Modeling the response of structured gyro-elastic systems

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Wave Propagation in Complex and Microstructured Media,
Cargése

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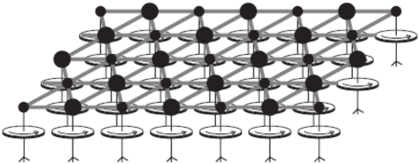
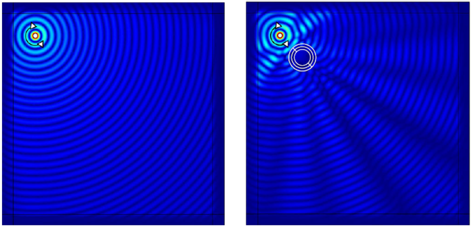
Background: Steady state analysis of chiral periodic structures

1. Brun M., Jones I.S. and Movchan A.B. (2012), *Vortex-type elastic structured media and dynamic shielding*, Proceedings of the Royal Society of London A, **468** , 3027-3046



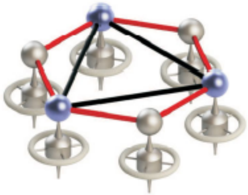


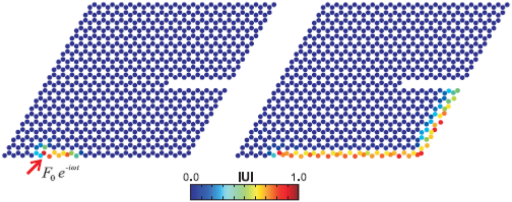
2. Carta G., Brun M., Movchan A.B., Movchan N.V. and Jones I.S. (2014), *Dispersion properties of vortex-type monatomic lattices, International Journal of Solids and Structures*, **51**, 2213-2225

- wave polarisation
- dynamic anisotropy

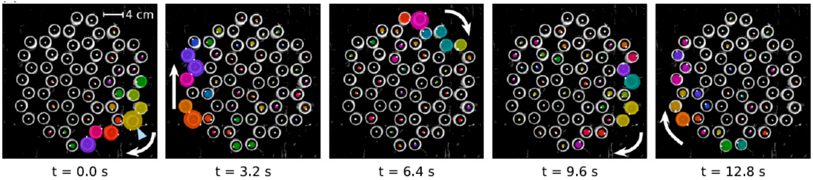
Background: Applications and experiments

3. Wang P., Lu L. and Bertoldi K. (2015), *Topological phononic crystals with one-way elastic edge waves*, Physics Review Letters, **115**, 104302

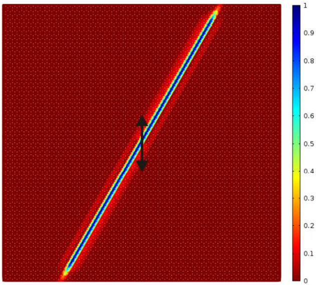


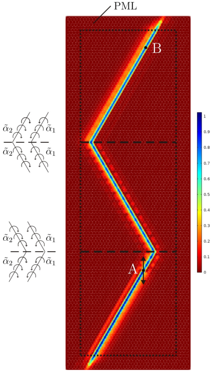


4. Nash L.M., Kleckner D., Read A., Vitelli V., Turner A.M. and Irvine W.T.M. (2015), *Topological mechanics of gyroscopic metamaterials*, PNAS, **112**, 14495-14500



