Evanescent Bloch waves

Why complex frequencies and wavenumbers?

- Complex frequencies describe the decay (or attenuation) of vibrations in time.
  - They are most useful for localized resonances.
  - The quality factor of a resonance, Q, measures the number of oscillations a resonator can support before the vibration energy has decreased by a factor $e^{-2\pi} \approx 0.2\%$.
  - Define a complex frequency $\omega_0 = \omega_r(1 + \frac{\imath}{2Q})$, with $\omega_r = \frac{2\pi}{T}$.
- Complex wavenumbers describe the decay in space of outgoing waves.
  - They are well suited for monochromatic waves in periodic media.
  - The complex band structure $\boldsymbol{k}(\omega)$ can describe frequency-dependent material loss [30, 31].
  - It also describes evanescent Bloch waves in relation to periodicity, both in the direction of propagation (Bragg band gaps, local resonances) and in the transverse direction (orders of diffraction).

Evanescent Bloch waves of sonic crystals

- Write the pressure wave equations as

$$\begin{aligned} -\frac{1}{\rho}\nabla \bar{p} + \imath k \boldsymbol{\alpha} \frac{1}{\rho}\bar{p} &= \imath\omega \bar{\mathbf{v}}, \\ \omega^2 \frac{1}{B}\bar{p} &= \nabla \cdot (\imath\omega\bar{\mathbf{v}}) - \imath k \boldsymbol{\alpha} \cdot (\imath\omega\bar{\mathbf{v}}). \end{aligned}$$

- Normal acceleration (or acceleration along the propagation direction):

$$\phi = \imath\omega\boldsymbol{\alpha}\cdot\bar{\mathbf{v}},$$

- Considering a vector of two test functions $(\phi', \bar{p}')$ in the same finite element space as $(\phi, \bar{p})$. Generalized eigenvalue problem [2]:

$$\int_\Omega d\mathbf{x} A(\phi', \bar{p}'; \phi, \bar{p}) = (\imath k) \int_\Omega d\mathbf{x} B(\phi', \bar{p}'; \phi, \bar{p}), \forall(\phi', p'),$$

$$\begin{aligned} A(\phi', \bar{p}'; \phi, \bar{p}) &= \phi'^*\phi + \phi'^*\frac{1}{\rho}(\boldsymbol{\alpha}\cdot\nabla\bar{p}) + \omega^2\bar{p}'^*\frac{1}{B}\bar{p} - (\nabla\bar{p}')^\dagger\frac{1}{\rho}\nabla\bar{p}, \\ B(\phi', \bar{p}'; \phi, \bar{p}) &= \phi'^*\frac{1}{\rho}\bar{p} - (\boldsymbol{\alpha}\cdot\nabla\bar{p}')^*\frac{1}{\rho}\bar{p} - \bar{p}'^*\phi. \end{aligned}$$

Sonic crystal of steel rods in water, 2D SQ I

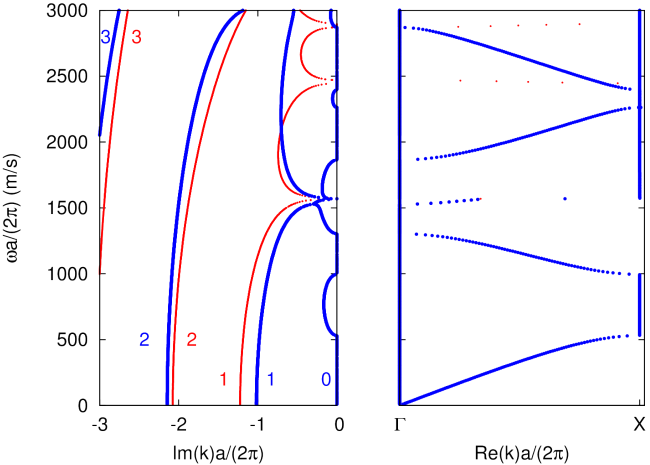


Figure: **Complex band structure.** $d/a = 0.8$. Small dots: antisymmetric (deaf) bands. Large dots: symmetric (non deaf) bands.

Sonic crystal of steel rods in water, 2D SQ II

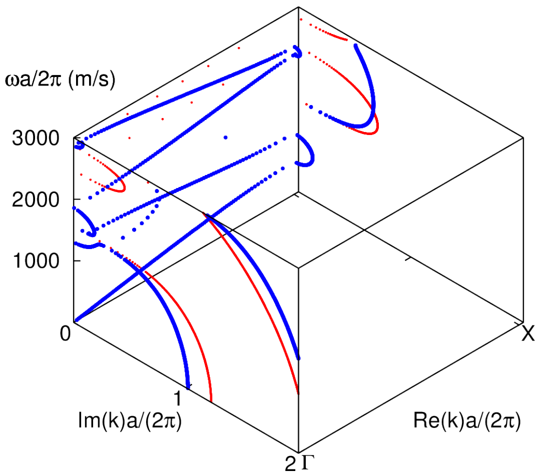


Figure: **Complex band structure.** $d/a = 0.8$. Small dots: antisymmetric (deaf) bands. Large dots: symmetric (non deaf) bands.

