Bulk waves in sonic crystals

Examples of sonic crystals

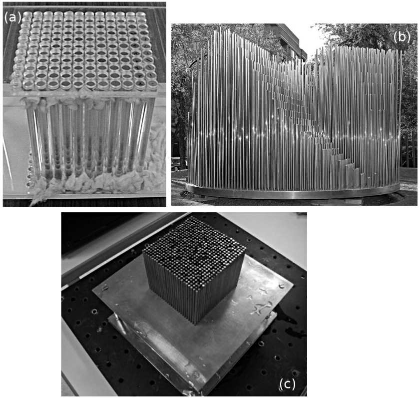


Figure: **Some sonic crystals.** (a) Square lattice crystal of acrylic rods in air [3]. (b) Sculpture by Eusebio Sempere (Fundación Juan March, Madrid) [4]. (c) Phononic crystal of steel rods in water [5].

Basic equations for pressure waves

- Newton's first law:

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\nabla p \quad \text{or} \quad \rho \frac{\partial^2 u_i}{\partial t^2} = -p_{,i}\,.$$

- For a linear and compressible fluid, we have the constitutive relation $p = -BS$ ($S = \nabla . \boldsymbol{u}$ is the strain).
- The material constants are discontinuous.
- Wave equations:

$$\boxed{\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{\partial^2}{\partial t^2} \left(\frac{p}{B} \right)}$$

$$\nabla \left(B \nabla \cdot \boldsymbol{u} \right) = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\,.$$

- Celerity $c = \sqrt{B/\rho}$; impedance $Z = \rho c$

Materials constants

Table: **Material constants for acoustic waves in fluids, at room temperature ($T = 293$ K).** Isotropic solids are considered as effective fluids, i.e., values for longitudinal waves are given ($B = c_{11}$).

	Density ρ $\mathrm{kg/m^3}$	Bulk modulus B GPa	Sound velocity c $\mathrm{m/s}$	Impedance Z $\mathrm{N.s/m^3}$
Air	1.2041	$1.42\ 10^{-4}$	343.4	413.5
Water	1000	2.2	1483	$1.483\ 10^{6}$
Propanol	786	1.076	1170	$1.16\ 10^{6}$
Mercury	13500	28.38	1450	$19.6\ 10^{6}$
Nylon	1150	6.6	2400	$2.76\ 10^{6}$
Plexiglas	1190	9.0	2750	$3.27\ 10^{6}$
Steel	7780	264	5825	$45.3\ 10^{6}$
PVC	1560	7.8	2236	$3.49\ 10^{6}$

A simple loss model

Phenomenologically, modify the constitutive relation:

$$p = -B\left(S + \tau \frac{\partial S}{\partial t}\right),$$

where τ is some time constant. The propagation equation becomes

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla\left(B\nabla\cdot\boldsymbol{u} + \tau\frac{\partial \nabla\cdot\boldsymbol{u}}{\partial t}\right).$$

This is no more a wave equation. Anyway, for monochromatic waves we obtain the complex dispersion relation

$$\omega^2 = c^2(1 + \imath\omega\tau)k^2.$$

In practice, $\boxed{B \longrightarrow B(1 + \imath\omega\tau)}$ becomes complex and dispersive.

Finite element modeling

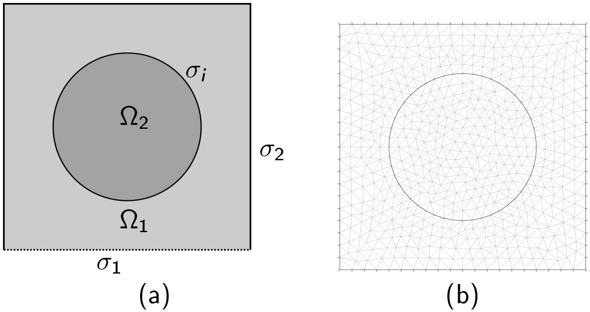


Figure: **A computation domain and its mesh.** (a) The domain Ω is the union of sub-domains Ω_1 and Ω_2. The outer boundary is $\sigma = \sigma_1 \bigcup \sigma_2$. σ_i is an internal boundary. (b) The inner parts of the sub-domains are meshed with triangles. Inside each element, $u(\boldsymbol{x})$ is approximated by $u^e(\boldsymbol{x}) = \sum_j N_j^e(\boldsymbol{x}) u_j^e$ where the $N_j^e(\boldsymbol{x})$ are basis functions.