Recent advances: Gaussian beams, localised waveforms



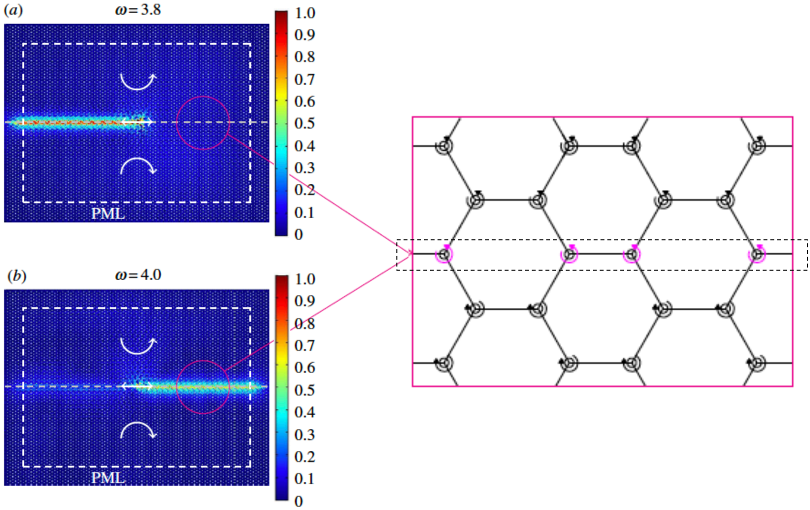


The DASER

Dynamic Amplification by Spinners in Elastic Reticulated systems

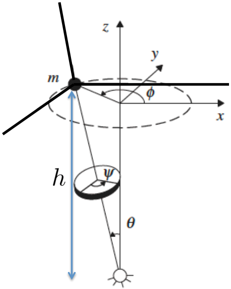
5 Carta, G., Jones, I.S., Movchan, N.V., Movchan, A.B. and Nieves, M.J. (2017): *"Deflecting elastic prism" and unidirectional localisation for waves in chiral elastic systems*, Scientific Reports **7**, no. 1, 1–26.

Recent advances: Uni-directional interfacial waveforms



6. Garau, M., Carta, G., Nieves, M.J., Jones, I.S., Movchan, N.V., Movchan, A.B., (2018): *Interfacial waveforms in chiral lattices with gyroscopic spinners*, Proc. R. Soc. Ser A **474**: 20180132.

Steady state model for arrays of gyroscopes interacting with trusses



Assuming

(i) small nutation θ and a constant spin rate $\dot{\psi}$.

(ii) Moments applied to the gyroscope about the x and y axes are zero.

$\implies$ $\dot{\phi}$ is constant and $\dot{\phi} \propto \omega$

In the time-harmonic regime, the equation of motion can be written as

$$-m\omega^2 \mathbf{u} = -\mathbf{K}\mathbf{u} + \mathrm{i}\beta\omega^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{u} \qquad (1)$$

$\beta = I_z/h^2$ – Spinner constant

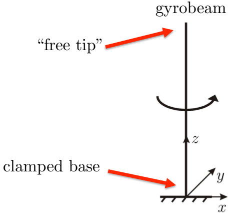
I_z – Moment of Inertia of gyro about z

h – "height" mass from base of gyroscope

1 Brun M., Jones I.S. and Movchan A.B. (2012), *Vortex-type elastic structured media and dynamic shielding*, Proceedings of the Royal Society of London A, **468** , 3027-3046

Gyrobeams

These are flexural elements with “additional stored angular momentum".



PDEs governing a gyrobeam

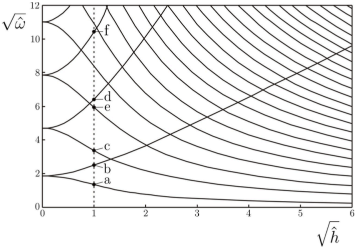
($0 < z < L, t > 0$):

$$EJ\frac{\partial^4 u(z,t)}{\partial z^4} + \rho A\frac{\partial^2 u(z,t)}{\partial t^2} + h\frac{\partial^3 v(z,t)}{\partial z^2 \partial t} = 0$$