Evanescent Bloch waves of phononic crystals

- Assuming monochromaticity and the Bloch-Floquet theorem, we can write for the periodic part of the solution

$$\bar{T}_{ij} = c_{ijkl}(\bar{u}_{k,l} - \imath k \alpha_l \bar{u}_k),$$
$$\bar{T}_{ij,j} - \imath k \alpha_j \bar{T}_{ij} = -\rho\omega^2 \bar{u}_i$$

- Considering a vector of $2r$ test functions (u'_i, τ'_i) living in the same finite element space as (u_i, τ_i), $i = 1, \cdots, r$. Generalized eigenvalue problem [2]

$$\int_\Omega d\mathbf{r} A(\boldsymbol{\tau}', \boldsymbol{u}'; \boldsymbol{\tau}, \boldsymbol{u}) = (\imath k) \int_\Omega d\mathbf{r} B(\boldsymbol{\tau}', \boldsymbol{u}'; \boldsymbol{\tau}, \boldsymbol{u}), \forall(\boldsymbol{\tau}', \boldsymbol{u}')$$

with

$$A(\boldsymbol{\tau}', \boldsymbol{u}'; \boldsymbol{\tau}, \boldsymbol{u}) = (\tau')^*_i \tau_i - c_{ijkl}\alpha_j(\tau')^*_i \bar{u}_{k,l} + \rho\omega^2 (\bar{u}')^*_i \bar{u}_i - c_{ijkl}(\bar{u}')^*_{i,j}\bar{u}_{k,l},$$
$$B(\boldsymbol{\tau}', \boldsymbol{u}'; \boldsymbol{\tau}, \boldsymbol{u}) = -c_{ijkl}\alpha_j\alpha_l(\tau')^*_i \bar{u}_k - c_{ijkl}(\bar{u}')^*_{i,j}\alpha_l \bar{u}_k + (\bar{u}')^*_i \tau_i.$$

Phononic crystal of holes in silicon, 2D SQ, ΓX direction I

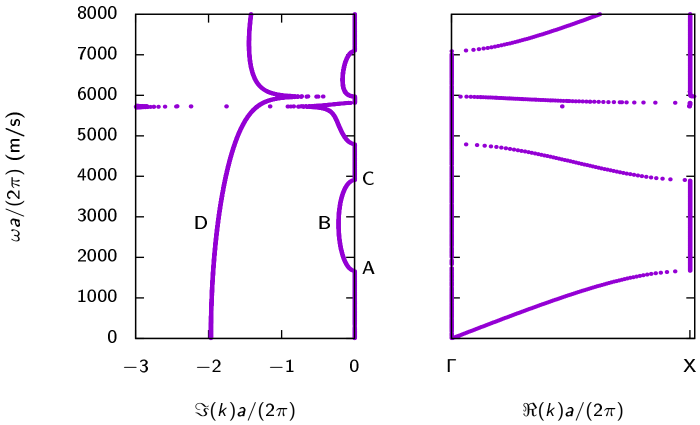


Figure: **Complex band structure** $d/a = 0.85$. Pure shear waves.

Phononic crystal of holes in silicon, 2D SQ, ΓX direction II

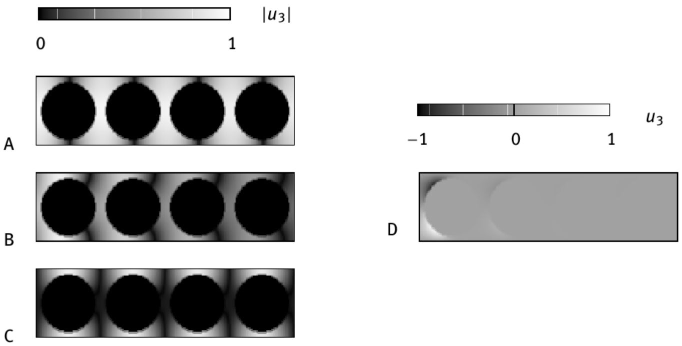


Figure: **Evolution of the polarization across the lower band gap for some pure shear Bloch waves.** Points A, B, and C are for the entrance, the middle, and the exit of the lower band gap, respectively. Point D is for an evanescent Bloch wave at the same frequency as point B.