Weak form of the pressure equation, boundary conditions

- Consider all possible test functions $q(t, \boldsymbol{x})$ belonging to the same finite element space as the pressure and form the scalar products

$$-\int_\Omega \mathrm{d}\boldsymbol{x}\, q \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) + \int_\Omega \mathrm{d}\boldsymbol{x}\, q \frac{1}{B} \frac{\partial^2 p}{\partial t^2} = \int_\Omega \mathrm{d}\boldsymbol{x}\, q f\,.$$

- Using the divergence theorem, the weak form is

$$\int_\Omega \mathrm{d}\boldsymbol{x}\, \nabla q \cdot \left(\frac{1}{\rho} \nabla p\right) - \int_\sigma \mathrm{d}s\, q \left(\frac{1}{\rho} \nabla p\right) \cdot \boldsymbol{n} + \int_\Omega \mathrm{d}\boldsymbol{x}\, q \frac{1}{B} \frac{\partial^2 p}{\partial t^2} = \int_\Omega \mathrm{d}\boldsymbol{x}\, q f\,.$$

- External boundary conditions – free: $\left(\frac{1}{\rho} \nabla p\right) \cdot \boldsymbol{n} = 0$; Dirichlet: $p = p_0$
- Continuity between elements of p and $\left(\frac{1}{\rho} \nabla p\right) \cdot \boldsymbol{n}$ (normal acceleration)

Example of an internal source and radiation BC

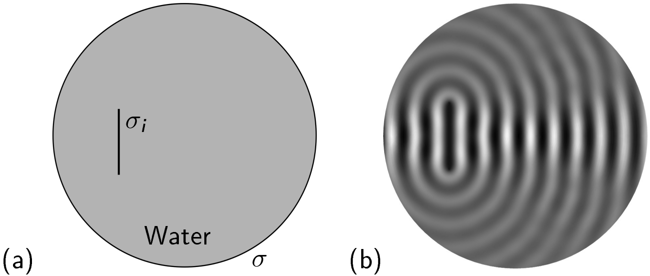


Figure: **Internal source and radiation boundary condition.** (a) The computational domain is a disk of water inside which a linear source is added by prescribing $p = 1$ Pa along internal boundary σ_i. A radiation boundary condition $\left(\frac{1}{\rho}\nabla p\right) \cdot \boldsymbol{n} = -\imath \frac{\omega p}{\rho c}$ is applied at boundary σ. (b) The solution shows the natural diffraction of the acoustic beam radiated from the source. The source dimension is slightly less than 3 wavelengths in water.

FEM for a unit-cell: the band structure of sonic crystals

- Look for Bloch waves in the form $p(\boldsymbol{r}) = \exp(-\imath \boldsymbol{k} \cdot \boldsymbol{r})\tilde{p}(\boldsymbol{r})$, and consider $\tilde{p}(\boldsymbol{r})$ as the unknown field
- In order to obtain the band structure, it is enough to solve the eigenproblem

$$\omega^2 \int_\Omega d\mathbf{r} \left(\tilde{q}^* \frac{1}{B} \tilde{p} \right) = \int_\Omega d\mathbf{r} \left((\nabla \tilde{q} - \imath \mathbf{k} \tilde{q})^\dagger \frac{1}{\rho} (\nabla \tilde{p} - \imath \mathbf{k} \tilde{p}) \right), \forall \tilde{q}$$

- There is no source term and the boundary integral vanishes identically because of the periodic boundary conditions.
- The wave vector $\boldsymbol{k}$ enters directly the variational formulation, and more precisely the stiffness matrix.