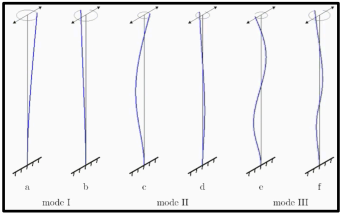
$$EJ\frac{\partial^4 v(z,t)}{\partial z^4} + \rho A\frac{\partial^2 v(z,t)}{\partial t^2} - h\frac{\partial^3 u(z,t)}{\partial z^2 \partial t} = 0$$

h - the gyricity parameter

eigenvalues vs. gyricity h



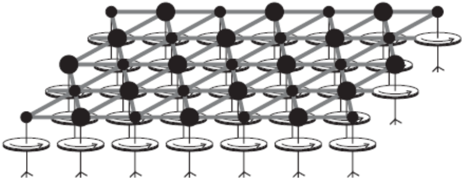
eigenmodes



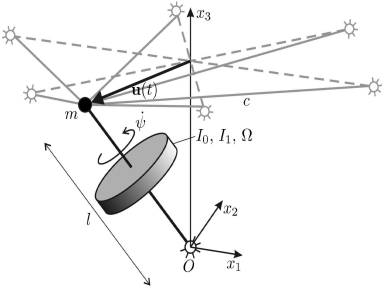
- D’Eleuterio G.M.T. and Hughes P.C. (1984), *Dynamics of gyroelastic continua*, Journal of Applied Mechanics 51, 415-422.

Transient analysis of a gyro-elastic lattice

Gyro-elastic lattice

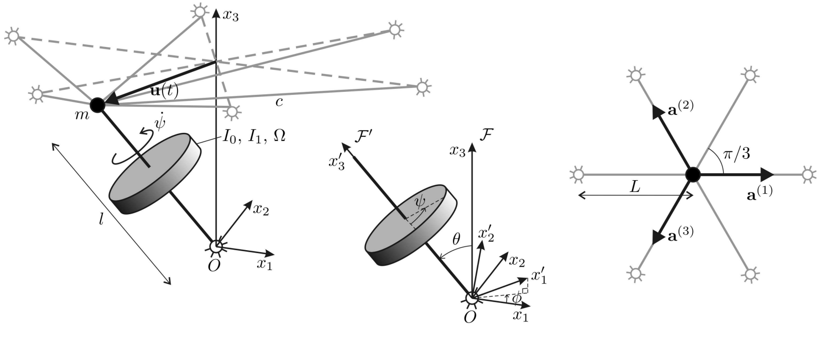


Model of a mass connected to several rods and a gyroscopic spinner



- M. Garau, M.J. Nieves, G. Carta, and M. Brun (2019), *Transient analysis of a gyro-elastic structured medium: unidirectional waveforms, preferential directionality and cloaking*, International Journal of Engineering Science, 143, 115-141.

Modelling the interaction between a gyroscope and the mass



(a) (b) (c)

We aim to derive a linearised transient model describing the motion of the mass in this system. This requires:

- **i)** Linear momentum balance for the mass.
- **ii)** Angular momentum balance of the gyroscope.
- **iii)** Assumptions governing the motion of the system.

Angular momentum balance for the gyroscope

Angular momentum balance:

$$\mathbf{M}_\mathrm{e} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_\mathrm{g}\boldsymbol{\omega}_\mathrm{g})$$

External moments: $\mathbf{M}_\mathrm{e} = \boldsymbol{l} \times \mathbf{F}$

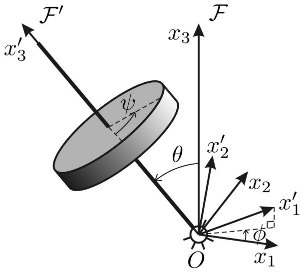
($\boldsymbol{l}$ is the vector representing the arm of $\mathbf{F}$)

Moment of inertia tensor:

$$\mathbf{I}_\mathrm{g} = I_0(\mathbf{e}'_1 \otimes \mathbf{e}'_1 + \mathbf{e}'_2 \otimes \mathbf{e}'_2) + I_1 \mathbf{e}'_3 \otimes \mathbf{e}'_3 \ .$$

Angular velocity vector of gyroscope:

