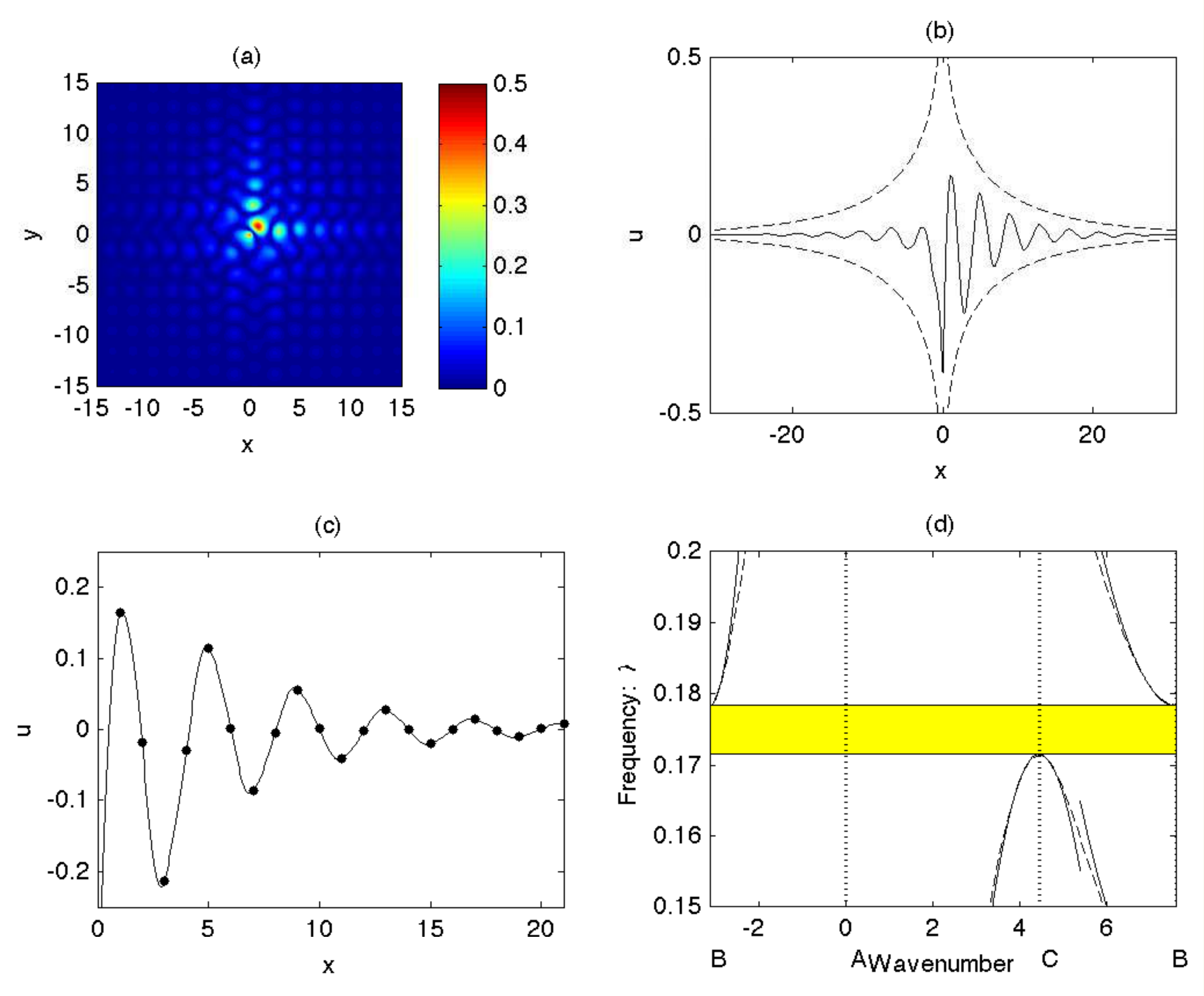
(iii) Simple explicit expressions for the coefficients T_{ij} are available only in the case of the checkerboard structures with piece-wise parameters governed by (see R.V.Craster, J.Kaplunov, E.Nolde & S.Guenneau, JOSA 2011)

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\omega^2}{c^2}\left[1 + g(x_1) + g(x_2)\right] u = 0$$

where $g_i(x_i) = 0$ for $-1 \leq x_i < 0$; $g_i(x_i) = r^2$ for $0 \leq x_i < 1$