FEM for a unit-cell: the mesh

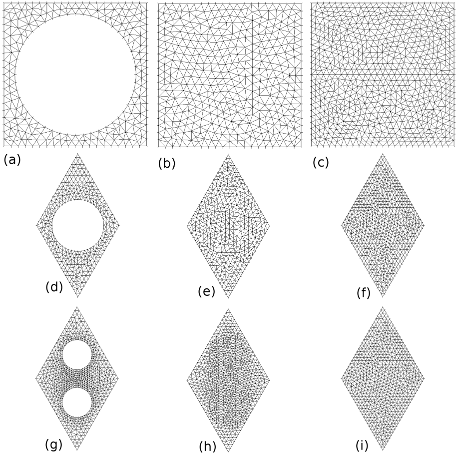


Figure: **Some examples of FEM meshes for various unit cells.**

Rigid cylinders in air (SQ, $d/a = 0.85$)

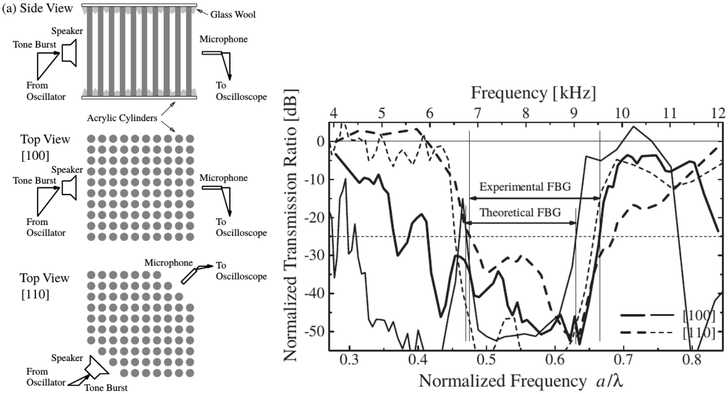
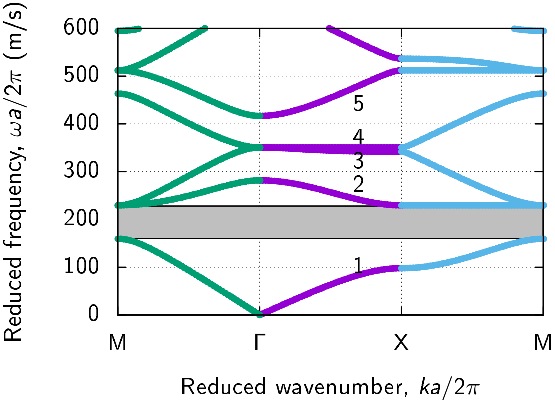
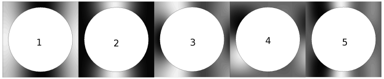


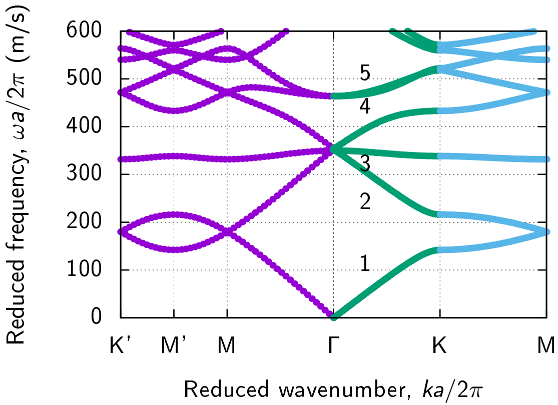
Figure: **A 2D sonic crystal of acrylic cylinders in air.** $a = 24$ mm and $r = 10.2$ mm. The filling fraction is 0.567. [3]

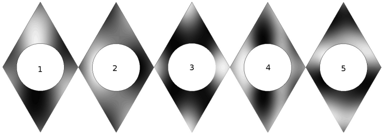
Rigid cylinders in air (SQ, $d/a = 0.85$)