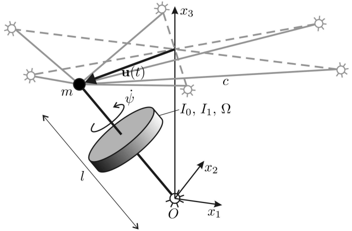
$$\boldsymbol{\omega}_\mathrm{g} = \dot{\theta}\mathbf{e}'_1 + \dot{\phi}\sin(\theta)\mathbf{e}'_2 + (\dot{\psi} + \dot{\phi}\cos(\theta))\mathbf{e}'_3 \ ,$$



In frame $\mathcal{F}'$ we can use Euler's equations to represent this balance. In the fixed frame $\mathcal{F}$, those equations read as:

$$\begin{aligned}
M_1 &= I_0 \frac{\mathrm{d}}{\mathrm{d}t}[-\dot{\phi}\sin(\phi)\sin(\theta)\cos(\theta) + \dot{\theta}\cos(\phi)] + I_1 \frac{\mathrm{d}}{\mathrm{d}t}[\sin(\theta)\sin(\phi)(\dot{\phi}\cos(\theta) + \dot{\psi})] \ , \\
M_2 &= I_0 \frac{\mathrm{d}}{\mathrm{d}t}[\dot{\phi}\cos(\phi)\sin(\theta)\cos(\theta) + \dot{\theta}\sin(\phi)] - I_1 \frac{\mathrm{d}}{\mathrm{d}t}[\sin(\theta)\cos(\phi)(\dot{\phi}\cos(\theta) + \dot{\psi})] \ , \\
M_3 &= I_0 \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\phi}\sin^2(\theta)) + I_1 \frac{\mathrm{d}}{\mathrm{d}t}[\cos(\theta)(\dot{\psi} + \dot{\phi}\cos(\theta))] \ .
\end{aligned}$$

Linear momentum balance for the mass



Linear momentum balance:

$$\mathbf{F} = -c\,\mathbf{h}[\mathbf{u}(t)] - m\ddot{\mathbf{u}}(t)$$

Here, $\mathbf{F}$ represents the force supplied to the mass by the gyroscope, that we should determine.

Nonlinear restoring force of the rods

$$\mathbf{h}[\mathbf{u}(t)] = \sum_{i=1}^{3}\left[(|\mathbf{u}(t) - L\mathbf{a}^{(i)}| - L)\frac{\mathbf{u}(t) - L\mathbf{a}^{(i)}}{|\mathbf{u}(t) - L\mathbf{a}^{(i)}|} + (|\mathbf{u}(t) + L\mathbf{a}^{(i)}| - L)\frac{\mathbf{u}(t) + L\mathbf{a}^{(i)}}{|\mathbf{u}(t) + L\mathbf{a}^{(i)}|}\right] ,$$

In going forward we use the following normalisations:

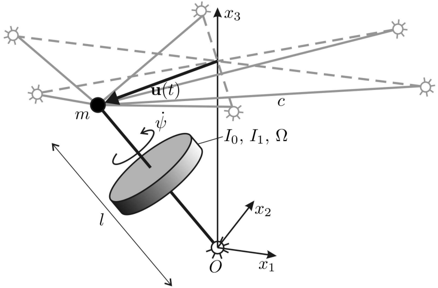
$$\tilde{\mathbf{h}} = \frac{\mathbf{h}}{L} , \quad \tilde{\mathbf{F}} = \frac{\mathbf{F}}{cL} , \quad \tilde{\mathbf{M}}_{\mathrm{e}} = \frac{\mathbf{M}_{\mathrm{e}}}{cLl} , \quad \tilde{I}_j = \frac{I_j}{mlL} \quad (j = 0, 1) ,$$

and

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{L} , \quad \tilde{\boldsymbol{l}} = \frac{\boldsymbol{l}}{l} , \quad \tilde{t} = \sqrt{\frac{c}{m\gamma}}\,t , \quad \gamma = 1 + \frac{I_0}{ml^2} , \quad \delta = \frac{l}{L} .$$

to obtain all quantities in the dimensionless form.

