Guided waves in phononic crystals

2019 Cargese Summer School on guided waves

Vincent Laude

Institut FEMTO-ST
Centre National de la Recherche Scientifique (CNRS),
Univ. Bourgogne Franche-Comté, Besançon, France

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Introduction to phononic crystals and waves in periodic media

Artificial crystals

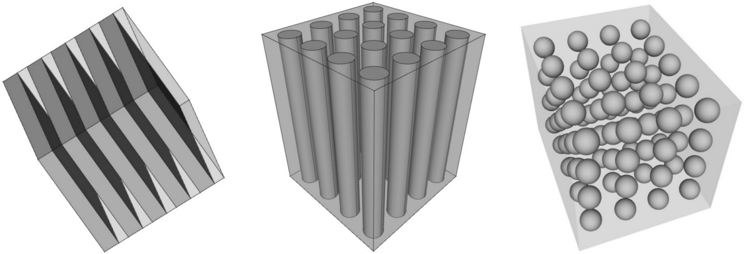
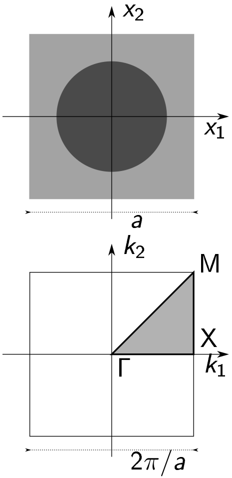


Figure: **Artificial crystals for waves with 1D, 2D, or 3D periodicity**

- Sonic crystal: matrix is a fluid (e.g., water or air)
- Phononic crystal: matrix is a solid (e.g., steel, silicon, quartz...) [1]
- Inclusions can be void, solid, or fluid

Sonic crystal of cylindrical steel rods in water



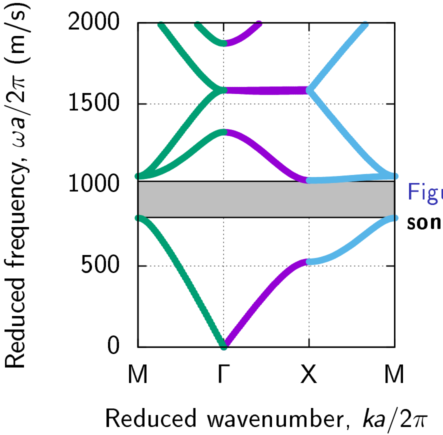
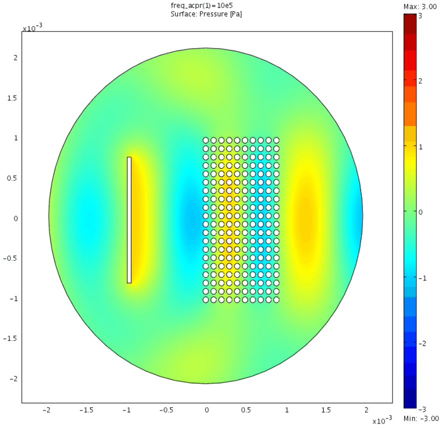


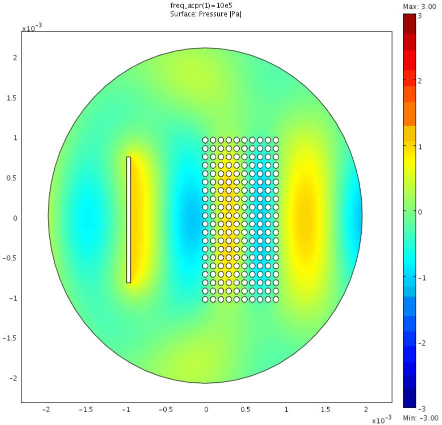
Figure: **2D square-lattice sonic crystal.** $d/a = 0.83$

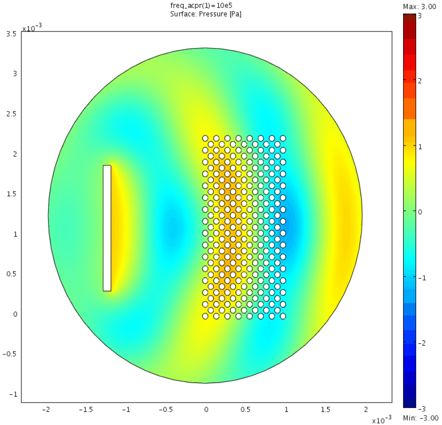
A square-lattice phononic crystal of steel rods in water I