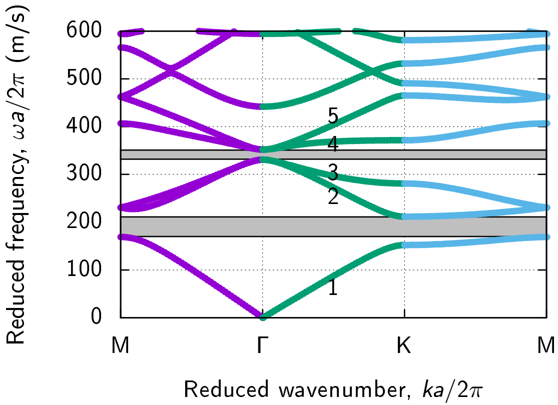


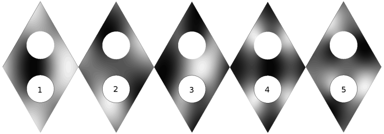
Rigid cylinders in air (HEX, $d/a = 0.63$)





Rigid cylinders in air (HC, $d/a = 0.3636$)





Steel rods in water (SQ, $d/a = 0.82$)

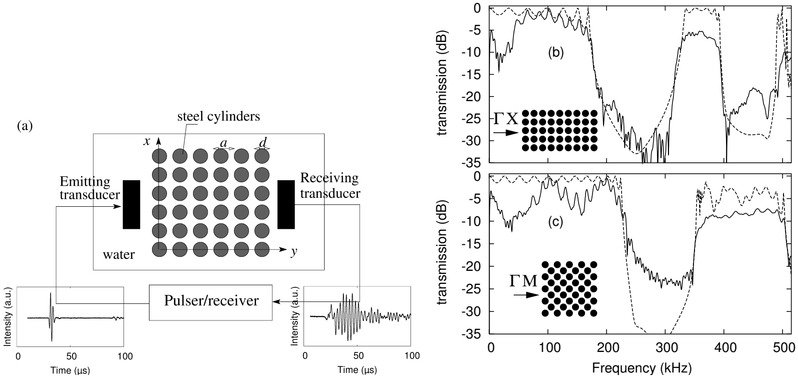


Figure: **A square-lattice sonic crystal of steel rods in water.** $a = 3$ mm and $d = 2.5$ mm (filling fraction 0.54) [6]

Steel rods in water (HEX)

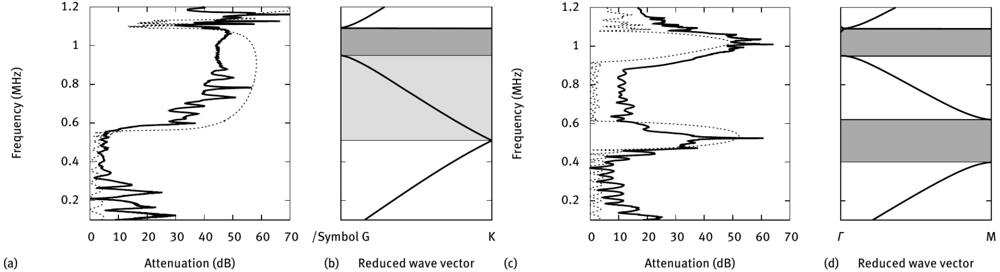


Figure: **A hexagonal-lattice sonic crystal of steel rods in water.** $a = 1.5$ mm and $d = 1.2$ mm (filling fraction 0.58) [7]

Steel rods in water (HC)

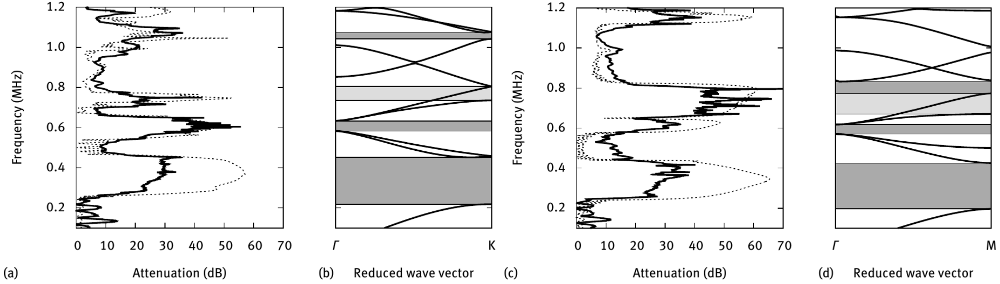


Figure: **A honeycomb-lattice sonic crystal of steel rods in water.** $a = 1.5$ mm and $d = 1.2$ mm (filling fraction 0.387) [7]