Assumptions on the motion of the system



Position vector of gyroscope tip:

$$
\begin{aligned}
\boldsymbol{l}(t) \;=\; & \delta \sin(\theta(t)) \sin(\phi(t)) \mathbf{e}_1 \\
& - \delta \sin(\theta(t)) \cos(\phi(t)) \mathbf{e}_2 \\
& + \delta \cos(\theta(t)) \mathbf{e}_3 \ ,
\end{aligned}
$$

We assume

I. The connection of the gyroscope with the mass is such that

$$
\begin{aligned}
\mathbf{u}(t) \;&=\; \boldsymbol{l}(t) - \boldsymbol{l}(0) \\
&=\; \delta \sin(\theta(t)) \sin(\phi(t)) \mathbf{e}_1 - \delta \sin(\theta(t)) \cos(\phi(t)) \mathbf{e}_2 + \delta(\cos(\theta(t)) - 1) \mathbf{e}_3 \ ,
\end{aligned}
$$

II. The nutation angle of the gyroscope and its derivatives satisfy

$$
\left| \frac{\mathrm{d}^j \theta(t)}{\mathrm{d}t^j} \right| \leq \text{Const}\ \varepsilon \ , \qquad j = 0, 1, 2 \ ,
$$

Governing equation for the mass

Application of assumption II (nutation angle is small) we find that

$$F_3 = O(\varepsilon^2)\ , \qquad u_3 = O(\varepsilon^2)\ , \qquad M_3 = O(\varepsilon^2)$$

and from the linear momentum balance we have

$$\mathbf{F} = -\mathbf{K}\mathbf{u}(t) - \gamma^{-1}\ddot{\mathbf{u}}(t) + O(\varepsilon^2)\ , \qquad \mathbf{K} = 2\sum_{j=1}^{3} \mathbf{a}^{(j)} \otimes \mathbf{a}^{(j)} = 3\mathbf{I}_2\ ,$$

now $\mathbf{F} = (F_1, F_2)^{\mathrm{T}}$ and $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$. On the other hand, from assumptions I and II with the angular momentum balance give

$$\begin{aligned} F_1 &= \frac{I_0}{\gamma\delta}\ddot{u}_1 + \frac{I_1}{\gamma\delta}[(\ddot{\psi} + \ddot{\phi})u_2 + (\dot{\psi} + \dot{\phi})\dot{u}_2] + O(\varepsilon^3)\ , \\ F_2 &= \frac{I_0}{\gamma\delta}\ddot{u}_2 - \frac{I_1}{\gamma\delta}[(\ddot{\psi} + \ddot{\phi})u_1 + (\dot{\psi} + \dot{\phi})\dot{u}_1] + O(\varepsilon^3)\ , \end{aligned} \qquad \text{and} \qquad \ddot{\psi}(t) + \ddot{\phi}(t) \sim 0$$

We have to leading order the sum of the precession and nutation rate is constant. We define

$$\alpha = \frac{I_1}{(\delta + I_0)} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \Omega = \dot{\psi}(0) + \dot{\phi}(0)\ .$$

where Ω is the gyricity of the gyroscope. Combining the above we obtain

Governing equation for the mass:

$$\ddot{\mathbf{u}}(t) + \alpha\Omega\mathbf{R}\dot{\mathbf{u}}(t) + 3\mathbf{u}(t) = \mathbf{0}$$

Eigenmode analysis of the system

Equation of motion of the mass is:

$$\ddot{\mathbf{u}}(t) + \alpha\Omega\mathbf{R}\dot{\mathbf{u}}(t) + 3\mathbf{u}(t) = \mathbf{0} \quad \text{with} \quad \mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