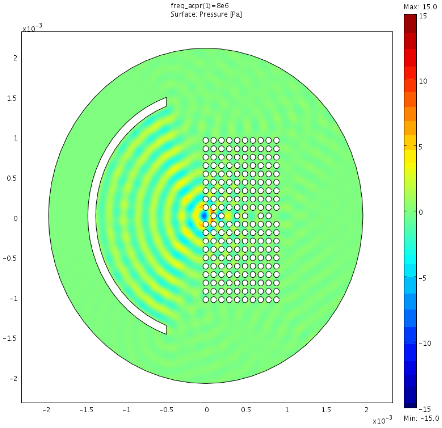
- Pitch: 100 μm
- Diameter: 70 μm
- Complete band gap: 8-9 MHz
- Plane source emits 1 Pa

A square-lattice phononic crystal of steel rods in water II





A coupled-cavity phononic crystal waveguide



- Coupled-resonator acoustic waveguide
- Works inside complete band gap only
- Complete band gap: 8-9 MHz
- Arc circle source emits 1 Pa

Bravais Lattices, 2D

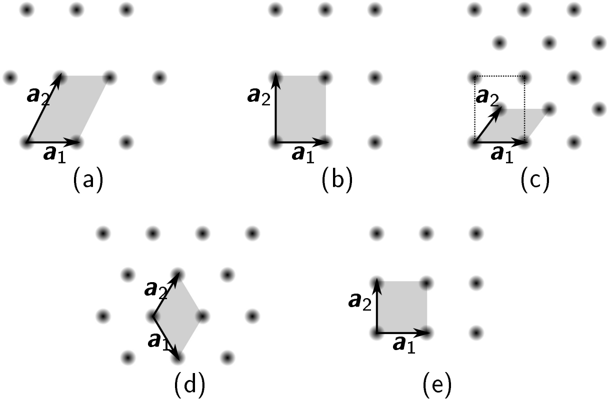


Figure: **The five two-dimensional Bravais lattices.** (a) Oblique (b) Rectangular (c) Centered rectangular (d) Hexagonal (e) Square. $\boldsymbol{R} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + n_3\boldsymbol{a}_3$

Bravais lattices, 3D

There are 14 possible Bravais lattices in 3D space: Triclinic, Monoclinic, Orthorhombic, Tetragonal, Rhomboedral, Hexagonal, Cubic.

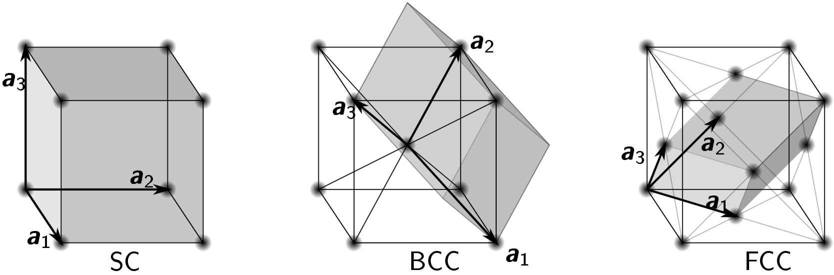


Figure: **Three-dimensional cubic Bravais lattices.** Simple cubic (SC), body-centered cubic (BCC), and face-centered cubic (FCC) lattices.

Primitive cell

The unit cell is any geometric “box” containing “atoms” arranged in 2- or 3-dimensions. Unit cells stacked periodically form the crystal without leaving any empty space.

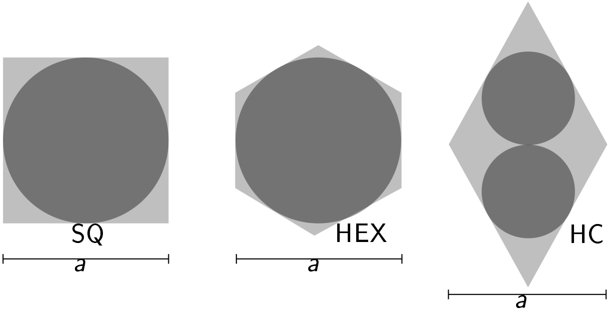


Figure: **Wigner-Seitz cells for the square lattice, the hexagonal lattice, and the honeycomb lattice.** Close packing condition: inclusions are touching but non-overlapping.