FEM meshes for 3D sonic crystals

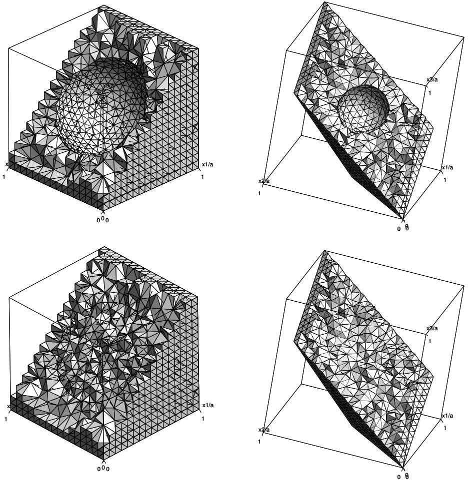


Figure: 3D meshes for simple cubic (SC, left) and face centered cubic (FCC, right) lattice sonic crystals.

Air bubbles in water I

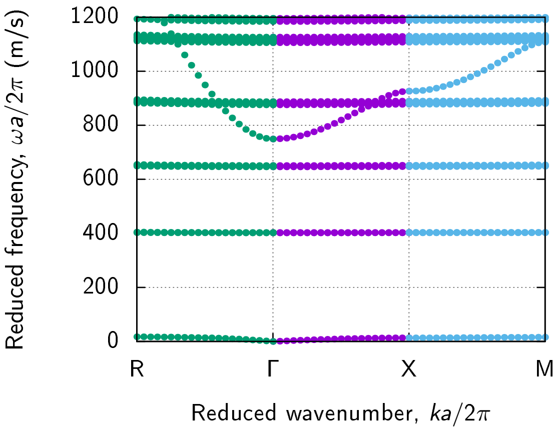


Figure: **Band structure of simple cubic lattice sonic crystal of air bubbles in water.** $d/a = 0.576$.

Air bubbles in water II

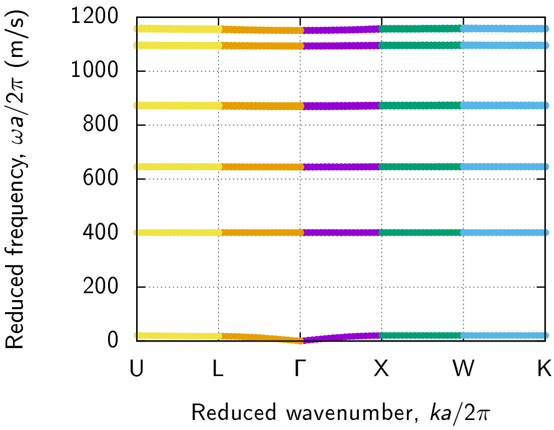


Figure: **Band structure of fcc lattice sonic crystal of air bubbles in water.** $d/a = 0.3628$.

Tungsten carbide beads in water

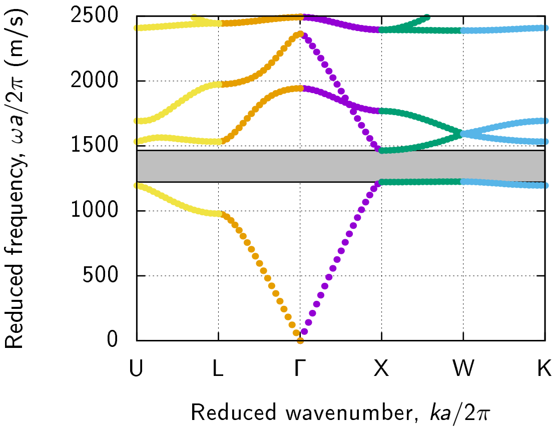


Figure: **Band structure of face centered cubic lattice sonic crystal of tungsten carbide beads in water.** $d/a = 0.707$ (close-packed).