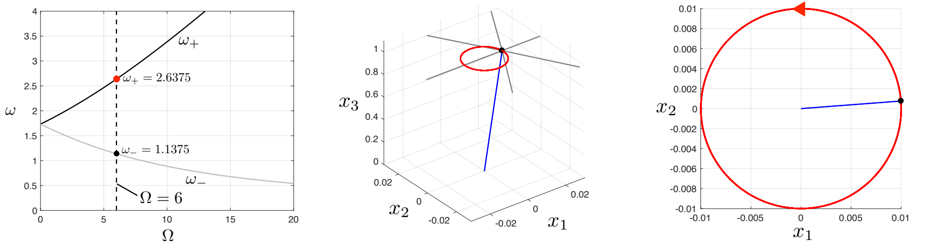
Introduce the time harmonic solution:

$$\mathbf{u}(t) = \mathbf{A}e^{\mathrm{i}\omega t} ,$$

The non-trivial eigenfrequencies and eigenvectors are then:

$$\omega_\pm = \frac{1}{2}\left[\pm\alpha\Omega + \sqrt{(\alpha\Omega)^2 + 12}\right] , \qquad \mathbf{A}(\omega) = \begin{pmatrix} 1 \\ \mathrm{i}\dfrac{3-\omega^2}{\alpha\Omega\omega} \end{pmatrix} .$$

Example: We set $\alpha = 0.25$ and observe dependency of the eigenfrequencies on Ω and some of the modes.



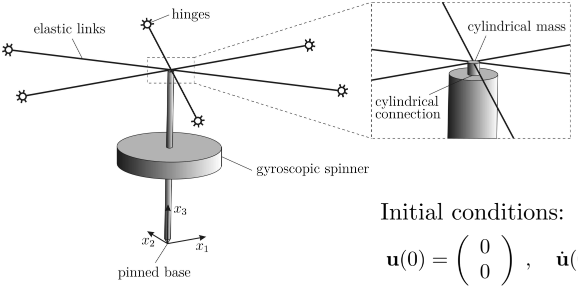
Predicting the motion of the system

$$\mathbf{u}(t) = c_1 \mathbf{A}(\omega_+) e^{\mathrm{i}\omega_+ t} + c_2 \overline{\mathbf{A}(\omega_+)} e^{-\mathrm{i}\omega_+ t} + c_3 \mathbf{A}(\omega_-) e^{\mathrm{i}\omega_- t} + c_4 \overline{\mathbf{A}(\omega_-)} e^{-\mathrm{i}\omega_- t} ,$$

Example: We take $\Omega = 6$, $\alpha = 0.25$ and determine the behaviour of the system after release from some initial configuration.

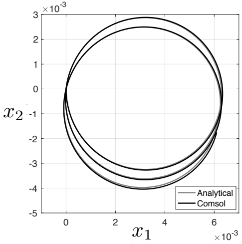
$$\mathbf{u}(0) = \begin{pmatrix} 0 \\ -0.05 \end{pmatrix} , \quad \dot{\mathbf{u}}(0) = \begin{pmatrix} 0.05 \\ 0 \end{pmatrix} .$$

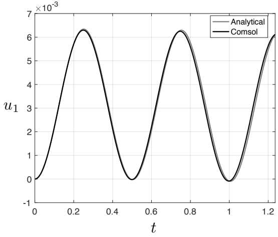
Validation against independent FE simulations

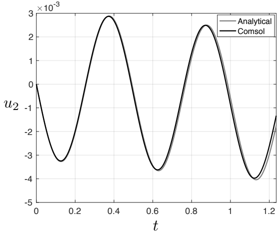


Initial conditions:

$$\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{u}}(0) = \begin{pmatrix} 0 \\ -0.040 \end{pmatrix}$$