Reciprocal lattice, Brillouin zones

- Reciprocal lattice = Bravais lattice in which the Fourier transform of the wavefield is represented, i.e. $\exp(-i\boldsymbol{K}\cdot\boldsymbol{R}) = 1$.
- Reciprocal lattice vectors are $\boldsymbol{K} = m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2 + m_3\boldsymbol{b}_3$ with $\boldsymbol{b}_j \cdot \boldsymbol{a}_i = 2\pi\delta_{ij}$.
- First Brillouin zone = Wigner-Seitz cell of the reciprocal lattice.

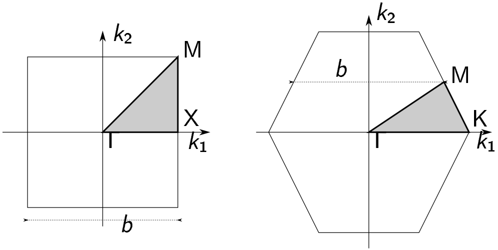


Figure: **First Brillouin zones for the square and the hexagonal lattices.** $b = 2\pi/a$

Linear chain I

Linear chain of punctual masses m connected by springs: $m\frac{\partial^2 u_n}{\partial t^2} = C(u_{n+1} - u_n) + C(u_{n-1} - u_n)$.

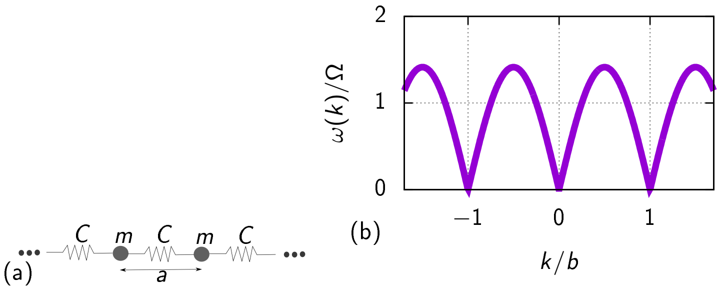


Figure: **A linear chain of masses coupled by springs.** $\Omega^2 = 2C/m$.

Linear chain II

Discrete Fourier transform (DFT) of the sequence u_n

$$\check{u}(q) = \sum_{n=-\infty}^{\infty} u_n \exp(2\imath\pi q n), q \in [-1/2, 1/2],$$

with the inverse formula

$$u_n = \int_{-1/2}^{1/2} \mathrm{d}q \, \check{u}(q) \exp(-2\imath\pi q n).$$

Taking the DFT of the linear chain equation,

$$m\frac{\partial^2 \check{u}(q)}{\partial t^2} = C(\exp(2\imath\pi q) + \exp(-2\imath\pi q) - 2)\check{u}(q) = 2C(\cos(2\pi q) - 1)\check{u}(q).$$

Time-harmonic solutions follow the dispersion relation

$$\boxed{\omega^2 = \Omega^2(1 - \cos(2\pi q)) = \Omega^2(1 - \cos(ka))}$$

with $\Omega^2 = 2C/m$ and $2\pi q = ka$.

Bilinear chain I

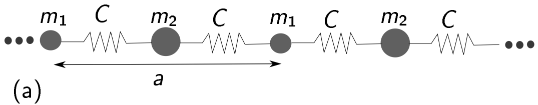


Figure: **A bilinear chain of masses coupled by springs.**

Linear chain with two types of punctual masses, or two types of atoms:

$$m_1 \frac{\partial^2 u_n}{\partial t^2} = C(v_n - u_n) + C(v_{n-1} - u_n),$$
$$m_2 \frac{\partial^2 v_n}{\partial t^2} = C(u_{n+1} - v_n) + C(u_n - v_n).$$