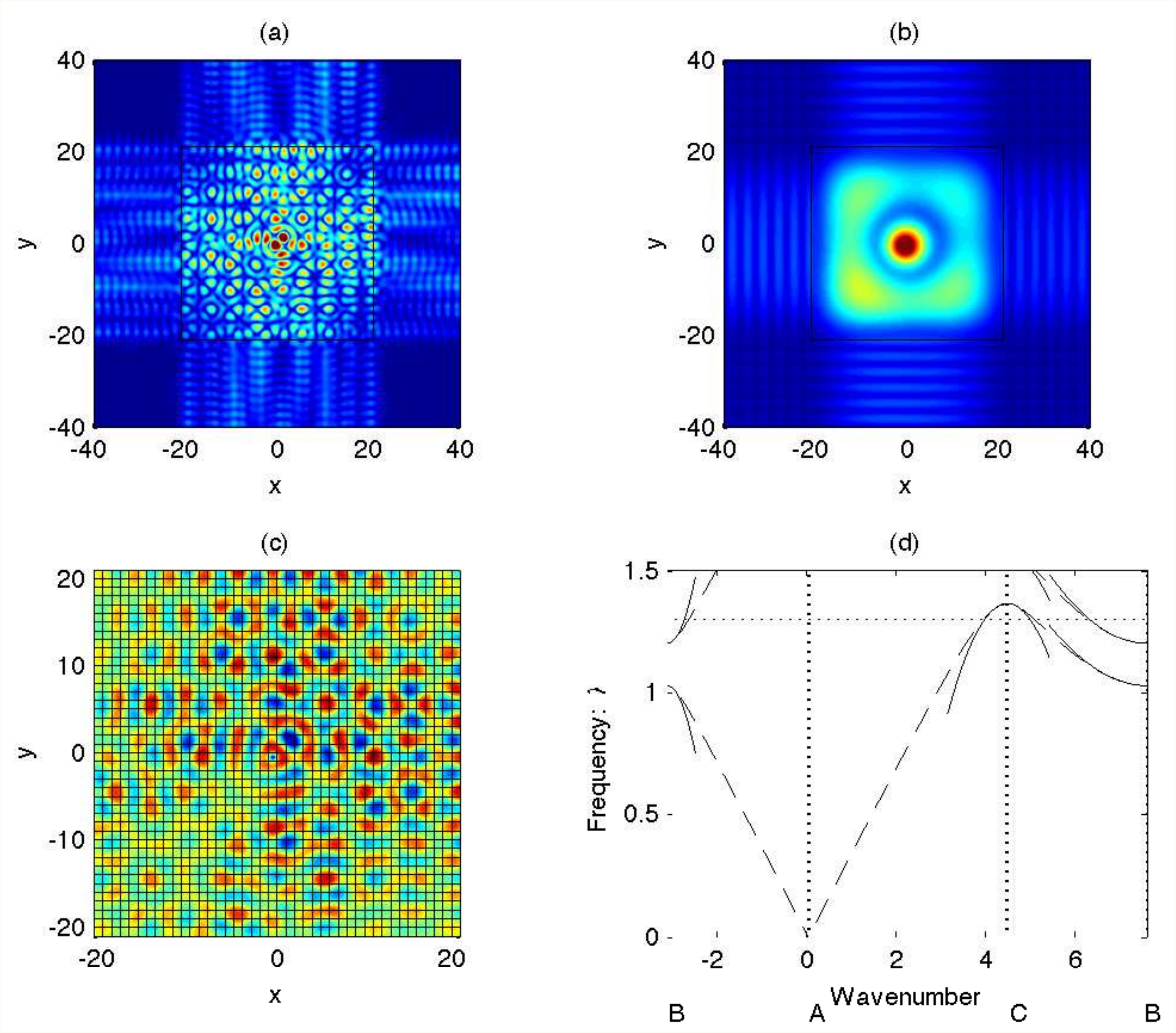
Applications of high frequency homogenization theory
for checkerboards in optics