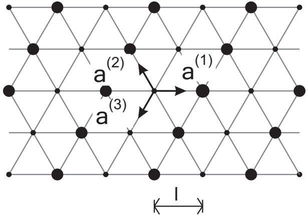
Gyroelastic lattice. Equation of motion

$$
\begin{aligned}
-\frac{m\omega^2}{c}\mathbf{u}^{(\mathbf{n})} = {} & \mathbf{a}^{(1)}\cdot(\mathbf{u}^{(\mathbf{n}+\mathbf{e}_1)}-\mathbf{u}^{(\mathbf{n})})\mathbf{a}^{(1)} + (-\mathbf{a}^{(1)})\cdot(\mathbf{u}^{(\mathbf{n}-\mathbf{e}_1)}-\mathbf{u}^{(\mathbf{n})})(-\mathbf{a}^{(1)}) \\
& +\mathbf{a}^{(2)}\cdot(\mathbf{u}^{(\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2)}-\mathbf{u}^{(\mathbf{n})})\mathbf{a}^{(2)} + (-\mathbf{a}^{(2)})\cdot(\mathbf{u}^{(\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2)}-\mathbf{u}^{(\mathbf{n})})(-\mathbf{a}^{(2)}) \\
& +\mathbf{a}^{(3)}\cdot(\mathbf{u}^{(\mathbf{n}-\mathbf{e}_2)}-\mathbf{u}^{(\mathbf{n})})\mathbf{a}^{(3)} + (-\mathbf{a}^{(3)})\cdot(\mathbf{u}^{(\mathbf{n}+\mathbf{e}_2)}-\mathbf{u}^{(\mathbf{n})})(-\mathbf{a}^{(3)}) \\
& +\frac{\alpha i\omega^2}{c}\mathbf{R}\mathbf{u}^{(n)}
\end{aligned}
$$

where $\mathbf{R}$ is the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and α is the spinners constant



Dispersion equation

$$\det\left[\mathbf{C}(\mathbf{k})-\omega^2(\mathbf{M}-\mathbf{\Sigma})\right]=0$$

where $\mathbf{M}=\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ and $\mathbf{\Sigma}=\begin{pmatrix} 0 & -i\alpha \\ i\alpha & 0 \end{pmatrix}$

Stiffness matrix

$$\mathbf{C}(\mathbf{k})=c\begin{pmatrix} 3-2\cos k_1 l-\dfrac{(\cos\zeta+\cos\xi)}{2} & \dfrac{\sqrt{3}(\cos\xi-\cos\zeta)}{2} \\ \dfrac{\sqrt{3}(\cos\xi-\cos\zeta)}{2} & 3-\dfrac{3(\cos\zeta+\cos\xi)}{2} \end{pmatrix}$$

where $\zeta=\frac{k_1 l}{2}+\frac{\sqrt{3}}{2}k_2 l$ and $\xi=\frac{k_1 l}{2}-\frac{\sqrt{3}}{2}k_2 l$

Explicitly

$$\omega^4(m^2-\alpha^2)-\omega^2 m\,\mathrm{tr}\mathbf{C}+\mathrm{det}\mathbf{C}=0$$

$$\omega_1(\mathbf{k})=\sqrt{\frac{\mathrm{tr}(\mathbf{C})-\sqrt{\mathrm{tr}^2(\mathbf{C})-4(1-(\alpha/m)^2)\mathrm{det}(\mathbf{C})}}{2(1-(\alpha/m)^2)}}$$

$$\omega_2(\mathbf{k})=\sqrt{\frac{\mathrm{tr}(\mathbf{C})+\sqrt{\mathrm{tr}^2(\mathbf{C})-4(1-(\alpha/m)^2)\mathrm{det}(\mathbf{C})}}{2(1-(\alpha/m)^2)}}$$

Three Regimes.

- $\alpha^2 < m^2$ *subcritical*: two dispersion surfaces.
- $\alpha^2 > m^2$ *supercritical*: one dispersion surface.
- $\alpha^2 = m^2$ *critical*: one dispersion surface

$$\omega=(m^{-1}\,\mathrm{det}\mathbf{C}/\mathrm{tr}\mathbf{C})^{1/2}$$