Taking the DFT, and with $\Omega_1^2 = 2C/m_1$ and $\Omega_2^2 = 2C/m_2$,

$$\frac{\partial^2 \check{u}(q)}{\partial t^2} = \frac{\Omega_1^2}{2}(1 + \exp(-2\imath\pi q))\check{v}(q) - \Omega_1^2 \check{u}(q),$$
$$\frac{\partial^2 \check{v}(q)}{\partial t^2} = \frac{\Omega_2^2}{2}(1 + \exp(2\imath\pi q))\check{u}(q) - \Omega_2^2 \check{v}(q).$$

Bilinear chain II

Time-harmonic solutions are such that

$$\begin{aligned}(\omega^2 - \Omega_1^2)\check{u}(q) + \frac{\Omega_1^2}{2}(1 + \exp(-2\imath\pi q))\check{v}(q) &= 0,\\ (\omega^2 - \Omega_2^2)\check{v}(q) + \frac{\Omega_2^2}{2}(1 + \exp(2\imath\pi q))\check{u}(q) &= 0.\end{aligned}$$

These equations are compatible only if the determinant vanishes, leading to the implicit dispersion relation

$$\boxed{(\omega^2 - \Omega_1^2)(\omega^2 - \Omega_2^2) = \frac{1}{2}\Omega_1^2\Omega_2^2(1 + \cos(ka))}$$

Solving for ω as a function of k we obtain the explicit dispersion relation

$$\boxed{\omega^2 = \frac{1}{2}(\Omega_1^2 + \Omega_2^2) \pm \frac{1}{2}\sqrt{\Omega_1^4 + \Omega_2^4 + 2\Omega_1^2\Omega_2^2\cos(ka)}}$$

Bilinear chain III

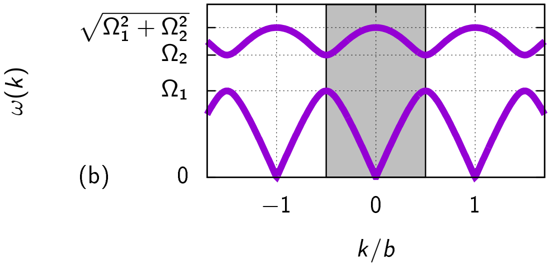


Figure: **Band structure**

- Band gap for frequencies between Ω_1 and Ω_2: there are no propagating Bloch waves.
- Same for $\omega > \Omega_2$.

Bilinear chain IV

We can obtain the explicit dispersion relation $k(\omega)$, with $k = k_r + \imath k_i$:

- First band gap, $k_i a = \pm \cosh^{-1}\left(1 - \left(\frac{\omega^2}{\Omega_1^2} - 1\right)\left(\frac{\omega^2}{\Omega_2^2} - 1\right)\right)$.
- Above the second band, $k_i a = \pm \cosh^{-1}\left(\left(\frac{\omega^2}{\Omega_1^2} - 1\right)\left(\frac{\omega^2}{\Omega_2^2} - 1\right) - 1\right)$.

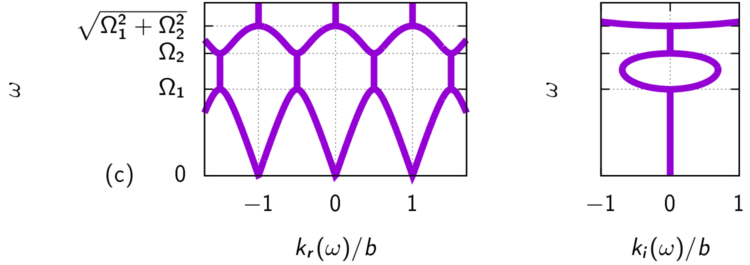


Figure: **Complex band structure**

Bloch theorem

Helmholtz equation with periodic coefficients: $-\nabla \cdot (c(\boldsymbol{r})\nabla u(\boldsymbol{r})) = \omega^2 u(\boldsymbol{r})$

Theorem (Bloch)

The eigenmodes of the periodic Helmholtz equation are Bloch waves of the form

$$u(\boldsymbol{r}) = \exp(-\imath \boldsymbol{k} \cdot \boldsymbol{r})\tilde{u}(\boldsymbol{r})$$

where $\tilde{u}(\boldsymbol{r})$ is a periodic function with the same periodicity as the crystal and $\boldsymbol{k}$ is the Bloch wave vector.