Defect modes



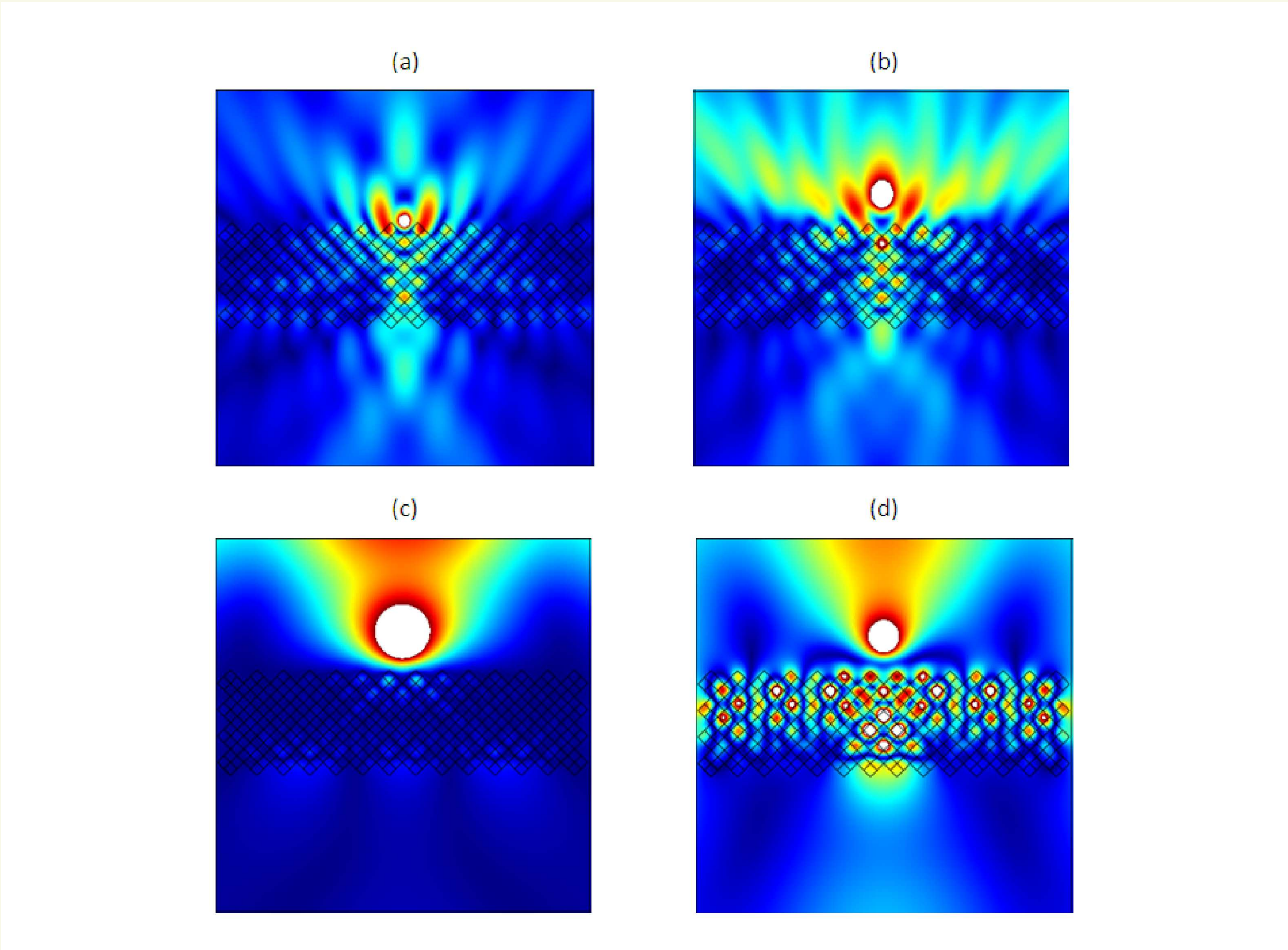
Dynamic homogenization

Ultra-refraction



Dynamic homogenization

All-angle negative refraction



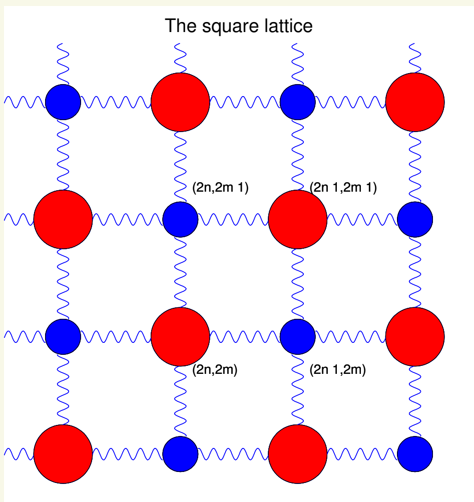
For further detail see *R.V.Craster, J.Kaplunov, E.Nolde & S.Guenneau, JOSA 2011*

Dynamic homogenization

High frequency homogenization for lattice structures

e.g. see *R.V.Craster, J.Kaplunov & J.Postnova, QJMAM 2010* for spring mass structures

and *E.Nolde, R.V.Craster & J.Kaplunov, JMPS 2011* for frame and truss structures