Phononic crystal of holes in silicon, 2D SQ, ΓX direction III

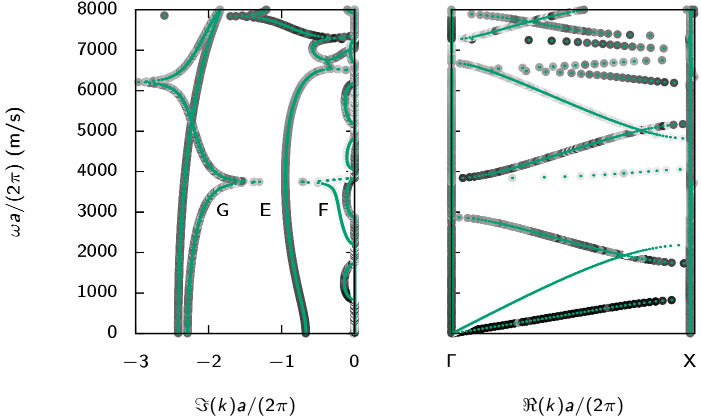


Figure: **Complex band structure.** $d/a = 0.85$. In-plane waves.

Phononic crystal of holes in silicon, 2D SQ, ΓM direction I

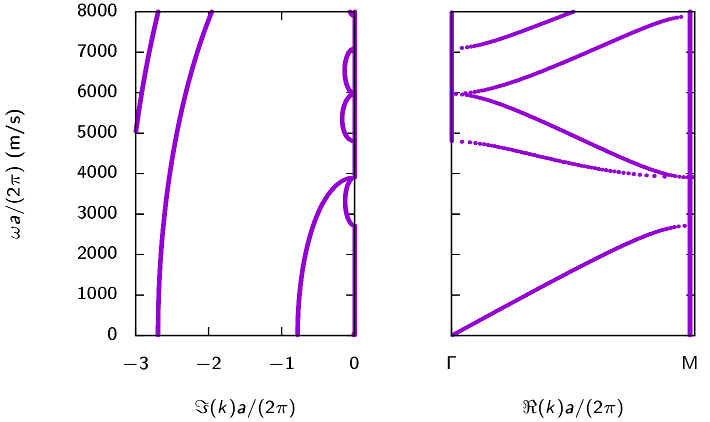


Figure: **Complex band structure.** $d/a = 0.85$. Pure shear waves.

Phononic crystal of holes in silicon, 2D SQ, ΓM direction II

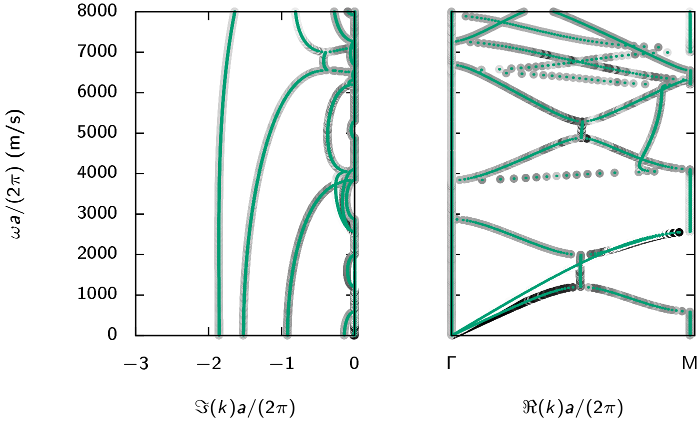
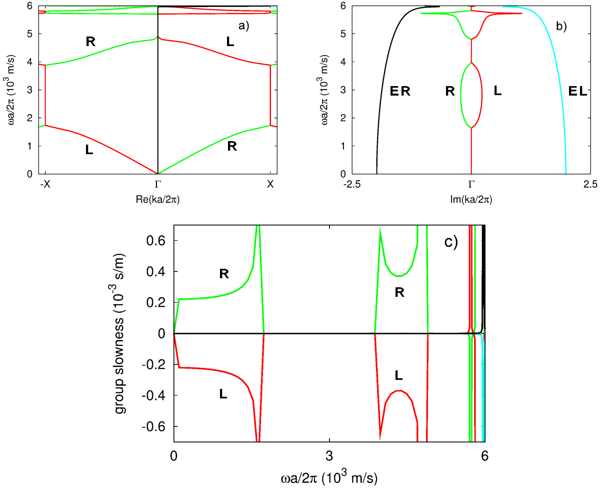


Figure: **Complex band structure.** $d/a = 0.85$. In-plane waves.

Phononic crystal of holes in silicon, 2D SQ, lossless



Phononic crystal of holes in silicon, 2D SQ, lossy