Demonstration: see [2]

One-dimensional sinusoidal grating I

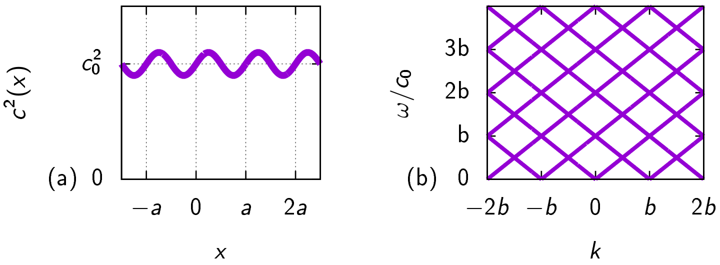


Figure: (a) Sinusoidal modulation. (b) Empty lattice model.

Sinusoidal modulation of the celerity in the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \quad c^2(x) = c_0^2 + c_1^2 \sin(2\pi x/a)$$

One-dimensional sinusoidal grating II

Bloch waves in the form $u(t,x) = \tilde{u}(x)\exp(\imath(\omega t - kx))$. Fourier series representation:

$$\tilde{u}(x) = \sum_{p=-\infty}^{\infty} \tilde{u}_p \exp(-2\imath\pi p x/a).$$

$$\frac{\partial^2 u(t,x)}{\partial t^2} = -\omega^2 \exp(\imath\omega t)\sum_p \tilde{u}_p \exp(-\imath(k+pb)x)$$

$$\frac{\partial^2 u(t,x)}{\partial x^2} = -\exp(\imath\omega t)\sum_p (k+pb)^2 \tilde{u}_p \exp(-\imath(k+pb)x).$$

$$(c_0^2(k+pb)^2 - \omega^2)\tilde{u}_p + \frac{c_1^2}{2\imath}(k+(p+1)b)^2\tilde{u}_{p+1} - \frac{c_1^2}{2\imath}(k+(p-1)b)^2\tilde{u}_{p-1} = 0.$$

$$\begin{pmatrix} \ddots & \ddots & 0 & \\ -\frac{c_1^2}{2\imath}(k-2b)^2 & c_0^2(k-b)^2-\omega^2 & \frac{c_1^2}{2\imath}k^2 & 0 \\ 0 & -\frac{c_1^2}{2\imath}(k-b)^2 & c_0^2k^2-\omega^2 & \frac{c_1^2}{2\imath}(k+b)^2 \\ & 0 & -\frac{c_1^2}{2\imath}k^2 & c_0^2(k+b)^2-\omega^2 \\ & & 0 & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \tilde{u}_{-1} \\ \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \end{pmatrix} = 0.$$

One-dimensional sinusoidal grating III

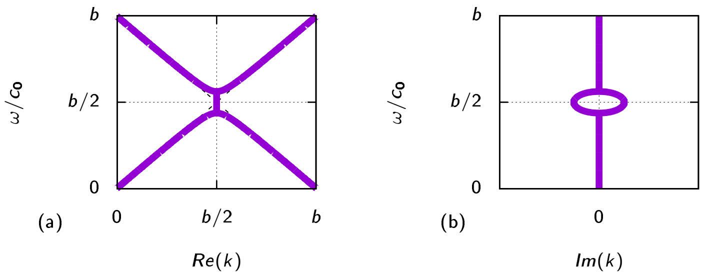


Figure: **Complex band structure $k(\omega)$**

Around the crossing $\omega = c_0 b/2$ and $k = b/2$:

$$\begin{pmatrix} c_0^2(k-b)^2 - \omega^2 & \frac{c_1^2}{2\imath} k^2 \\ -\frac{c_1^2}{2\imath}(k-b)^2 & c_0^2 k^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \tilde{u}_{-1} \\ \tilde{u}_0 \end{pmatrix} = 0,$$

Dispersion relation: $\boxed{(c_0^2(k-b)^2 - \omega^2)(c_0^2 k^2 - \omega^2) - \frac{c_1^4}{4} k^2 (k-b)^2 = 0}$