Two-scale approach

$$\mathbf{u} = \mathbf{u}\left(\mathbf{X}, \boldsymbol{\xi}\right)$$

continuous

discrete

$(\boldsymbol{\xi} = \{(0,0), (0,1), (1,0), (1,1)\})$

By applying Taylor series in $\mathbf{X}$ and periodicity - antiperiodicity conditions we arrive at a matrix-differential problem $[4 \times 4]$. It is

$$\Big[\underbrace{(A_0 - \lambda^2 M)}_{\textit{linear algebra}} + \varepsilon A_1(\partial_i, \lambda) + \varepsilon^2 A_2(\partial_i \partial_j, \lambda) + \ldots\Big] \mathbf{u}(\mathbf{X}, \boldsymbol{\xi}) = 0$$

with $\varepsilon = 1/N \ll 1, \quad \partial_i = \partial/\partial X_i, \quad M =$ diag (M_1, M_1, M_2, M_2)

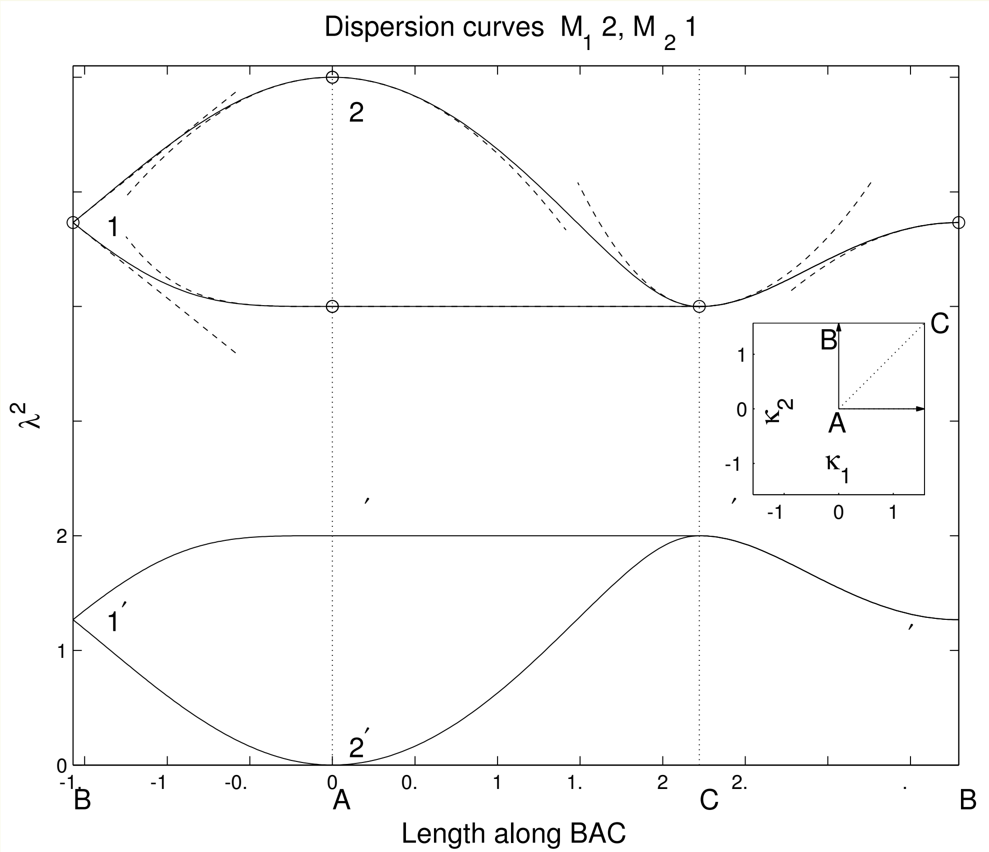
Variety of homogenized models

e.g. for periodicity in both directions when $\mathbf{u}(\mathbf{X}, \boldsymbol{\xi}) = v_0(\mathbf{X})\,[0, 0, -1, 1]^T$

The result is

$$\frac{l^4}{4(M_2 - M_1)}\left(\nabla_x^4 v_0 - 4\partial_{x_1}^2 \partial_{x_2}^2 v_0\right) + \left(\lambda^2 - \frac{4}{M_2}\right) v_0 = 0$$

which is particularly handy for analyzing localized phenomena.



Concluding remarks

- ✓ Multi-component high-contrast waveguides require specialised theory
- ✓ High contrast may lead to unexpectedly low natural frequencies
- ✓ Stronger components subject to Neumann conditions, perform almost rigid body motions
- ✓ Two-mode theories for long-wave low-frequency motion of layered plates (asymptotically uniform or composite)
- ✓ Deep parallels between long-wave dispersion of thin and periodic structures